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DERIVATIONS OF AW^*-ALGEBRAS ARE INNER

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Using the theory of spectral subspaces associated with a group of isometries of a Banach space it is proved that each derivation of an AW^*-algebra is inner. This constructive method of proof yields a generator b for the case of a skew-adjoint derivation which is seen to be the unique positive generator such that $\|bp\| = \|\delta \mid Ap\|$ for each central projection p in the AW^*-algebra A.

Introduction. The problem of whether derivations of AW^*-algebras are inner was first studied by I. Kaplansky in [9] and settled in the affirmative for the case of a type I algebra. Later the result was extended to type III algebras and type II factors by G. A. Elliott, and to type II_1 algebras with central trace by J. C. Deel, ([3], [4]). It is not known whether this covers all cases.

The purpose of the present note is to show that each derivation of an AW^*-algebra is inner, avoiding type classification. The method employed is a modification of the one developed by W. B. Arveson in [1], where he proves the corresponding theorem for W^*-algebras. (See also Borchers [2].)

Specifically, we prove that the group of $*$-automorphisms $e^{it\delta}$, where δ is a derivation of the AW^*-algebra A satisfying the condition $\delta(a^*) = -(\delta(a))^*$, is implemented by a unitary group e^{itb} with b a positive element of A.

In § 2 we prove a lemma which establishes a sufficient condition that an element of a C^*-algebra belong to a spectral subspace of the group $u_t \cdot u_t^*$, where $u_t = \int_\alpha^\beta e^{itx} dp(x)$ with $p(x)$ a given increasing family of projections on $[\alpha, \beta]$. This lemma is a corollary of [1, Theorem 2.3], formulated to suit the present context.

In § 3, we use Lemma 1 and the fact that each subset of an AW^*-algebra has a largest left-annihilating projection inside the algebra to construct an implementing group of unitaries for $e^{it\delta}$. The constructive method of proof yields a generator b for δ, which is seen to be the unique positive generator for δ such that $\|bp\| = \|\delta \mid Ap\|$ for each projection p in the center of A, an observation not made in [1].

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1. Notation. The notation is taken from [1]. For a brief recapitulation, let us look at the special case in which we are interested, where α_t is a norm-continuous one-parameter group of isometries of a Banach space X. For each f in $L^1(\boldsymbol{R})$ let $\pi_\alpha(f)$ denote the bounded operator on X given by

$$\pi_\alpha(f) = \int \alpha_t f(t) dt \ ,$$

where the integral exists in the Bochner sense. With $\hat{f}(s) = \int f(t) e^{ist} dt$ and $-\infty \leqq t \leqq w \leqq \infty$ we denote by $R^\alpha(t, w)$ the norm-closed subspace in X generated by the vectors $\pi_\alpha(f)x$ where $x \in X$ and $\hat{f}$ has compact support in (t, w). Note that since every norm-closed convex set in X is $\sigma(X, X')$-closed, with X' the dual of X, these subspaces are in fact identical to the ones defined in [1]. The spectral subspace associated with $[t, w]$ is

$$M^\alpha[t, w] = \bigcap_{n \in N} R^\alpha\left(t - \frac{1}{n},\ w + \frac{1}{n}\right).$$

It follows immediately from this definition that

$$\bigcap_{s<t} M^\alpha[s, w] = M^\alpha[t, w]$$

and that the spectral subspaces are invariant under α_t. As shown in [1] we have

$$M^\alpha[t, w] = \{x \in X \mid \pi_\alpha(f)x = 0 \ \ \forall f \in I_0[t, w]\}$$

where $I_0[t, w]$ denotes the set of function f in $L^1(\boldsymbol{R})$ such that $\hat{f}$ has support disjoint from $[t, w]$. The existence of an approximate unit (f_λ) in $L^1(\boldsymbol{R})$ where $(\hat{f}_\lambda)$ consist of functions with compact support ensures that the above relation also holds if we define $I_0[t, w]$ to be those L^1-functions f such that $\hat{f}$ has compact support disjoint from $[t, w]$.

