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ORTHOGONAL GROUPS OF DYADIC UNIMODULAR QUADRATIC FORMS II

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Let $O(M)$ be the orthogonal group of a unimodular quadratic form over the integers in a dyadic local field. The subgroups of $O(M)$ normalized by the commutator subgroup are classified when the rank $r(M) \geqq 9$, or when $r(M) \geqq 7$ and the residue class field has at least 8 elements.

Classifications of the subgroups of an orthogonal group normalized by the commutator subgroup have been given by many authors. For isotropic nonsingular quadratic forms over fields there is the fundamental result of Dickson [3] and Dieudonné [4]: The projective commutator subgroup is simple when the form has dimension at least 5. Other proofs of this, which allow the field to have characteristic two, have been given by Eichler [5] and Tamagawa [17]. In [12], Klingenberg generalized this result to nondegenerate quadratic forms over local rings, provided the residue class field is not of characteristic two, and classified the subgroups normalized by the commutator subgroup by using congruence subgroups and mixed commutator subgroups. Klingenberg's work has been further extended in [1, 2, 7-10, 13, 16, 18, 19] by relaxing the restrictions either on the form or on the ring. In particular, I studied this problem for unimodular quadratic forms over the ring of integers in a dyadic local field with 2 an unramified prime and the residue class field having at least 8 elements [9, 10]. These last two restrictions will now be removed, that is, 2 may ramify and there is no restriction on the residue class field (except only that it is perfect).

An outline of the paper follows. Denote by $\mathfrak{o}$ the ring of integers in a dyadic local field F and by M a free $\mathfrak{o}$-module of finite rank $r(M) \geqq 3$ endowed with an isotropic symmetric bilinear form $B: M \times M \rightarrow \mathfrak{o}$ with determinant a unit in $\mathfrak{o}$. After introducing some basic isometries, the commutator subgroup $\Omega(M)$ of the orthogonal group $O(M)$ is determined. Apart for a few exceptional modules M with small rank, $\Omega(M)$ is equal to the spinorial kernel of $O(M)$ and is generated by the Siegel transformations. Next, the "primitive" submodules M_ξ, $\xi \in \Xi$(Ξ a suitable indexing set), invariant under the action of the commutator subgroup are determined. For each ideal $\mathfrak{a}$ in $\mathfrak{o}$, the submodules $\mathfrak{a}M_\xi$ are still invariant and are used to define the subgroups $\mathscr{E}(\mathfrak{a}M_\xi)$ and $\mathscr{F}(\mathfrak{a}M_\xi)$. The main result is:

If $r(M) \geqq 9$, a subgroup $\mathscr{N}$ of the orthogonal group $O(M)$ is *normalized by the commutator subgroup $\Omega(M)$ if and only if it satisfies a ladder relation of the form*

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N} \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$$

for some ideal $\mathfrak{a}$ in $\mathfrak{o}$ and some $\xi \in \Xi$.

The restriction $r(M) \geqq 9$ can be weakened to $r(M) \geqq 7$ if the residue class field has at least 8 elements. If $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$, the subgroups $\mathscr{E}(\mathfrak{a}M_\xi)$ can be characterized as mixed commutator subgroups with the help of congruence subgroups obtained from $\mathfrak{a}M_\xi$. In a subsequent paper we shall indicate how the local structure obtained here can be injected into orthogonal groups over Dedekind domains. In particular, some of the structure of $\mathscr{F}(\mathfrak{a}M_\xi)/\mathscr{E}(\mathfrak{a}M_\xi)$ that transfers to the global situation will be given.

The notation for subgroups in [9, 10] has been slightly modified in the present paper. In particular, $\mathscr{E}_*(\mathfrak{a})$ will now be written as $\mathscr{E}(\mathfrak{a}M_*)$ and the subgroups $\mathscr{E}(\mathfrak{a}, \zeta)$ are now included amongst those denoted $\mathscr{E}(\mathfrak{a}M_\xi)$. Similarly, $\mathscr{F}^*(\mathfrak{a})$ becomes $\mathscr{F}(\mathfrak{a}M^*)$ and corresponding changes will be made for the congruence subgroups.

1. **Preliminaries.** Let V be a finite dimensional vector space over the dyadic local field F of characteristic zero and $q: V \to F$ a quadratic form on V, that is, $q(\alpha x) = \alpha^2 q(x)$ for $\alpha \in F$, $x \in V$ and the symmetric mapping $B: V \times V \to F$ defined by

$$B(x, y) = q(x + y) - q(x) - q(y)$$

is bilinear. Denote by $\mathfrak{o}$ the ring of integers in F, by $\mathfrak{p}$ the maximal ideal in $\mathfrak{o}$ and by $\mathfrak{u}$ the group of units. Assume V supports a unimodular lattice M; thus M is a free $\mathfrak{o}$-module spanning V over F with $B(M, M) = \mathfrak{o}$ and $\det_B(M)$ a unit. Unimodular lattices are discussed in [14; §93D]; we summarize below the main results required.

Fix a prime π in $\mathfrak{o}$ and a normalized valuation ord on F. Thus $\operatorname{ord}\pi = 1$ and $\operatorname{ord} 2 = e \geqq 1$. In

$$q(M) = \{q(x) \mid x \in M\} \subseteqq \frac{1}{2}\mathfrak{o}$$

choose $q(w) = (1/2)a$ such that $\operatorname{ord} a$ is minimal. O'Meara calls a a *norm generator* of M. The *norm group* is

$$\mathfrak{g}M = 2(q(M) + \mathfrak{o}) .$$

Let $\mathfrak{m}M$ denote the largest ideal of $\mathfrak{o}$ in $\mathfrak{g}M$ and define the *weight* $\mathfrak{w}M$ by the equation

$$\mathfrak{w}M = \mathfrak{p}\mathfrak{m}M + 2\mathfrak{o} .$$

Then $\mathfrak{b} \in \mathfrak{o}$ is called a *weight generator* of M if $\mathfrak{b}\mathfrak{o} = \mathfrak{w}M$.

If $r(M) \geqq 5$, or if $r(M) \geqq 3$ and $\operatorname{ord}(\mathfrak{a}\mathfrak{b})$ is even, M is split by a hyperbolic plane. Thus

$$M = H \perp K$$

where $H = \mathfrak{o}u + \mathfrak{o}v$ with $q(u) = q(v) = 0$ and $B(u, v) = 1$. In this manner we can reduce the general unimodular lattice to the form

$$M = H \perp N \perp L$$

where $r(L) \leqq 4$ and $q(N) \subseteqq \mathfrak{o}$. Here N will be an orthogonal sum of hyperbolic planes plus, possibly, the anisotropic binary plane $\langle A(2, 2\rho)\rangle$. In general, $\langle A(\alpha, \beta)\rangle$ denotes a binary unimodular lattice $\mathfrak{o}x + \mathfrak{o}y$ where $2q(x) = B(x, x) = \alpha$, $2q(y) = B(y, y) = \beta$ and $B(x, y) = 1$. For $\alpha \in \mathfrak{u}$, denote by $\langle \alpha \rangle$ a lattice $\mathfrak{o}x$ where $B(x, x) = \alpha$. If $r(L) \geqq 1$, the lattice L obtained above in the splitting of M can be taken as one of the forms given in the following table. Here $\mathfrak{a}$ and $\mathfrak{b}$ are norm and weight generators, $\operatorname{ord} c \geqq \operatorname{ord} \mathfrak{b}$ and $\zeta \in \mathfrak{o}$ (see [14; 93:17–18]). Moreover, when $r(L) = 2$ and $\operatorname{ord}(\mathfrak{a}\mathfrak{b})$ is even, we may take $\mathfrak{b} = 2$. It will be apparent later that the subgroup structure of the orthogonal group $O(M)$ is determined mainly by L.

