PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 1, 1975

CLOSE-TO-STARLIKE HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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Let X be a finite dimensional complex normed linear space with unit ball $B = \{x \in X: \| x \| < 1\}$. In this paper the notion of a close-to-starlike holomorphic mapping from B into X is defined. The definition is a direct generalization of W. Kaplan's notion of one dimensional close-to-convex functions. It is shown that close-to-starlike mappings of B into X are univalent and these mappings are given an alternate characterization in terms of subordination chains.

1. Introduction. In 1952 [2] W. Kaplan defined the class of close-to-convex functions: $f(z) = z + \cdots$ analytic and

$$Re\,\{f'(z)/\phi'(z)\} > 0 \tag{1.1}$$

in $|z| < 1$, for some univalent convex function $\phi(z) = az + \cdots$ ($|z| < 1$). Subsequent interest in this class stems from Kaplan's observation that (1.1) implies $f(z)$ is univalent in $|z| < 1$. In this paper we present the natural generalization of close-to-convex vector valued functions in finite dimensional complex spaces. This is a continuation of recent work on vector valued holomorphic starlike and convex mappings [7], [8]. We use the notions of subordination chains of holomorphic maps in C^n and the generalized Loewner differential equation [5] to elucidate the geometry of the mappings.

2. Statement of main results. Let X be a finite dimensional complex normed linear space with dual X^* and $\mathscr{L}(X)$ the set of continuous linear operators from X into X. We let $\mathscr{H}(B)$ denote the set of functions $f(x)$ that are holomorphic in the unit ball $B = \{x \in X: ||x|| < 1\}$ with values in X. The notation $f(x) = ax + \cdots$, $a \in C$, for $f \in \mathscr{H}(B)$ indicates that $Df(0) = aI$ where I is the identity in $\mathscr{L}(X)$.

For $0 \neq x \in X$ we define

$$T(x) = \{x^* \in X^*: x^*(x) = ||x|| \quad \text{and} \quad ||x^*|| = 1\}\,,$$

and note that $T(x)$ is nonempty by the Hahn-Banach theorem. We let $\mathscr{M}$ denote the class of functions $h(x) = x + \cdots \in \mathscr{H}(B)$ such that $Re\, x^*(h(x)) > 0$ for each $x \in B - \{0\}$ and $x^* \in T(x)$. A mapping $g(x) = x + \cdots \in \mathscr{H}(B)$ is called starlike if g is univalent in B and $tg(B) \subset g(B)$ for all $0 \leqq t \leqq 1$.

DEFINITION 1. A mapping $f(x) = x + \cdots \in \mathscr{H}(B)$ is said to be *close-to-starlike* if there exist $h(x) \in \mathscr{M}$ and a starlike map $g \in \mathscr{H}(B)$ such that

$$Df(x)(h(x)) = g(x),\ x \in B\,. \tag{2.1}$$

REMARK. By this definition, close-to-starlike maps in X are a generalization of Kaplan's close-to-convex functions in C. Indeed when $X = C$ a function $h \in \mathscr{M}$ has the form $h(z) = zP(z)$ where

$$P(0) = 1,\ Re\, P(z) > 0(|z| < 1)$$

(see [8, p. 576]) and (2.1) is equivalent to the condition (1.1) for the convex function $\phi(z) = \int_0^z g(x)/x\, dx$. By using the criteria for starlikeness and convexity of vector valued maps established in [7] one can easily construct examples showing that Alexander's theorem ($\phi(z)$ is convex if and only if $g(z) = z\phi'(z)$ is starlike) fails in spaces of dimension greater than one. Hence the name close-to-starlike seems most natural in our work.

A mapping $v(x) \in \mathscr{H}(B)$ is called a *Schwarz function* if $\|v(x)\| \leqq \|x\|$ for all $x \in B$. A *subordination chain* ([5], [6]) is a function $f(x, t)$ from $B \times [0, \infty)$ into X such that for each $t \geqq 0$, $f_t(x) = f(x, t) = e^t x + \cdots$ is in $\mathscr{H}(B)$ and there exist Schwarz functions $v(x, s, t)$ such that

$$f(x, s) = f(v(x, s, t), t),\ 0 \leqq s \leqq t,\ x \in B\,, \tag{2.2}$$

for all $0 \leqq s \leqq t < \infty$. A *univalent subordination chain* is a subordination chain $f(x, t)$ such that for each $t \geqq 0$, $f_t(x)$ is univalent in B.

