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CONTINUOUS SPECTRA OF A SINGULAR SYMMETRIC DIFFERENTIAL OPERATOR ON A HILBERT SPACE OF VECTOR-VALUED FUNCTIONS

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Let H be the Hilbert space of complex vector-valued functions $f: [a, \infty) \to C^2$ such that f is Lebesgue measurable on $[a, \infty)$ and $\int_a^\infty f^*(s)f(s)ds < \infty$. Consider the formally self adjoint expression $\iota(y) = -y'' + Py$ on $[a, \infty)$, where y is a 2-vector and P is a 2×2 symmetric matrix of continuous real valued functions on $[a, \infty)$. Let D be the linear manifold in H defined by

$$D = \{f \varepsilon H: f, f' \text{ are absolutely continuous on compact subintervals of } [a, \infty),\ f \text{ has compact support interior to } [a, \infty) \text{ and } \iota(f) \varepsilon H\}\,.$$

Then the operator L defined by $L(y) = \iota(y)$, $y \varepsilon D$, is a real symmetric operator on D. Let L_0 be the minimal closed extension of L. For this class of minimal closed symmetric operators this paper determines sufficient conditions for the continuous spectrum of self adjoint extensions to be the entire real axis. Since the domain, D_0, of L_0 is dense in H, self adjoint extensions of L_0 do exist.

A general background for the theory of the operators discussed here is found in [1], [3], and [5]. The theorems in this paper are motivated by the theorems of Hinton [4] and Eastham and El-Deberky [2]. In [4], Hinton gives conditions on the coefficients in the scalar case to guarantee that the continuous spectrum of self adjoint extensions covers the entire real axis. Eastham and El-Deberky [2] study the general even order scalar operator.

DEFINITION 1. Let $\tilde{L}$ denote a self adjoint extension of L_0. Then we define the *continuous spectrum*, $C(\tilde{L})$, of $\tilde{L}$ to be the set of all λ for which there exists a sequence $\langle f_n \rangle$ in $D_{\tilde{L}}$, the domain of $\tilde{L}$, with the properties:

(i) $\|f_n\| = 1$ for all n,

(ii) $\langle f_n \rangle$ contains no convergent subsequence (i.e., is not compact), and

(iii) $\|(\tilde{L} - \lambda)f_n\| \to 0$ as $n \to \infty$.

For the self adjoint operator $\tilde{L}$ we have the following well-known lemma.

LEMMA 1. *The continuous spectrum of $\tilde{L}$ is a subset of the real numbers.*

Proof. Let $\lambda = \alpha + i\beta$ where $\beta \neq 0$. Then for all $f \in D_{\tilde{L}}$ we can see by expanding $\|(\tilde{L} - \lambda)f\|^2$ that

$$\|(\tilde{L} - \lambda)f\|^2 \geqq |\beta|^2 \|f\|^2 ,$$

which implies $\lambda \notin C(\tilde{L})$.

THEOREM 2. *Let $L(y) = y'' + P(t)y$ for $a \leqq t < \infty$, where $P(t) = \begin{bmatrix} \alpha(t) & \gamma(t) \\ \gamma(t) & \beta(t) \end{bmatrix}$ where $\gamma(t)$ is positive and has two continuous derivatives. Let $g(t) > 0$ be one of $\alpha(t)$ or $\beta(t)$, where both α and β are continuous on $[\alpha, \infty)$ and $g(t)$ has a continuous derivative. Then if for some sequence of intervals $\{A_m\}$ where $A_m \subseteqq [a, \infty)$, $A_m = [c_m - a_m, c_m + a_m]$ and $a_m \to \infty$, the following are satisfied:*

(i) $\min_{x \in A_m} \{g(x)\} \to \infty$,

(ii) $\int_{A_m} ((g'(x))^2)/(g(x))\, dx = o(a_m)$,

(iii) $\int_{A_m} g(x)dx = o(a_m^3)$,

(iv) $\int_{A_m} [\gamma(x)]^2 dx = o(a_m)$,

we can conclude that $C(\tilde{L})$ is $(-\infty, \infty)$.