TABLE I

$r(L)$	L
1	$\mathfrak{o}w = \langle \mathfrak{a} \rangle$
2	$\mathfrak{o}w + \mathfrak{o}z = \langle A(\mathfrak{a}, c)\rangle$
3	$\mathfrak{o}w \perp (\mathfrak{o}x + \mathfrak{o}y) = \langle \mathfrak{a} \rangle \perp \langle A(\mathfrak{b}, 2\zeta)\rangle$
4	$(\mathfrak{o}w + \mathfrak{o}z) \perp (\mathfrak{o}x + \mathfrak{o}y) = \langle A(\mathfrak{a}, c)\rangle \perp \langle A(\mathfrak{b}, 2\zeta)\rangle$

Denote by M_* the sublattice of M consisting of all $r \in M$ with $q(r)$ in $\mathfrak{o}$. Let p and r in M_* be such that $q(p) = B(p, r) = 0$. Then $E(p, r)$ denotes the Siegel transformation defined for $s \in M$ by

$E(p, r)(s) = s - B(p, s)r + B(r, s)p - q(r)B(p, s)p$.

Now assume that M is split by a hyperbolic plane $H = \mathfrak{o}u + \mathfrak{o}v = \langle A(0, 0)\rangle$. In future H denotes this fixed hyperbolic plane. Then $M = H \perp K$ with K unimodular. Denote by $\mathscr{E}$ the subgroup of $O(M)$ generated by the Siegel transformations $E(u, r)$ and $E(v, r)$ with r ranging over $K_* = K \cap M_*$. The isometries Δ and $\Phi(\varepsilon)$, where ε is a unit, are defined by

$$\Delta\colon u \longmapsto v,\ v \longmapsto u,\ s \longmapsto s \quad \text{for} \quad s \in K$$

and

$$\Phi(\varepsilon)\colon u \longmapsto \varepsilon u,\ v \longmapsto \varepsilon^{-1} v,\ s \longmapsto s \quad \text{for} \quad s \in K\,.$$

Finally, if $r \in M$ is such that $q(r) \notin \mathfrak{p}$, denote by $\Psi(r)$ the symmetry about r defined by

$$\Psi(r)(s) = s - q(r)^{-1}B(r, s)r .$$

PROPOSITION 1.1. *The following relations hold.*

$$\Delta\Phi(\varepsilon)\Delta^{-1} = \Phi(\varepsilon^{-1}) .$$

For $\theta \in O(M)$, $q(p) = B(p, r) = 0$ *and* $r \in M_*$

$$\theta E(p, r)\theta^{-1} = E(\theta(p), \theta(r)) .$$

In particular,

$$\begin{aligned} \Delta E(u, r)\Delta^{-1} &= E(v, r) \\ \Phi(\varepsilon)E(u, r)\Phi(\varepsilon^{-1}) &= E(u, \varepsilon r) \\ \Phi(\varepsilon)E(v, r)\Phi(\varepsilon^{-1}) &= E(v, \varepsilon^{-1}r) . \end{aligned}$$

Also, for $r, s \in M_*$ *with* $B(p, r) = B(p, s) = 0$,

$$E(p, r)E(p, s) = E(p, r + s) .$$

Proof. These are well-known and easy to verify.

Perhaps less well-known are the following two identities.

PROPOSITION 1.2. *Let* $M = H \perp K$. *Let* $r \in K_*$ *and* $\beta \in \mathfrak{o}$ *be such that* $\varepsilon = 1 - \beta q(r)$ *is a unit. Then*

$$E(v, \beta r)E(u, r) = E(u, \varepsilon^{-1}r)E(v, \beta\varepsilon r)\Phi(\varepsilon^{-2}) .$$

Let $s \in K$ *have* $q(s)$ *a unit. Then*

$$\Delta\Psi(s) = \Phi(-q(s))E(v, s)E(u, q(s)^{-1}s)E(v, s) .$$

Proof. These can be verified by checking the images of u, v and $t \in K$. Alternatively (see [10]), they can be established by making suitable calculations in the Clifford algebra of V.

COROLLARY 1.3. *Let* $M = H \perp K$ *and* $q(K)$ *contain a unit. Then for all units* ε *in* $\mathfrak{u}$, $\Phi(\varepsilon^2) \in \mathscr{E}$. *In particular, the hypothesis is satisfied if* $r(M) \geqq 5$.

Proof. Take $r \in K$ with $q(r)$ a unit and put $\beta = q(r)^{-1}(\varepsilon - 1)$. The result now follows from the first identity in Proposition 1.2. If $r(K) \geqq 3$ and $\operatorname{ord}(ab)$ is even, then K is split by a hyperbolic plane and consequently represents units. If $\operatorname{ord}(ab)$ is odd, either

$\operatorname{ord}(2a^{-1})$ or $\operatorname{ord}(2b^{-1})$ is even, and K clearly represents units.

The *characteristic set* $\mathscr{M}(s)$ of a primitive element s in M is defined by

$$\mathscr{M}(s) = \{t \in M \mid B(s, t) = 1\} .$$

Since M is unimodular, $\mathscr{M}(s)$ is not empty. For any isometry $\varphi \in O(M)$,

$$q(\mathscr{M}(\varphi(s))) \equiv q(\mathscr{M}(s)) \bmod \mathfrak{o} .$$

This relation controls the equivalence of elements of M under the action of the orthogonal group (see Hsia [6]).

If $M = M_1 \perp M_2$, denote by $O(M_1)$ the subgroup of isometries in $O(M)$ that act identically on M_2. Let $i(M)$ denote the Witt index of M.

PROPOSITION 1.4. *Let $M = H \perp K$ and $q(K)$ contain a unit. If* card $\mathfrak{o}/\mathfrak{p} = 2$, *assume also $r(M) \geqq 7$, or $i(M) = 1$, or $M = H \perp H' \perp K'$ and $q(K')$ contains a unit. Then for each $\varphi \in O(M)$ there exists an isometry $\psi \in \mathscr{E}$ such that*

$$\psi\varphi\psi^{-1} = E(u, x)E(v, y)\Phi(\varepsilon)\theta$$

where $x, y \in K_*$, ε is a unit and $\theta \in O(K)$.

Proof. The proof of Lemma 3.6 (3), (4) in [9] is modified as follows.