THEOREM 2.3. [1] *states the following relation: Let α_t, β_t be groups of isometries on X. Denote by φ_t the group on $B(X)$ such that $\varphi_t(a) = \alpha_t \cdot a \cdot \beta_t^{-1}$, all a in $B(X)$. Then*

$$aM^\beta[t, \infty) \subseteqq M^\alpha[s + t, \infty) \forall t \Longleftrightarrow a \in M^\varphi[s, \infty) \ .$$

2. On some unitary groups. Let $t \longrightarrow p(t)$ be an increasing projection-valued map from $\boldsymbol{R}$ into the C^*-algebra A, and assume that there exist α and β in $\boldsymbol{R}$, $\alpha \leqq \beta$, such that $p(t) = 0$ for all $t \leqq \alpha$ and $p(t) = 1$ for all $t \geqq \beta$. Let f be a continuous map from $[\alpha, \beta]$ into $\boldsymbol{C}$. Put

$$s_\pi(f, p) = \sum_{i=1}^{n} f(t_i)(p(u_i) - p(u_{i-1}))$$

where π denotes the division $\alpha = u_0 \leqq u_1 \leqq \cdots \leqq u_n = \beta$ and $t_i \in [u_{i-1}, u_i]$. Then by a well-known theorem the limit of s_π exists and is

$$\int_\alpha^\beta f(t)dp(t) = \lim_{|\pi|\to 0} s_\pi$$

where $|\pi| = \max |u_i - u_{i-1}|$. Take $f(t) = e^{itx}$ with x in $\boldsymbol{R}$, and set

$$\int_\alpha^\beta e^{itx}dp(t) = u_x .$$

Then $x \longrightarrow u_x$ is a norm-continuous group of unitary elements of A. In the case where $p(t) = 1$ for all $t > \beta$, u_x as above denotes the common value of the integrals from α to $\beta + \varepsilon$, all $\varepsilon > 0$.

LEMMA 1. *Let* $u_x = \int_\alpha^\beta e^{itx}dp(t)$ *and put* $\varphi_x = u_x \cdot u_x^*$. *Then for* a *in* A *and* s *in* $\boldsymbol{R}$

$$p(t+s)a\,p(t) = p(t+s)a \;\forall t \in \boldsymbol{R} \Longrightarrow a \in M^\varphi[s, \infty) .$$

Proof. Assume A to be represented faithfully on a Hilbert space H. By Stone's theorem we know the existence of a unique increasing left-continuous spectral measure $q(t)$ such that

$$u_x = \int e^{itx}dq(t) ,$$

and from the relations

$$M^u[t_0, \infty) \subset R^u(t, \infty) \subset [(1 - q(t))H] \subset M^u[t, \infty)$$

for all $t_0 > t$ we see that

$$M^u[t, \infty) = [(1 - q(t))H] .$$

Now $p(t)$ tends strongly to $q(t_0)$ as $t \nearrow t_0$, and so $p(t+s)a(1-p(t))$ tends strongly to $q(t_0+s)a(1-q(t_0))$ for all a in A. From this it follows that if a satisfies the hypothesis of the lemma, it also satisfies the relation

$$q(t+s)a\,q(t) = q(t+s)a \;\forall t \in \boldsymbol{R} ,$$

but this is equivalent to

$$aM^u[t, \infty) \subset M^u[t+s, \infty) \;\forall t \in \boldsymbol{R} ,$$

which by [1, Theorem 2.3] implies that $a \in M^\varphi[s, \infty)$.

3. Construction of the generator for δ. Recall that a C^*-algebra A is an AW^*-algebra, (see [8]) if for any subset S of A

there is a unique projection p in A such that

$$\{a \in A \mid as = 0 \ \forall s \in S\} = Ap .$$

p is called the left-annihilating projection of S.

THEOREM 2. *If δ is a derivation of the AW^*-algebra A there is an element b in A such that $\delta = ad_b$. If $\delta = -\delta^*$, b can be chosen positive and with norm equal to the norm of δ.*

Proof. Since each derivation δ has a unique decomposition $\delta = \delta_1 + i\delta_2$, with $\delta_i = \delta_i^*$, it suffices to prove the last statement.