Proof. We will establish the theorem for $g(t) = \alpha(t)$ since the other case follows in exactly the same way.

Note that to prove the theorem then we need only show that for any real number μ there is a sequence $\langle f_m \rangle$ in $D(\tilde{L})$ such that $\|f_m\| = 1$, $f_m \to 0$ a.e., f_m vanishes outside A_m and $\|(\tilde{L} - \mu)f_m\| \to 0$ as $m \to \infty$.

Let $\langle h_m \rangle$ be defined by

$$h_m(t) = \begin{cases} [1 - \{(t - c_m)/a_m\}^2]^3 & \text{for } |t - c_m| \leqq a_m \\ 0 & \text{for } |t - c_m| > a_m \end{cases} . \tag{1}$$

Then define $\langle f_m(t) \rangle$ by

$$f_m(t) = h_m(t)\begin{bmatrix} b_{m1} e^{iQ_1(t)} \\ b_{m2} e^{iQ_2(t)} \end{bmatrix} , \tag{2}$$

where Q_1, Q_2 are real functions with two continuous derivatives and b_{m1}, b_{m2} are normalization constants.

To find $|b_m| = \sqrt{b_{m1}^2 + b_{m2}^2}$ we have

$$1 = \|f_m\|^2 = \int_{c_m - a_m}^{c_m + a_m} |b_m|^2 h_m^2(t)dt = |b_m|^2 \int_{-a_m}^{a_m} \left[1 - \left(\frac{x}{a_m}\right)^2\right]^6 dx$$

$$= |b_m|^2 \int_{-1}^{1} a_m[1 - y^2]^6 dy = |b_m|^2 (2a_m)\left[1 + \sum_{r=1}^{6} \binom{6}{r}(2r + 1)^{-1}\right] .$$

Hence for some positive constant K

(3) $$|b_m|^2 = K(2a_m)^{-1},$$

and

(4) $$|f_m(t)| \leqq |b_m| = \sqrt{K}/\sqrt{2}a_m .$$

Hence

(5) $$f_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty ,$$

(6) $$|h_m^{(r)}(t)| \leqq K_r(a_m)^{-r} ,$$

where K_r does not depend on t or m.

Since $f_m \in D(\tilde{L})$, we have

$$(\tilde{L} - \mu I)f_m = f_m'' + (P - \mu I)f_m$$
$$= \begin{bmatrix} f_{m1}'' + (\alpha - \mu)f_{m1} + \gamma f_{m2} \\ f_{m2}'' + (\beta - \mu)f_{m2} + \gamma f_{m1} \end{bmatrix}$$
$$(\tilde{L} - \mu I)f_m = \begin{bmatrix} \{-Q_1'^2 + (\alpha - \mu)\}f_{m1} + \gamma f_{m2} + iQ_1''f_{m1} \\ \{-Q_2'^2 + (\beta - \mu)\}f_{m2} + \gamma f_{m1} + iQ_2''f_{m2} \end{bmatrix} + \begin{bmatrix} b_{m1}e^{iQ_1}h_m'' + 2iQ_1'b_{m1}e^{iQ_1}h_m' \\ b_{m2}e^{iQ_2}h_m'' + 2iQ_2'b_{m2}e^{iQ_2}h_m' \end{bmatrix}.$$

Now if Q_1 is chosen so that

$$Q_1'^2 = \alpha - \mu , \qquad Q_1'' = \frac{\alpha'}{2\sqrt{\alpha - \mu}} ,$$

and b_{m2} is chosen to be identically zero we have that

$$(\tilde{L} - \mu I)f_m = \begin{bmatrix} iQ_1''f_{m1} \\ \gamma f_{m1} \end{bmatrix} + \begin{bmatrix} b_{m1}e^{iQ_1}h_m'' + 2iQ_1'b_{m1}e^{iQ_1}h_m' \\ 0 \end{bmatrix}.$$