(3) Assume $\alpha, \beta \in \mathfrak{p}$. Then s is primitive in K. The characteristic set of v is

$$\mathscr{M}(v) = \{z \in M \mid B(v, z) = 1\} = u + (K \perp \mathfrak{o}v) .$$

Since $\varphi(\mathscr{M}(v)) = \mathscr{M}(\varphi(v))$, there exists $t_1 \in \mathscr{M}(\varphi(v))$ such that $q(t_1)$ is a unit. Let t be the component of t_1 in K. Then $q(t) \in \mathfrak{o}$. Also, since $B(t_1, \varphi(v)) = 1$ and $\alpha, \beta \in \mathfrak{p}$, it follows that $B(s, t)$ is a unit. Hence $B(s, t) + \alpha q(t)$ is a unit.

(4) Finally assume α is a unit and $\beta \in \mathfrak{p}$. If card $\mathfrak{o}/\mathfrak{p} \geqq 4$, the earlier version still holds. There remains the case card $\mathfrak{o}/\mathfrak{p} = 2$ and $B(s, t)$ a unit. Then $K = (\mathfrak{o}s + \mathfrak{o}t) \perp K'$. If $q(s)$ is a unit, replace t by s. Otherwise $\mathfrak{o}s + \mathfrak{o}t$ is a hyperbolic plane H'. Now choose a new $t \in K'$ with $q(t)$ a unit and $B(s, t) = 0$. This completes the proof.

2. Generators for $O(M)$. In this section we obtain generators for the orthogonal group $O(M)$ (see also O'Meara and Pollak [15]).

PROPOSITION 2.1. *Let* $M = H \perp N \perp L$ *where* $q(N) \subseteqq \mathfrak{o}$. *Then the orthogonal group* $O(M)$ *is generated by* $\mathscr{E}$ *and* $O(H \perp L)$.

Proof. The proof of Theorem 2.1(1) in [9] or of the lemma in [10; IV] generalizes without significant change.

REMARK 2.2. Let $w, z \in L$ be such that $B(w, z) = 1$ and $q(z) \in \mathfrak{o}$ (but not necessarily $q(w) \in \mathfrak{o}$). The argument in [9; Theorem 2.1(1)] also shows that $\varphi \in O(H \perp L)$ can be changed by isometries in $\mathscr{E}$, and Δ, to an isometry fixing w. This fact will be used later.

It is clear that $O(H)$ is generated by Δ and the isometries $\Phi(\varepsilon)$. We now obtain generators for $O(H \perp L)$ where L is as in Table I.

2.3. *Let* $L = \mathfrak{o}w = \langle a \rangle$. *Then* $O(M)$ *is generated by* $O(H)$ *and* $\mathscr{E}$, *together with* $\Psi(w)$ *if* 2 *is tamely ramified* (ord 2 *odd*).

Proof. By Proposition 2.1 it suffices to consider $O(H \perp \mathfrak{o}w)$. Let $\varphi \in O(H \perp \mathfrak{o}w)$ and

$$\varphi(w) = \beta u + \gamma v + \delta w$$

where δ is a unit. Then

$$(\mathfrak{o}\varphi(w))^{\perp} = \mathfrak{o}(a\delta v - \beta w) + \mathfrak{o}(a\delta u - \gamma w) \cong H .$$

It follows that $q(\beta w)$ and $q(\gamma w)$ are in $\mathfrak{o}$. Assume $2 \in \mathfrak{o}(1 - \delta)$ (a similar argument will hold if instead $2 \in \mathfrak{o}(1 + \delta)$). Using

$$\beta\gamma = \frac{1}{2}a(1 - \delta^2) ,$$

it follows that

$$E(u, \gamma^{-1}(1 + \delta)w)\varphi(w) = \gamma v - w .$$

Then

$$\Psi(w)E(v, a^{-1}\gamma w)(\gamma v - w) = w ,$$

and we have reduced φ to an isometry in $O(H)$. When 2 is wildly ramified, $\Psi(w)$ can be expressed in terms of the isometries in $O(H)$ and $\mathscr{E}$ using the second identity in Proposition 1.2.

2.4. *Let* $L = \mathfrak{o}w + \mathfrak{o}z = \langle A(a, c) \rangle$. *Then* $O(M)$ *is generated by* $O(H)$ *and* $\mathscr{E}$, *together with* $\Psi(w - az)$ *if* ord $(2a)$ *is odd, and with* $\Psi(z)$ *if* ord $(2c^{-1})$ *is odd and positive.*

Proof. We first change $\varphi \in O(M)$ by the given isometries to an isometry fixing w. If $q(z) = (1/2)c \in \mathfrak{o}$, Remark 2.2 gives this immediately. We therefore assume $\operatorname{ord}(ac)$ is odd (otherwise, change z and increase $\operatorname{ord} c$). Again, by Proposition 1.4, since $q(L)$ now contains a unit, assume $\varphi \in O(L)$.

Let $\operatorname{ord}(2c^{-1}) = g \geqq 1$ and $h = [(g+1)/2]$ (integral part). Then $q(\pi^h z) \in \mathfrak{o}$ and $E(u, \pi^h z)(w) = \pi^h u + w$. Let

$$\varphi(\pi^h u + w) = \pi^h u + w + \lambda w + \mu z .$$

If $x = \pi^{-h}(\lambda w + \mu z)$ is in L_*, replacing φ by

$$\psi = E(u, -\pi^h z)E(v, x)\varphi E(u, \pi^h z)$$

gives the desired reduction since $\psi(w) = w$.

From $q(w) = q(\varphi(\pi^h u + w))$, it follows that

$$\frac{1}{2}a\lambda^2 + \frac{1}{2}c\mu^2 = -a\lambda - (\lambda + 1)\mu \in \mathfrak{o}$$

and hence, since $\operatorname{ord}(ac)$ is odd, $l = \operatorname{ord}\lambda \geqq (1/2)\operatorname{ord}(2a^{-1})$ and $m = \operatorname{ord}\mu \geqq h \geqq (1/2)g$. Let f be the minimum order of the four terms in this equation, that is

$$f = \min\{2l - \operatorname{ord}(2a^{-1}),\ 2m - g,\ l + \operatorname{ord} a,\ m\} .$$

Assume $f < g$. If $f = m$, then $2m - g \geqq m$ leads to a contradiction. Similarly, $f = l + \operatorname{ord} a$ gives a contradiction with $2l - \operatorname{ord}(2a^{-1}) \geqq f$. Since there must be at least two terms with the minimum order, this leaves

$$f = 2m - g = 2l - \operatorname{ord}(2a^{-1})$$

which contradicts the hypothesis that $\operatorname{ord}(ac)$ is odd. Hence $f \geqq g$. This will now be strengthened to $f \geqq 2h$, which ensures that $x \in L_*$ as required.

If g is even, $2h = g$. Assume, therefore, $g = 2h - 1$ and $f = g$. Considering again the definition of f, both $f = l + \operatorname{ord} a$ and $f = 2l - \operatorname{ord}(2a^{-1})$ (which is even) lead to contradictions. Hence $f = m = 2m - g = g$ and $(1/2)c\mu + \lambda + 1 \equiv 0 \bmod \pi$. Replace φ by $\Psi(z)\varphi$ and the new coefficient of z lies in $\pi^{g+1}\mathfrak{o}$. Repeating the previous calculations now gives $f \geqq g + 1 = 2h$.