THEOREM 1. *If $f(x) = x + \cdots \in \mathscr{H}(B)$ is locally biholomorphic in B and close-to-starlike relative to the starlike function $g(x) = x + \cdots$ then*

$$F(x, t) = f(x) + (e^t - 1)g(x),\ 0 \leqq t,\ x \in B\,, \tag{2.3}$$

is a univalent subordination chain. Hence $f(x)$ is univalent in B.

We shall give the proof of Theorem 1 in §3 below. The subordination chain characterization (2.3) yields the linear accessibility of the images of the balls $B_r = \{x \in X: \|x\| < r\}$ $(0 < r < 1)$ (compare [1] and [3]).

COROLLARY 1. *If f and g satisfy the hypotheses of Theorem 1 then for each r, $0 < r < 1$, the complement (in X) of $f(B_r)$ is the union of nonintersecting rays.*

Proof. We assume that Theorem 1 holds and therefore the rays

$$L(t: x, r) = \{f(x) + tg(x): t \geqq 0 \; x \quad \text{fixed,} \quad \|x\| = r\}$$

are clearly disjoint and fill up the complement of $f(B_r)$.

THEOREM 2. *Suppose* $f(x) = x + \cdots$ *is holomorphic in* B *and that* $g(x) = x + \cdots \in \mathscr{H}(B)$ *is starlike. If*

$$F(x, t) = f(x) + (e^t - 1)g(x), \; 0 \leqq t, \; x \in B \tag{2.3}$$

is a univalent subordination chain then f *is close-to-starlike relative to* g.

We shall prove this theorem in §4 below. By the results in [8] a mapping $f(x) = x + \cdots \in \mathscr{H}(B)$ is starlike univalent if and only if it is close-to-starlike relative to itself, i.e., (2.1) holds with $g = f$. Thus from Theorems 1 and 2 we have immediately the

COROLLARY 2. *Let* $f(x) = x + \cdots$ *be locally biholomorphic in* B. *Then* f *is univalent and starlike in* B *if and only if* $F(x, t) = e^t f(x)$ *is a univalent subordination chain.*

This extends to higher dimensional spaces Pommerenke's one dimensional result in Folgerung 2 of [6].

3. *Proof of Theorem* 1. We shall give the proof in a sequence of three lemmas. We use the notation $f_r(x) = f(rx)/r$, $g_r(x) = g(rx)/r$ and $F_r(x, t) = f_r(x) + (e^t - 1)g_r(x)$ for $0 \leqq r \leqq 1$, $t \geqq 0$. Let $R = \{r: 0 \leqq r \leqq 1$ and $F_\rho(x, t)$ is a univalent subordination chain for $\rho < r\}$. Then $0 \in R$ and clearly R is closed. We wish to show that R is open so $R = [0, 1]$.

LEMMA 3.1. *If* $r \in R$ *then* $F_r(x, t)$ *is a univalent subordination chain.*

Proof. Since $f_0(x) = Df(0)(x) = x = g_0(x)$ we have $F_0(x, t) = x + (e^t - 1)x = e^t x$ which is clearly a univalent subordination chain. Now if $0 < r \in R$, $\rho \leqq \lambda < r$ then for $s \leqq t$ and $\|x\| < \rho/\lambda$ we have

$$\begin{aligned} F_\lambda(v_\lambda(x, s, t), t) &= F_\lambda(x, s) = f(\lambda x)/\lambda + (e^s - 1)g(\lambda x)/\lambda \\ &= (\rho/\lambda)[(1/\rho)f(\rho(\lambda x/\rho)) + (1/\rho)(e^s - 1)g(\rho(\lambda x/\rho))] \\ &= (\rho/\lambda)F_\rho(\lambda x/\rho, s) = (\rho/\lambda)F_\rho(v_\rho(\lambda x/\rho, s, t), t) \\ &= F_\lambda((\rho/\lambda)v_\rho(\lambda x/\rho, s, t), t) \; . \end{aligned}$$

Hence $(\rho/\lambda)v_\rho((\lambda/\rho)x, s, t)$ is independent of ρ when $\rho \leqq \lambda$ and $\|x\| <$

ρ/λ ($v_\rho(x, s, t)$, $v_\lambda(x, s, t)$ are the univalent Schwarz functions postulated by the fact that $F_\rho(x, t)$ and $F_\lambda(x, t)$ are univalent subordination chains). Hence we may define $v_r(x, s, t) = (\rho/r)v_\rho((r/\rho)x, s, t)$ where $\|x\| < 1$ and ρ satisfies $\|x\| < \rho/r < 1$. Then v_r is well defined in B, it is a univalent Schwarz function and

$$\begin{aligned} F_r(x, s) &= (\rho/r)F_\rho(rx/\rho, s) = (\rho/r)F_\rho(v_\rho(rx/\rho, s, t), t) \\ &= F_r((\rho/r)v_\rho(rx/\rho, s, t), t) = F_r(v_r(x, s, t), t) \end{aligned}$$

when $\|x\| < \rho/r < 1$, $0 \leqq s \leqq t$. Therefore $F_r(x, t)$ is a univalent subordination chain.