By the way Q_1 is chosen,

$$\|(\tilde{L} - \mu I)f_m\| \leqq \left\|\left(\frac{\alpha'}{2\sqrt{\alpha - \mu}}\right)f_m\right\| + \|\gamma f_m\| + \|b_m h_m''\| + \|2Q_1'b_m h_m'\| .$$

Now, by (ii)

$$\left\|\frac{\alpha'}{2\sqrt{\alpha - \mu}}f_m\right\| \leqq \left[\frac{K}{a_m}\int_{A_m}\left(\frac{\alpha'}{2\sqrt{\alpha - \mu}}\right)^2\right]^{1/2} = o(1) \quad \text{as} \quad m \longrightarrow \infty .$$

By condition (iv),

$$\|\gamma f_m\| \leqq \left(\frac{K}{a_m}\int_{A_m}|\gamma|^2\right)^{1/2} = o(1) \quad \text{as} \quad m \longrightarrow \infty .$$

Next, by (iii), (3) and (6)

$$\| Q_1' b_m h_m' \| = \left(\int_{A_m} (\alpha - \mu) \frac{K}{2a_m} \cdot \frac{K_1^2}{a_m^2} \right)^{1/2}$$
$$= K_1 K^{1/2} \left(\frac{1}{2a_m^3} \int_{A_m} (\alpha - \mu) \right)^{1/2} = o(1) \quad \text{as} \quad m \longrightarrow \infty .$$

Then, by (3), (6), and the Cauchy-Schwartz Inequality

$$\| b_m h_m'' \| \leqq \left(\int_{A_m} | b_m |^2 \right)^{1/2} \left(\int_{A_m} | h_m'' |^2 \right)^{1/2}$$
$$\leqq \sqrt{K/2} \left(\int_{A_m} (K_r^2 / a_m^2) \right)^{1/2} = o(1) \quad \text{as} \quad m \longrightarrow \infty .$$

Hence it follows that

$$\| (\tilde{L} - \mu I) f_m \| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty ,$$

which is what we were to show.

COROLLARY 3. *If* $P(t) = \begin{bmatrix} at^\sigma & ct^\tau \\ ct^\tau & bt^\beta \end{bmatrix}$ *on some half-line* $d \leqq t < \infty$ *in Theorem* 2 *and*

(i) $a, c > 0$ *with* $\delta < 0,\ 0 < \sigma < 2$, *or*

(ii) $b, c > 0$ *with* $\delta < 0,\ 0 < \eta < 2$

then $C(\tilde{L}) = (-\infty, \infty)$.

THEOREM 4. *Suppose* $L(y)$ *is as in Theorem* 2, *where* $\gamma(t)$ *is positive and has two continuous derivatives. If for some sequence of intervals* $\{A_m\}$, *where* $A_m = [c_m - a_m, c_m + a_m]$, $A_m \subseteqq [a, \infty)$ *and* $a_m \rightarrow \infty$, *the following are satisfied*:

(i) $\min_{t \in A_m} \{\gamma(t)\} \rightarrow \infty$,

(ii) $\int_{A_m} ((\gamma'(t))^2)/(\gamma(t))\ dt = o(a_m)$,

(iii) $\int_{A_m} \gamma(t) dt = o(a_m^3)$,

(iv) $\int_{A_m} \alpha^2(t) dt$ *and* $\int_{A_m} \beta^2(t) dt$ *are* $o(a_m)$,

then $C(\tilde{L}) = (-\infty, \infty)$.

Proof. In the proof of Theorem 2 choose $Q_1'^2 = Q_2'^2 = \gamma(t) - \mu$, so that $f_{m1} = f_{m2}$. Then $Q_1'' = Q_2'' = (\gamma'(t))/(2\sqrt{\gamma(t) - \mu})$ and applying conditions (i) – (iv) as before where $g(t)$ is replaced by $\gamma(t)$ we get that $\| (\tilde{L} - \mu I) f_m \| \rightarrow 0$ as $m \rightarrow \infty$.