We may now assume $\varphi(w) = w$. Modifying the argument in 2.3, we now reduce φ to an isometry in $O(H)$. Let $r = w - az$ so that $B(r, w) = 0$. Since $B(\varphi(z), w) = 1$,

$$\varphi(z) = \alpha u + \beta v + \gamma r + z$$

for some $\alpha, \beta, \gamma \in \mathfrak{o}$. Computing characteristic sets gives

$$\mathscr{M}(z) = w + H \perp \mathfrak{o}(z - cw)$$

and

$$2q(\mathscr{M}(\varphi(z))) \equiv 2q(\mathscr{M}(z)) \equiv \{a + \rho^2 c(1 + ac) \mid \rho \in \mathfrak{o}\} \bmod 2\mathfrak{o} .$$

Since $v + (1 - \alpha)w \in \mathscr{M}(\varphi(z))$, and either $c \in 2\mathfrak{o}$ or $\operatorname{ord}(ac)$ is odd, it follows that $a\alpha^2 \in 2\mathfrak{o}$. Similarly, $a\beta^2 \in 2\mathfrak{o}$. Let $\sigma = -B(r, z) - \gamma q(r)$. Then $q(z) = q(\varphi(z))$ gives $a\gamma\sigma = a\alpha\beta \in 2\mathfrak{o}$. But $\operatorname{ord}(\gamma q(r)) = \operatorname{ord}(1/2)(a\gamma)$ and hence $\sigma \in \mathfrak{o}$. If σ is a unit,

$$E(u, \alpha\sigma^{-1}r)\varphi(z) = \beta v + z .$$

Similarly, the coefficient of v can be made zero and we obtain an isometry fixing both w and z. If, however, σ is not a unit, then $\gamma q(r)$ is a unit (since $B(r, z)$ is a unit). In $\Psi(r)\varphi(z)$ the new coefficient of r becomes $\sigma q(r)^{-1}$. Now proceed as before. Note that if $\operatorname{ord}(2a)$ is even, $\Psi(r)$ can be expressed in terms of the elements of $\mathscr{E}$ and $O(H)$. This completes the reduction.

2.5. *Let* $L = \mathfrak{o}w \perp (\mathfrak{o}x + \mathfrak{o}y) = \langle a \rangle \perp \langle A(b, 2\zeta) \rangle$ *with* $\operatorname{ord} b$ *odd. Then* $O(M)$ *is generated by* $O(H)$ *and* $\mathscr{E}$, *together with one symmetry* $\Psi(r)$ *where* $\operatorname{ord}(q(r))$ *is odd.*

Proof. Let $\varphi \in O(M)$. Since $B(w + x, y) = 1$ and $q(y) \in \mathfrak{o}$, by Remark 2.2, φ can be assumed to have the property $\varphi(w + x) = w + x$. But

$$L = \mathfrak{o}(w + x) \perp (\mathfrak{o}(ay - w) + \mathfrak{o}(ax - bw)) .$$

The result now follows from 2.4 since $B(ay - w, ax - bw)$ is a unit, $2q(ay - w) = a(1 + 2a\zeta)$ and $2q(ax - bw) = ab(a + b)$.

2.6. *Let* $L = (\mathfrak{o}w + \mathfrak{o}z) \perp (\mathfrak{o}x + \mathfrak{o}y) = \langle A(a, c) \rangle \perp \langle A(b, 2\zeta) \rangle$ *with* $\operatorname{ord}(ab)$ *odd. Then* $O(M)$ *is generated by* $O(H)$ *and* $\mathscr{E}$, *together with one symmetry* $\Psi(r)$ *where* $\operatorname{ord}(q(r))$ *is odd.*

Proof. Let $\varphi \in O(M)$. By Remark 2.2, we may assume $\varphi(x) = x$. If it can also be arranged that $\varphi(y) = y$, invoking 2.4 will complete the proof. Changing z if necessary, we may assume that either $c \in 2\mathfrak{o}$ or $\operatorname{ord}(ac)$ is odd.

If $c \in 2\mathfrak{o}$, using 2.2 again, we also have $\varphi(w) = w$. When $c \in 2\mathfrak{o}$, let $g = \operatorname{ord}(2b^{-1})$ and $h = [(g + 1)/2]$. Now put $s = \pi^h(x - by)$ so that $q(s) \in \mathfrak{o}$. If, however, $c \notin 2\mathfrak{o}$ so that $\operatorname{ord}(ac) \equiv \operatorname{ord}(ab) \bmod 2$, let $2h = \operatorname{ord}(cb^{-1}) \geqq 0$. Since $\operatorname{ord}(ac)$ is odd, there exists $r \in \mathfrak{o}w + \mathfrak{o}z$ such that

$$s = r + \pi^h(x - by)$$

is in M_*. Moreover, $B(s, w)$ is a unit, and by Remark 2.2 we can change φ so that again $\varphi(w) = w$.

Examining the proof of Proposition 1.4(1), we find that either φ or $\Delta\varphi$ can be expressed in the form $E(u, p_1)E(v, p_2)\Phi(\varepsilon)\theta$ where $\theta \in O(L)$ and $B(p_i, x) = B(p_i, w) = 0$ for $i = 1, 2$ (since the conditions $\varphi(x) = x$ and $\varphi(w) = w$ ensure that the component of $\varphi(v)$ in H is primitive). To prove 2.6 it now suffices to show that any $\varphi \in O(L)$ with $\varphi(x) = x$ and $\varphi(w) = w$ can be expressed in terms of the given generators.

We still have h and s available as constructed. In both cases,

$$E(u, s)(y) = \alpha u + y$$

where $\alpha = B(s, y) = \pi^h(1 - 2b\zeta)$. Note that $E(u, s)$ leaves x fixed. Let

$$\varphi(\alpha u + y) = \alpha u + \beta(w - az) + \gamma(x - by) + y$$

where $\beta, \gamma \in \mathfrak{o}$. Since $q(y) = q(\varphi(\alpha u + y))$, it follows that $a\beta^2 + b\gamma^2 \in 2\mathfrak{o} + ac\mathfrak{o}$, and hence $\operatorname{ord}\beta \geqq h$ and $\operatorname{ord}\gamma \geqq h$ (in fact, $\operatorname{ord}\beta \geqq h + 1$ if $c \notin 2\mathfrak{o}$). Thus

$$\varphi(\alpha u + y) = \alpha u + \pi^h t + y$$

where $t \in L$ and $B(t, x) = 0$. Suppose that $q(t) \in \mathfrak{o}$. Then

$$E(u, -s)E(v, \alpha^{-1}\pi^h t)\varphi E(u, s)(y) = y$$

and changing φ by elements in $O(H)$ and $\mathscr{E}$ we have obtained an isometry acting identically on x and y. This, by 2.4, would complete the proof. If $c \notin 2\mathfrak{o}$ we need one symmetry in 2.4; this is also true if $c \in 2\mathfrak{o}$ and $\operatorname{ord}(2a)$ is odd. When $c \in 2\mathfrak{o}$ and $\operatorname{ord}(2a)$ is even, the symmetry will appear below.