Let $\delta = -\delta^*$. Denote by α_t the $*$-automorphism group $e^{it\delta}$, and let $p(t)$ be the left-annihilating projection of the spectral subspace $M^\alpha[t, \infty)$. The map $t \to p(t)$ taking $\boldsymbol{R}$ into the fixed-point algebra $M^\alpha[0]$ is increasing. As $1 \in M^\alpha[0]$, we have $p(0) = 0$. The claim $p(t) = 1$ for $t > \|\delta\|$ is seen as follows: We want to prove that whenever $f \in L^1(\boldsymbol{R})$ such that $\hat{f}$ has compact support in $(\|\delta\| + \varepsilon, \infty)$, then $\pi_\alpha(f) = 0$, or equivalently that for all $g \in L^1(\boldsymbol{R})$ where $\hat{g}$ has compact support in $(0, \infty)$, $\pi_\alpha(g \cdot e^{-i(\|\delta\|+\varepsilon)\cdot}) = 0$.

Now g extends to an H^1 function in the lower half plane if we define

$$g(z) = \frac{1}{2\pi}\int_0^\infty \hat{g}(t)e^{-itz}dt$$

and for the L^1-norms of $x \to g_y(x) = g(x + iy)$, y fixed, we have

$$\|g_y\|_1 \leqq \|g\|_1$$

(see [6], p. 124–128 and p. 131). Now

$$\begin{aligned} g_y(x) = g(x + iy) &= \frac{1}{2\pi}\int_0^\infty \hat{f}(t + \|\delta\| + \varepsilon)e^{-i(x+iy)t}dt \\ &= \frac{1}{2\pi}\int_{\|\delta\|+\varepsilon}^\infty \hat{f}(w)e^{-izw}e^{+iz(\|\delta\|+\varepsilon)}dw \\ &= e^{+iz(\|\delta\|+\varepsilon)}f(z) = e^{+ix\|\delta\|}e^{-y(\|\delta\|+\varepsilon)}f_y(x) \end{aligned}$$

so

$$\|g_y\|_1 = e^{-y(\|\delta\|+\varepsilon)}\|f_y\|_1$$

from which it follows that

$$e^{-y(\|\delta\|+\varepsilon)}\|f_y\|_1 \leqq \|f\|_1$$

and so we get (see [2])

$$\begin{aligned}\left\|\int f(x)\alpha_x dx\right\| &= \left\|\int f(x+iy)\alpha_{x+iy}dx\right\| \\ &\leqq e^{y(\|\delta\|+\varepsilon)}\|f\|_1 \cdot \|e^{i(x+iy)\delta}\| \leqq e^{y(\|\delta\|+\varepsilon)}\|f\|_1 \cdot e^{-y\|\delta\|} \\ &= e^{y\varepsilon}\|f\|_1 \longrightarrow 0 \quad \text{as} \quad y \longrightarrow -\infty .\end{aligned}$$

According to § 2, the group $u_t = \int e^{itx}dp(x)$ is well-defined. We want to show that it implements α_t, i.e., that

$$\alpha_t = u_t \cdot u_t^* .$$

Denoting the right side by φ_t, it suffices to see that

$$M^\alpha[t, \infty) \subseteqq M^\varphi[t, \infty) \; \forall t \in \boldsymbol{R} .$$

Indeed, as φ_t and α_t are both norm-continuous one-parameter groups of self-adjoint (i.e., adjoint-preserving) operators on A, the group $\beta_t(\gamma) = \varphi_t \cdot \gamma \cdot \alpha_t^{-1}$, $\gamma \in B(A)$, is a norm-continuous adjoint-preserving group on $B(A)$. It follows that $M^\beta[t, \infty)^* = M^\beta(-\infty, -t]$ for all t in $\boldsymbol{R}$, so whenever a self-adjoint element γ in $B(A)$ belongs to $M^\beta[t, \infty)$, γ belongs to $M^\beta[t, -t]$. We know by [1, Theorem 2.3] that the inclusion $M^\alpha[t, \infty) \subseteqq M^\varphi[t, \infty)$ implies that $id \in M^\beta[0, \infty)$. The preceding argument shows that id is then in $M^\beta[0]$, so $\varphi_t id \alpha_t^{-1} = id$ for all t, thus $\varphi_t = \alpha_t$.