COROLLARY 5. *Let* $P(t) = \begin{bmatrix} at^\sigma & ct^\delta \\ ct^\delta & bt^\eta \end{bmatrix}$ *in Theorem* 4. *If* $c > 0$, $0 < \delta < 2$ *and* $\sigma,\ \eta < 0$ *then* $C(\tilde{L}) = (-\infty, \infty)$.

Let H be the Hilbert space $\tilde{L}_2([a, \infty), w)$ of complex vector-valued functions $f\colon [a, \infty) \to C^2$ such that $\|f\|^2 = \int_a^\infty w(f^*f) < \infty$, where w is positive and $w \in C^{(2)}[a, \infty)$. Let $l(y) \equiv (1/w)y'' + Py$. Then define L_0 as before and let $\tilde{L}$ be a self adjoint extension of L_0.

THEOREM 6. *Suppose there is a sequence of intervals, $A_m \subseteqq [a, \infty)$, $A_m = [c_m - a_m, c_m + a_m]$ where $a_m \to \infty$ as $m \to \infty$, such that*

(i) $\int_{A_m} (\alpha(w')^2)/w^3 = o(|a_m|)$, $\int_{A_m} \alpha/w = o(|A_m|)^3$, $\min_{t \in A_m} \alpha(t) \to \infty$,

(ii) $\int_{A_m} (w')^4/w^6 = o(|A_m|)$, $\int_{A_m} (w''/w^2)^2 = o(|A_m|)$, $\int_{A_m} 1/w^2 = o(|A_m|^5)$,

(iii) $\int_{A_m} ([(wa)']^2)/(\alpha w^3) = o(|A_m|)$, *and*

(iv) $\int_{A_m} \gamma^2 = o(|A_m|)$

as $m \to \infty$. Then $C(\tilde{L}) = (-\infty, \infty)$.

Note that (ii) implies that $\int_{A_m} (w'/w^2)^2 = o(|A_m|^3)$ by $(w'/w^2)^2 = (w')^2/w^3 \cdot 1/w$ and Cauchy-Schwartz Inequality.

Proof. As is the previous theorem define

$$f_m = \begin{bmatrix} f_{m1} \\ f_{m2} \end{bmatrix} \quad \text{where} \quad f_{m2} = 0 \quad \text{and} \quad f_{m1} = (b_m e^{iQ} h_m) w^{-1/2} .$$

Then again $b_m^2 = K/a_m$ and $|f_{m1}| \leqq b_m w^{-1/2} = (K/(wa_m))^{1/2}$. Calculating

$$\begin{aligned} f'_{m1} &= w^{-1/2} b_m e^{iQ} h'_m + f_{m1}[iQ' - 1/2w^{-1}w'] \\ f''_{m1} &= f_{m1}[-(Q')^2 - iQ'w^{-1}w' + 3/4w^{-2}(w')^2 - 1/2w^{-1}w'' + iQ''] \\ &\quad + b_m e^{iQ}[2w^{-1/2}iQ'h'_m - w^{-3/2}w'h'_m + w^{-1/2}h''_m] . \end{aligned}$$

Then $(\tilde{L} - \mu I)f_m = (1/w)f''_m + Pf_m$, where the top element is

$$\begin{aligned} &\frac{1}{w}f''_{m1} + (\alpha - \mu)f_{m1} = \frac{f_{m1}}{w}[-(Q')^2 + (\alpha - \mu)w] \\ &+ \frac{f_{m1}}{w}\Big[-iQ'w^{-1}w' + \frac{3}{4}w^{-2}(w')^2 - \frac{1}{2}w^{-1}w'' + iQ''\Big] \\ &+ b_m e^{iQ}[w^{-3/2}][2iQ'h'_m - w^{-1}w'h'_m + h''_m] \\ &= \frac{f_{m1}}{w}[-(Q')^2 + (\alpha - \mu)w] + \frac{f_{m1}}{w^3}\Big[-iQ'ww' + \frac{3}{4}(w')^2 - \frac{1}{2}ww'' + w^2 iQ''\Big] \\ &+ b_m e^{iQ} w^{-3/2}[2iQ'h'_m - w^{-1}w'h'_m + h''_m] . \end{aligned}$$