It remains to show $q(t) \in \mathfrak{o}$. Since

$$\pi^{2h}q(t) = -\pi^h B(t, y) = \gamma(1 - 2b\zeta)\ ,$$

it suffices to show $\operatorname{ord}\gamma \geqq 2h$. Again, from $q(y) = q(\varphi(\alpha u + y))$,

$$a\beta^2 + b\gamma^2 + 2\gamma \equiv \begin{cases} 0 \bmod b\pi^{4h} & \text{if} \quad c \notin 2\mathfrak{o} \\ 0 \bmod 2\pi^{2h} & \text{if} \quad c \in 2\mathfrak{o}\ . \end{cases}$$

Except when $c \in 2\mathfrak{o}$, $\operatorname{ord}(2b^{-1}) = 2h - 1 = \operatorname{ord}\gamma$ and $2 + b\gamma \equiv 0 \bmod 2\pi$, we can conclude that $\operatorname{ord}\gamma \geqq 2h$. In the exceptional case, replace φ by $\Psi(x - by)\varphi$ and the new coefficient of x (the new γ) is divisible by π^{2h}. This completes the proof.

THEOREM 2.7. *Let M be a unimodular $\mathfrak{o}$-lattice split by a hyperbolic plane H. Then the orthogonal group $O(M)$ is generated by*

$O(H)$ and $\mathscr{E}$, together with at most one symmetry $\Psi(r)$. The symmetry $\Psi(r)$ is required if and only if M contains an element r with

$$\operatorname{ord}(q(r)^{-1}) \geqq 1$$

and odd.

Proof. This merely summarizes the results 2.3–2.6.

COROLLARY 2.8. *Any element $\varphi \in O(M)$ can be expressed in the form*

$$\varphi = \Delta^c \Phi(\varepsilon) \Psi(r)^d \theta$$

where $c, d \in \{0, 1\}$, ε is a unit and $\theta \in \mathscr{E}$. In particular, $d = 0$ if M does not contain an element r with $\operatorname{ord}(q(r)^{-1}) \geqq 1$ and odd.

Proof. This follows immediately from Theorem 2.7 and Proposition 1.1.

Let Θ denote the spinor norm on the special orthogonal group $SO(V)$ and $Sk(M)$ the spinorial kernel in $O(M)$,

$$Sk(M) = \{\varphi \in SO(M) \mid \Theta(\varphi) = 1\} .$$

THEOREM 2.9. *Let $M = H \perp K$ and assume $q(K)$ contains a unit of $\mathfrak{o}$. Then*

$$Sk(M) = \mathscr{E} .$$

Proof. It is well-known that the isometry $E(p, s)$ has spinor norm 1. Hence $\mathscr{E} \subseteqq Sk(M)$ always. Conversely, let

$$\varphi = \Delta^c \Phi(\varepsilon) \Psi(r)^d \theta$$

have spinor norm 1. Then $\det \varphi = 1$ gives $c = d$. Now $\Theta(\varphi) = (-q(r))^c \varepsilon$, since $\Delta = \Psi(u - v)$ and $\Phi(\varepsilon) = \Delta\Psi(u - \varepsilon v)$. If r exists, $\operatorname{ord}(q(r))$ is odd, and hence $c = 0$ and $\varepsilon = \eta^2$ for some unit η. Corollary 1.3 now gives $Sk(M) \subseteqq \mathscr{E}$.

3. **$\mathscr{E}$-invariant sublattices.** We now study the $\mathscr{E}$-invariant sublattices of M, that is, sublattices of M that are invariant under the action of $\mathscr{E}$. For $O(M)$-invariant sublattices, see [11].

Clearly, $M_* = \{s \in M \mid q(s) \in \mathfrak{o}\}$ is invariant under the action of both $O(M)$ and $\mathscr{E}$. Let M^* be the dual lattice of M_*, that is,

$$M^* = \{s \in V \mid B(s, M_*) \subseteqq \mathfrak{o}\} .$$

Then $2M^* \subseteqq M$ and $2M^*$ is invariant under the action of $O(M)$. Let P be a sublattice of M. Define

$$\mathfrak{a}(P) = \sum_{r \in P} B(r, M_*) .$$

Then $B(P, M_*) \subseteqq \mathfrak{a}(P)$ and $\mathfrak{a}(P)$ is an ideal in $\mathfrak{o}$, since each $B(r, M_*)$ is.

THEOREM 3.1. *Let $M = H \perp K$ where $q(K)$ contains a unit. If* card $\mathfrak{o}/\mathfrak{p} = 2$, *assume also $r(M) \geqq 7$. Then a sublattice P of M is $\mathscr{E}$-invariant if and only if*

$$M_* \subseteqq \mathfrak{a}(P)^{-1}P \subseteqq M^* .$$

Proof. Write $\mathfrak{a} = \mathfrak{a}(P)$. Since $B(\mathfrak{a}^{-1}P, M_*) \subseteqq \mathfrak{o}$, it follows that $\mathfrak{a}^{-1}P \subseteqq M^*$. Now let $x \in K_*$ and $y \in P$. Since $B(P, M_*) \subseteqq \mathfrak{a}$,

$$E(u, x)(y) \equiv y \bmod \mathfrak{a}M_* ,$$

and P is $\mathscr{E}$-invariant if $\mathfrak{a}M_* \subseteqq P$. It remains to show that if $r \in P$ and $B(r, M_*) = \mathfrak{b}$, then $\mathfrak{b}M_* \subseteqq P$ if P is $\mathscr{E}$-invariant.

Write $r = \beta u + \gamma v + s$ where $s \in K$. Then $\mathfrak{b} = \beta\mathfrak{o} + \gamma\mathfrak{o} + B(s, K_*)$. We may assume $\mathfrak{b} = \beta\mathfrak{o}$ (otherwise replace r by $E(u, t)(r)$ where $B(s, t)$ generates $\mathfrak{b}$, or interchange u and v). Take $y \in K_*$ such that $q(y)$ is a unit and, when card $\mathfrak{o}/\mathfrak{p} = 2$, also $B(s, y) = 0$. For a suitable unit ε, $(E(v, \varepsilon y) - I)(r)$ gives rise to an element $v + z$ in $\mathfrak{b}^{-1}P$ with $z \in K$ and $q(z)$ a unit. Then, for any unit η,

$$(E(u, \eta z) - I)(v + z) = -\eta z + \eta(2 - \eta)q(z)u$$

is in $\mathfrak{b}^{-1}P$. If card $\mathfrak{o}/\mathfrak{p} \geqq 4$, it follows that $u \in \mathfrak{b}^{-1}P$ and it is now easy to show that $M_* \subseteqq \mathfrak{b}^{-1}P$. If card $\mathfrak{o}/\mathfrak{p} = 2$, put $\eta = 1$ so that $v + q(z)u$ is in $\mathfrak{b}^{-1}P$. Take $p \in K$ primitive and isotropic. Then $E(u, p)(v + q(z)u)$ is in $\mathfrak{b}^{-1}P$. Hence $p \in \mathfrak{b}^{-1}P$ and consequently $M_* \subseteqq \mathfrak{b}^{-1}P$. This completes the proof.