Using the multiplicative property of α_t a rather straightforward calculation shows that for all t and s in $\boldsymbol{R}$

$$R^\alpha(t, \infty)R^\alpha(s, \infty) \subseteqq R^\alpha(t+s, \infty) .$$

Indeed, for $f, g \in L^1(\boldsymbol{R})$ such that $\hat{f}$, $\hat{g}$ have compact support

$$\begin{aligned}\pi_\alpha(f)x\pi_\alpha(g)y &= \iint f(t)g(u)\alpha_t(x\alpha_{u-t}y)dtdu \\ &= \iint f(t)g(w+t)\alpha_t(x\alpha_w y)dtdw \\ &= \int\left(\int f(t)g_w(t)\alpha_t(x\alpha_w y)dt\right)dw \\ &= \int\left(\int (\hat{f} * \hat{g}_w)^\vee(t)\alpha_t(x\alpha_w y)dt\right)dw \\ &= \int z_w dw \cdot .\end{aligned}$$

So if $\operatorname{supp}\hat{f} \subset (t, \infty)$ and $\operatorname{supp}\hat{g} \subset (s, \infty)$ we have $z_w \subset R^\alpha(t+s, \infty)$, as $\operatorname{supp}\hat{f} * \hat{g}_w \subset (t+s, \infty)(g_w(t) = g(t+w)$, so $\hat{g}_w(s) = e^{-isw}\hat{g}(s))$.

From this it follows immediately that for all t and s in $\boldsymbol{R}$

$$M^\alpha[t, \infty)M^\alpha[s, \infty) \subseteqq M^\alpha[t+s, \infty) ,$$

so if $a \in M^\alpha[t, \infty)$ and $d \in M^\alpha[s, \infty)$ we get that

$$p(t+s)ad = 0 .$$

However, this implies that $p(t+s)a$ belongs to the left-annihilator of $M^\alpha[s, \infty)$, thus

$$p(t+s)a\,p(s) = p(t+s)a .$$

The desired conclusion now follows from Lemma 1.

The generator for δ thus constructed is

$$b = \int_0^{\|\delta\|} t\,dp(t) .$$

It is obvious that $\|b\| \leqq \|\delta\|$. On the other hand, $b - (\|b\|)/2\cdot 1$ is also a generator for δ, and $\|b - (\|b\|)/2\cdot 1\| = \|b\|/2$. So we get that $\|b\| = \|\delta\|$.

In [5] it is shown that for each inner derivation δ of an AW^*-algebra A there is a unique generator a of norm $\|\delta\|/2$ such that

$$\|ap\| = \frac{1}{2}\|\delta \mid Ap\|$$

for each projection p in the center of A. This generalizes a result in [7] concerning self-adjoint derivations of von Neumann algebras. Here we have the following result:

PROPOSITION 3. *With $\delta = -\delta^*$, the element b in A as constructed above is the unique positive generator for δ such that*

$$\|bp\| = \|\delta \mid Ap\|$$

for each projection p in the center C of A. If $\delta = ad_c$, $c \geqq 0$, then $c \geqq b \geqq 0$, so b is the minimal positive generator for δ.

Proof. Let p denote a central projection in A. We want to see that $\|bp\| = \|\delta \mid Ap\|$. Since $\alpha_t = e^{it\delta}$ leaves C pointwise invariant it follows from the definition of spectral subspaces that

$$M^\alpha[t, w] \cap Ap = M^{\alpha p}[t, w] ,$$

where αp denotes the group of automorphisms of Ap obtained by restricting the α_t's to Ap. Consequently the construction carried out in the proof above will produce bp as a generator for δ on Ap. Thus

$$\|bp\| = \|\delta \mid Ap\| .$$

Now assume c to be another positive generator for δ. Using nothing but the fact that b is a positive generator for δ satisfying the above condition on the norm we can prove $c \geqq b$, as follows: Since

b and c are both positive generators for δ, the difference $b - c$ is in C. Suppose λ was a positive scalar in $sp(b - c)$. Given a sufficiently small $\varepsilon > 0$ we could then find a nonzero projection p in C such that

$$(b - c)p \geqq \varepsilon p \,.$$

But as cp is a positive generator for $\delta \mid Ap$ we have, arguing as before

$$\| bp \| = \| \delta \mid Ap \| \leqq \| cp \| \,,$$

and combining we get

$$0 \leqq cp \leqq bp - \varepsilon p \leqq (\| bp \| - \varepsilon)p \leqq (\| cp \| - \varepsilon)p \,,$$

a contradiction. Therefore, $sp(b - c) \subseteqq (-\infty, 0)$, i.e., $c - b \geqq 0$. The uniqueness of the positive generator satisfying the above norm condition follows from the fact that it is the smallest positive generator.

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