Of course, the second element of $(L - \mu I)f_m$ is γf_{m1}. By choosing $(Q')^2 = (\alpha - \mu)w$ we have that by (i)

$$Q' = [(\alpha - \mu)w]^{1/2} = O((\alpha w)^{1/2}) \quad \text{as} \quad t \longrightarrow \infty .$$

$$Q'' = O\left(\frac{[\alpha w]'}{\sqrt{\alpha w}}\right) \quad \text{as} \quad t \longrightarrow \infty .$$

Then by the calculations above

$$\begin{aligned} \|(\tilde{L} - \mu I)f_m\| \leqq & \left\|\frac{f_{m1}}{w^2}Q'w'\right\| + \frac{3}{4}\left\|\frac{f_{m1}}{w^3}(w')^2\right\| + \frac{1}{2}\left\|f_{m1}\frac{w''}{w^2}\right\| \\ & + \left\|\frac{f_{m1}Q''}{w}\right\| + 2\|b_m w^{-3/2}Q'h'_m\| \\ & + \|b_m w^{-5/2}w'h'_m\| \\ & + \|b_m w^{-3/2}h''_m\| + \|\gamma f_{m1}\| . \end{aligned} \tag{7}$$

Since $|f_{m1}|^2 \leqq K/(wa_m)$ and $(Q')^2 = (\alpha - \mu)w$,

$$\|f_{m1}w^{-2}Q'w'\| \leqq \left(\frac{K}{a_m}\int_{A_m}(\alpha - \mu)w^{-3}(w')^2\right)^{1/2} = o(1) \quad \text{as} \quad m \longrightarrow \infty \quad \text{by (i)} .$$

Similarly,

$$\|f_{m1}w^{-3}(w')^2\| \leqq \left(\frac{K}{a_m}\int_{A_m}[(w')^2w^{-3}]^2\right)^{1/2} = o(1) \quad \text{by (ii)} .$$

By the definition of Q and f_{m1},

$$\|f_{m1}w^{-1}Q''\| = O\left(\int_{A_m}\frac{K[(\alpha w)']^2}{a_m\alpha w^3}\right)^{1/2} = o(1) \quad \text{by (iii)} .$$

And by condition (ii),

$$\|f_{m1}w^{-2}w''\| \leqq \left(\frac{K}{a_m}\int_{Am}[(w'')^2w^{-4}]\right)^{1/2} = o(1) .$$

Since $|b_m|^2 = K/a_m$ and $|h'_m| \leqq K_1/a_m$,

$$\|b_m w^{-3/2}Q'h'_m\| \leqq \left((KK_1^2/a_m^3)\int_{A_m}\left(\frac{\alpha - \mu}{w}\right)\right)^{1/2} = o(1) \quad \text{by (i)} .$$

Similarly, by the remark at the end of the theorem,

$$\|b_m w^{-5/2}w'h'_m\| \leqq \left((KK_1^2/a_m^3)\int_{A_m}(w')^2w^{-4}\right)^{1/2} = o(1) .$$

Since $|h''_m| \leqq K_2/a_m^2$,

$$\|b_m w^{-3/2}h''_m\| \leqq \left((KK_2^2/a_m^5)\int_{A_m}w^{-2}\right)^{1/2} = o(1) \quad \text{by (ii)} .$$

By (iv),

$$\| \gamma f_{m1} \| \leqq \left((K/a_m) \int_{A_m} \gamma^2 \right)^{1/2} = o(1) \quad \text{as} \quad m \longrightarrow \infty \ .$$

Hence, by the above calculations and (7),

$$\| (\tilde{L} - \mu I) f_m \| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty \ .$$

Since this is what we were to show, this conclude the proof.

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