COROLLARY 3.2. *Let $r \in M$ and $B(r, M_*) = \mathfrak{a}$. Under the hypotheses of the theorem, $\mathfrak{a}M_* + \mathfrak{o}r$ is the smallest $\mathscr{E}$-invariant sublattice in M containing r.*

Proof. Clear.

Introduce an indexing set Ξ so that the lattices M_ξ, $\xi \in \Xi$, are all the distinct lattices on V satisfying

$$M_* \subseteqq M_\xi \subseteqq M^* .$$

If $\mathfrak{o}/\mathfrak{p}$ is finite, then Ξ is also finite. Let $\mathfrak{a}$ be an ideal such that $\mathfrak{a}M_\xi \subseteqq$

M. Then $\mathfrak{a}M_\xi$ is an $\mathscr{E}$-invariant lattice.

4. **The subgroups $\mathscr{E}(\mathfrak{a}M_\xi)$ and $\mathscr{F}(\mathfrak{a}M_\xi)$.** For $\xi \in \Xi$, let $\mathfrak{a}$ be an ideal such that $\mathfrak{a}M_\xi \subseteqq M_*$. Define $\mathscr{E}(\mathfrak{a}M_\xi)$ to be the subgroup of $O(M)$ generated by isometries of the form $\psi E(u, z)\psi^{-1}$ and $\psi E(v, z)\psi^{-1}$ where $\psi \in \mathscr{E}$ and $z \in K \cap \mathfrak{a}M_\xi$. The subgroups $\mathscr{E}(\mathfrak{a}M_\xi)$ are obviously normalized by $\mathscr{E}$. Also, let

$$\mathscr{F}(\mathfrak{a}M_\xi) = \{\varphi \in O(M) \mid [\varphi, \mathscr{E}] \subseteqq \mathscr{E}(\mathfrak{a}M_\xi)\} .$$

Then any subgroup $\mathscr{N}$ of $O(M)$ satisfying

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N} \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$$

for some $\mathfrak{a}M_\xi \subseteqq M_*$ is normalized by $\mathscr{E}$ since

$$[\mathscr{N}, \mathscr{E}] \subseteqq [\mathscr{F}(\mathfrak{a}M_\xi), \mathscr{E}] \subseteqq \mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N} .$$

For $\mathfrak{a}M_\xi \subseteqq M_*$ define the congruence subgroup $O(\mathfrak{a}M_\xi)$ by $O(\mathfrak{a}M_\xi) = \{\varphi \in O(M) \mid \varphi(x) \equiv x \bmod \mathfrak{a}M_\xi$ for all $x \in M_*\} \times \{\pm I\}$. These subgroups are normalized by $\mathscr{E}$ since M_* and $\mathfrak{a}M_\xi$ are $\mathscr{E}$-invariant. If $\mathfrak{a}M_\xi = M_*$, then $O(M_*) = O(M)$. Now let $\mathfrak{a} \subseteqq \mathfrak{p}$. If $\varepsilon \equiv 1 \bmod \mathfrak{a}$, then $\Phi(\varepsilon) \in O(\mathfrak{a}M_\xi)$. Also, for $z \in K \cap \mathfrak{a}M_\xi$, both $E(u, z)$ and $E(v, z)$ are in $O(\mathfrak{a}M_\xi)$ provided $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$. Hence $\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq O(\mathfrak{a}M_\xi)$, provided $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$.

LEMMA 4.1. *Let $\mathfrak{a} \subseteqq \mathfrak{p}$ and $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$. Then $\varphi \in O(\mathfrak{a}M_\xi)$ can be expressed in the form*

$$\varphi = \pm E(u, x)E(v, y)\Phi(\varepsilon)\theta$$

where x and y are in $K \cap \mathfrak{a}M_\xi$, $\varepsilon \equiv 1 \bmod \mathfrak{a}$ and $\theta \in O(K) \cap O(\mathfrak{a}M_\xi)$.

Proof. Let $\varphi(v) = \alpha u + \beta v + s$ where $s \in K \cap \mathfrak{a}M_\xi$ and

$$\beta \equiv \pm 1 \bmod \mathfrak{a} .$$

If $\beta \equiv -1 \bmod \mathfrak{a}$, replace φ by $-\varphi$. Now put $\varphi_1 = \Phi(\beta)E(u, \beta^{-1}s)\varphi \in O(\mathfrak{a}M_\xi)$ so that $\varphi_1(v) = v$. Let $\varphi_1(u) = u - q(t)v + t$ where $t \in K \cap \mathfrak{a}M_\xi$. Put $\theta = E(v, t)\varphi_1 \in O(\mathfrak{a}M_\xi)$. Then $\theta \in O(K)$ and φ can be rewritten in the desired form.

THEOREM 4.2. *Let $M = H \perp K$ where $q(K)$ contains a unit. If* $\operatorname{card} \mathfrak{o}/\mathfrak{p} = 2$, *assume that $r(M) \geqq 9$. Then, if $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$,*

$$\mathscr{E}(\mathfrak{a}M_\xi) = [O(\mathfrak{a}M_\xi), \mathscr{E}]$$

and hence

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq O(\mathfrak{a}M_\xi) \subseteqq \mathscr{F}(\mathfrak{a}M_\xi) .$$

Proof. We first show that $[O(\mathfrak{a}M_\xi), \mathscr{E}] \subseteqq \mathscr{E}(\mathfrak{a}M_\xi)$. If $\mathfrak{a}M_\xi = M_*$, then $\mathscr{E}(M_*) = \mathscr{E}$ and $[O(M), \mathscr{E}] \subseteqq \mathscr{E}$ by Corollary 2.8. Now assume $\mathfrak{a} \subseteqq \mathfrak{p}$. Consider first $[\varphi, E(u, t)]$ where $\varphi \in O(\mathfrak{a}M_\xi)$ and $t \in M_*$. By Lemma 4.1,

$$\varphi = \pm E(u, x)E(v, y)\Phi(\varepsilon)\theta$$

with $E(u, x)$ and $E(v, y)$ in $\mathscr{E}(\mathfrak{a}M_\xi)$. From Proposition 1.1,

$$[\varphi, E(u, t)] \equiv E(u, \varepsilon\theta(t) - t) \bmod \mathscr{E}(\mathfrak{a}M_\xi) .$$

But $[\varphi, E(u, t)]$ is in $O(\mathfrak{a}M_\xi)$ and hence

$$E(u, \varepsilon\theta(t) - t)(v) \equiv v \bmod \mathfrak{a}M_\xi$$

so that $\varepsilon\theta(t) - t \in \mathfrak{a}M_\xi$. Hence $[\varphi, E(u, t)] \in \mathscr{E}(\mathfrak{a}M_\xi)$. From the properties of commutators, it follows that

$$[O(\mathfrak{a}M_\xi), \mathscr{E}] \subseteqq \mathscr{E}(\mathfrak{a}M_\xi) .$$

For the converse inclusion we must show $E(u, z)$ and $E(v, z)$ are in $[O(\mathfrak{a}M_\xi), \mathscr{E}]$ for all $z \in K \cap \mathfrak{a}M_\xi$. If $\operatorname{card} \mathfrak{o}/\mathfrak{p} \geqq 4$, there exists a unit ζ such that $\eta^{-1} = \zeta^2 - 1$ is also a unit. Then

$$E(u, z) = [\Phi(\zeta^2), E(u, \eta z)] \in [\mathscr{E}, O(\mathfrak{a}M_\xi)] .$$

Finally, let $\operatorname{card} \mathfrak{o}/\mathfrak{p} = 2$. Since now $r(M) \geqq 9$, $M = H \perp H' \perp K'$ where $z \in K'$ and $H' = \mathfrak{o}u' + \mathfrak{o}v'$ is a second hyperbolic plane. Then

$$\begin{aligned}[E(u', z), E(v', u)] &= E(E(u', z)(v'), u)E(v', -u) \\ &= E(u, -E(u', z)(v'))E(u, v') \\ &= E(u, z + q(z)u')\end{aligned}$$

is in $[O(\mathfrak{a}M_\xi], \mathscr{E}]$. Since $\mathfrak{a}q(M_\xi) \subseteqq \mathfrak{o}$, we have $q(z)u' \in \mathfrak{a}M_\xi$. A similar argument shows that $E(u, q(z)u')$ is also in $[O(\mathfrak{a}M_\xi), \mathscr{E}]$. The result now follows immediately.

THEOREM 4.3. *Let $M = H \perp K$ where $q(K)$ contains a unit. If $\operatorname{card} \mathfrak{o}/\mathfrak{p} = 2$, assume that $r(M) \geqq 9$. Then*

$$\mathscr{E} = \Omega(M) .$$

Proof. Take $\mathfrak{a}M_\xi = M_*$ in Theorem 4.2. Then

$$\mathscr{E} = \mathscr{E}(M_*) = [O(M), \mathscr{E}] \subseteqq \Omega(M) \subseteqq \mathscr{E} ,$$

the final inclusion following from Theorem 2.9.

REMARK 4.4. With greater effort, a stronger result may be obtained when card $\mathfrak{o}/\mathfrak{p} = 2$ (see, for example, [9; Theorem 2.6]). Also, when $\mathfrak{a}q(M_\xi) \not\subseteq \mathfrak{o}$, anormalous behaviour may occur (see [10; Table II]).

5. **Subgroups normalized by $\Omega(M)$.** Let $\mathscr{N}$ denote a subgroup of $O(M)$ normalized by $\mathscr{E}$. We now prove, under suitable hypotheses, that there exists a sublattice $\mathfrak{a}M_\xi$ invariant under $\mathscr{E}$ such that

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N} \subseteqq \mathscr{F}(\mathfrak{a}M_\xi) .$$

The method is as follows. Assume $\varphi \in \mathscr{N}$ and choose $\psi \in \mathscr{E}$ as in Proposition 1.4 such that

$$\psi\varphi\psi^{-1} = E(u, x)E(v, y)\Phi(\varepsilon)\theta$$

is also in $\mathscr{N}$. We shall show that $E(u, x)$, $E(v, y)$ and $\Phi(\varepsilon)\theta$ are all in $\mathscr{N}$ and use these isometries (varying φ in $\mathscr{N}$) to obtain a maximal subgroup of the form $\mathscr{E}(\mathfrak{a}M_\xi)$ in $\mathscr{N}$. It then remains to prove $\mathscr{N} \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$. We prepare for this theorem with a number of lemmas.

LEMMA 5.1. *Let $M = H \perp K$ where $q(K)$ contains a unit. Then if* card $\mathfrak{o}/\mathfrak{p} \geqq 8$ *and*

$$\varphi = E(u, x)E(v, y)\Phi(\varepsilon)\theta$$

is in a subgroup $\mathscr{N}$ normalized by $\mathscr{E}$, there exist units ζ and η (independent of φ) such that $E(u, \zeta x)$ and $E(v, \eta y)$ are also in $\mathscr{N}$.

Proof. Modify Lemma 3.8 in [9].

LEMMA 5.2. *Assume $r(M) \geqq 7$ and $E(u, x)$ is in $\mathscr{N}$. Then $E(u, \alpha x)$ is in $\mathscr{N}$ for all $\alpha \in \mathfrak{o}$.*

Proof. $x \in K$ can be embedded in a binary (or unary) sublattice B of K with $K = B \perp C$. Then $r(C) \geqq 3$. From [14; 93: 20], $\Theta(SO(C))$ contains all units. Let ε be any unit and take $\theta \in SO(C)$ such that $\Theta(\theta) = \varepsilon$. Then $\Phi(\varepsilon)\theta \in Sk(M) = \mathscr{E}$. Conjugating $E(u, x)$ in $\mathscr{N}$ with $\Phi(\varepsilon)\theta$ gives $E(u, \varepsilon\theta(x)) = E(u, \varepsilon x)$ is in $\mathscr{N}$ for all units ε. If $\alpha \in \mathfrak{o}$ is not a unit, then $\alpha = 1 + \varepsilon$ with ε unit and now $E(u, \alpha x)$ is also in $\mathscr{N}$. This proves the lemma.

The previous two lemmas show that for $r(M) \geqq 7$ and card $\mathfrak{o}/\mathfrak{p} \geqq 8$ that if $E(u, x)E(v, y)\Phi(\varepsilon)\theta$ lies in a subgroup $\mathscr{N}$ normalized by $\mathscr{E}$, then so do $E(u, x)$, $E(v, y)$ and $\Phi(\varepsilon)\theta$. We show now that this is still true for card $\mathfrak{o}/\mathfrak{p} = 2$ or 4 provided the rank of M is at least 9.

LEMMA 5.3. *Let $M = H \perp K$ with $r(M) \geqq 9$ and $\varphi = E(u, x)E(v,$*

$y)\Phi(\varepsilon)\theta \in \mathscr{N}$ where $x, y \in K_*$ and $\theta \in O(K)$. Then $E(u, x)$, $E(v, y)$ and $\Phi(\varepsilon)\theta$ are all in $\mathscr{N}$.

Proof. Since $r(M) \geqq 9$, we have $M = H \perp H' \perp K'$ where $H' = \mathfrak{o}u' + \mathfrak{o}v'$ is a hyperbolic plane and $y \in K'$. Then

$$E(u, -x)[\varphi, E(u, u')]E(u, x) = E(v, y)E(u, \varepsilon\theta(u'))E(v, -y)E(u, -u')$$

is in $\mathscr{N}$. Hence

$$\begin{aligned} &E(u, \varepsilon\theta(u'))E(E(v, -y)(u), -u') \\ &\quad = E(u, \varepsilon\theta(u'))E(u + y - q(y)v, -u') \\ &\quad = E(u, \varepsilon\theta(u') - u')E(u', y - q(y)v) \end{aligned}$$

is also in $\mathscr{N}$. Let $t = \varepsilon\theta(u') - u'$. Take $s \in K'$ with $q(s)$ a unit and $B(s, y) = 0$. Then $[E(u, s), E(u, t)E(u', y - q(y)v]$, and hence also $E(u', q(y)(s + q(s)u))$, are in $\mathscr{N}$. But $\mathfrak{o}v + \mathfrak{o}(s + q(s)u)$ is a hyperbolic plane, so that both $E(u', q(y)v)$ and $E(u, t)E(u', y)$ are in $\mathscr{N}$. This already completes the proof in the special case where $\varepsilon = 1$ and θ is the identity map, since then $t = 0$. Returning to the general case, since $r(K') \geqq 5$, there exists $\psi \in O(K')$ such that $\psi(y) = y$ and $\Delta\psi \in \mathscr{E}$. Conjugating $E(u, t)E(u', y)$ with $\Delta\psi$, shows that $E(v, \psi(t))E(u', y)$ is in $\mathscr{N}$. Hence $E(u, t)E(v, -\psi(t)) \in \mathscr{N}$ and, by the special case noted above, it follows that $E(u, t)$ is in $\mathscr{N}$. Finally, $E(u', y) \in \mathscr{N}$ and the result now follows.

LEMMA 5.4. *Let $M = H \perp K$ with $r(M) \geqq 7$. If $\operatorname{card} \mathfrak{o}/\mathfrak{p} \leqq 4$, assume also $r(M) \geqq 9$. Let $E(u, x) \in \mathscr{N}$ where $x \in K_*$ and $B(x, M_*) = \mathfrak{a}$. Then*

$$\mathscr{E}(\mathfrak{a}M_*) \subseteqq \mathscr{N} .$$

Proof. Take $z \in K_*$ such that $B(x, z) = \alpha$ where $\mathfrak{o}\alpha = \mathfrak{a}$. We may assume $q(z)$ is a unit, for if not, take $z_1 \in K_*$ with $B(z, z_1) = 0$ and $q(z_1)$ a unit; if $B(x, z_1) \in \alpha\mathfrak{u}$, replace z by z_1, otherwise, replace z by $z + z_1$. Moreover, there exists $y \in K_*$ with $B(x, y) = 0$ and $q(y)$ a unit. Let $\varepsilon = q(z)q(y)$. Conjugating $E(u, x) \in \mathscr{N}$ with $\Phi(\varepsilon)\Psi(z)\Psi(y)$ from $Sk(M) = \mathscr{E}$ gives $E(\varepsilon u, x - \alpha q(z)^{-1}z) \in \mathscr{N}$. From Lemma 5.2 it follows that $E(u, \alpha z)$ is in $\mathscr{N}$. If $w \in K_*$ and $\operatorname{card} \mathfrak{o}/\mathfrak{p} \geqq 8$, there is a unit η such that $q(z + \eta w) \in \mathfrak{u}$ and $B(x, z + \eta w) \in \alpha\mathfrak{u}$. A similar argument shows $E(u, \alpha(z + \eta w))$, and hence also $E(u, \alpha w)$, are in $\mathscr{N}$. Conjugating with $\Delta\Phi(-q(z))\Psi(z) \in \mathscr{E}$ gives now $\mathscr{E}(\mathfrak{a}M_*) \subseteqq \mathscr{N}$.

Now assume $\operatorname{card} \mathfrak{o}/\mathfrak{p} \leqq 4$ so that $r(M) \geqq 9$. Then $M = H \perp H' \perp K'$ with $x \in K'$. Conjugating $E(u, x)$ with $E(u', z)$ leads to $E(u, \alpha u') \in \mathscr{N}$. Similarly, $E(u, \alpha v') \in \mathscr{N}$. Take $t \in K'_*$. Finally, conjugating

$E(u, \alpha u')$ with $E(v', t) \in \mathscr{E}$ shows that $E(u, \alpha t)$ is in $\mathscr{N}$ and hence again, $\mathscr{E}(\mathfrak{a}M_*) \subseteqq \mathscr{N}$.

THEOREM 5.5. *Let M be a unimodular lattice with $r(M) \geqq 7$, and $r(M) \geqq 9$ if* card $\mathfrak{o}/\mathfrak{p} = 2,4$. *Then a subgroup $\mathscr{N}$ of the orthogonal group $O(M)$ is normalized by the commutator subgroup $\Omega(M)$ if and only if it satisfies*

$$\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N} \subseteqq \mathscr{F}(\mathfrak{a}M_\xi)$$

for some ideal $\mathfrak{a}$ in $\mathfrak{o}$ and an invariant sublattice M_ξ with $\mathfrak{a}M_\xi \subseteqq M_*$.

Proof. We have already observed that subgroups satisfying these ladder relations are normalized by $\mathscr{E} = \Omega(M)$. Now assume $\mathscr{N}$ is a subgroup normalized by $\Omega(M)$ and choose $\mathfrak{a}M_\xi \subseteqq M_*$ maximal such that $\mathscr{E}(\mathfrak{a}M_\xi) \subseteqq \mathscr{N}$. Clearly, at least $\{I\} = \mathscr{E}(\{O\}M_*) \subseteqq \mathscr{N}$; moreover, if both $\mathscr{E}(\mathfrak{a}_1 M_{\xi_1})$ and $\mathscr{E}(\mathfrak{a}_2 M_{\xi_2})$ are contained in $\mathscr{N}$, these two subgroups generate $\mathscr{E}(\mathfrak{a}_3 M_{\xi_3}) \subseteqq \mathscr{N}$ where $\mathfrak{a}_3 = \mathfrak{a}_1 + \mathfrak{a}_2$ (see §4).

Now let $\varphi \in \mathscr{N}$; we must prove $\varphi \in \mathscr{F}(\mathfrak{a}M_\xi)$. By Proposition 1.4 there exists $\psi \in \Omega(M)$ such that

$$\psi\varphi\psi^{-1} = E(u, x)E(v, y)\Phi(\varepsilon)\theta$$

where $\theta \in O(K)$. By Lemmas 5.1-5.3 we know that $E(u, x)$ and $E(v, y)$ are in $\mathscr{N}$ and hence by Lemma 5.4 and §4 they are even in $\mathscr{E}(\mathfrak{a}M_\xi)$. It therefore suffices to prove that $\Phi(\varepsilon)\theta$ is in $\mathscr{F}(\mathfrak{a}M_\xi)$. For $s \in K_*$,

$$[\Phi(\varepsilon)\theta, E(u, s)] = E(u, \varepsilon\theta(s) - s)$$

is in $\mathscr{N}$. Again, from Lemma 5.4 and §4, it follows that $[\Phi(\varepsilon)\theta, E(u, s)]$ is in $\mathscr{E}(\mathfrak{a}M_\xi)$. Hence

$$[\Phi(\varepsilon)\theta, \mathscr{E}] \subseteqq \mathscr{E}(\mathfrak{a}M_\xi)$$

and, therefore, $\Phi(\varepsilon)\theta \in \mathscr{F}(\mathfrak{a}M_\xi)$. This proves the theorem.

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