LEMMA 3.2. *If $r \in R$, $r < 1$ then there exists $\varepsilon_0 > 0$ such that $F_{r+\varepsilon}(x, t)$ is a univalent function of $x \in B$ for each $t \geqq 0$ and $0 \leqq \varepsilon < \varepsilon_0$.*

Proof. Since $F_r(x, t)$ is a univalent subordination chain, $f_r(x)$ is univalent in the closed ball $\bar{B}$ (for otherwise, there exist $\rho < r$, x, y, t, $x \neq y$, $\|x\| = \|y\| < \rho/r$, $t > 0$ such that $v_r(x, 0, t) = v_r(y, 0, t)$). Let $G(x, y)$ be the $n \times n$ determinant whose kth column is

$$A_k = \begin{cases} (x_k - y_k)^{-1}[f_r(y_1, \cdots, y_{k-1}, x_k, \cdots, x_n) - f(y_1, \cdots, y_k, \\ \quad x_{k+1}, \cdots, x_n)], \; x_k \neq y_k \\ \dfrac{\partial}{\partial x_k} f_r(y_1, \cdots, y_{k-1}, x_k, \cdots, x_n), \quad \text{if} \quad x_k = y_k \end{cases}$$

and define $H(x, y) = |G(x, y)| + \|f_r(x) - f_r(y)\|$ where $x, y \in B_{1+\varepsilon}$, $1 + \varepsilon < 1/r$. If $x = y$ we have $H(x, x) = |\det Df_r(x)| > 0$ since f_r is biholomorphic. If $x \neq y$ and $f_r(x) \neq f_r(y)$ then $H(x, y) > 0$. If $x \neq y$ and $f_r(x) = f_r(y)$ then $\sum_{k=1}^n (x_k - y_k)A_k = f_r(x) - f_r(y) = 0$ and $H(x, y) = 0$ since the columns of $G(x, y)$ are dependent. Thus $H(x, y) = 0$ if and only if $f_r(x) = f_r(y)$ and $x \neq y$. We conclude that $H(x, y)$ has a positive minimum on $\bar{B} \times \bar{B}$ and in fact $H(x, y) > 0$ if $(x, y) \in B_{1+\varepsilon} \times B_{1+\varepsilon}$ when $0 \leqq \varepsilon < \varepsilon'$ for some $\varepsilon' > 0$. This implies that $f_{r+\varepsilon}$ is univalent in B for $0 \leqq \varepsilon < \varepsilon''$ for some $\varepsilon'' > 0$.

For small $\varepsilon > 0$, $e^{-t}F_{r+\varepsilon}(x, t)$ converges to $g_{r+\varepsilon}(x)$ uniformly in B as $t \to \infty$. Hence $F_{r+\varepsilon}(x, t)$ is univalent and starlike for $t > t_0$ for some $t_0 > 0$.

Now assume the lemma is false. Then there exist sequences $\{\varepsilon_k\}$, $\{t_k\}$ of positive numbers and points $\{x_k\}$, $\{y_k\}$ in $\bar{B}$ such that $\varepsilon_k \to 0$, $x_k \neq y_k$, $\|x_k\| = \|y_k\| = 1$, $t_k < t_0$ and $F_{r+\varepsilon_k}(x_k, t_k) = F_{r+\varepsilon_k}(y_k, t_k)$. (We may assume $\|x_k\| = \|y_k\| = 1$ since by the reasoning of Ono in [4] univalence on the boundary of B implies univalence in the interior.) By choosing subsequences we may find limit points s, u, v, $0 < s \leqq t_0$, $\|u\| = \|v\| = 1$ such that $F_r(u, s) = F_r(v, s)$. Since $F_r(x, t)$ is a univalent subordination chain we must have $u = v$. Hence

$$\begin{aligned} 0 = \frac{F_{r+\varepsilon_k}(x_k, t_k) - F_{r+\varepsilon_k}(y_k, t_k)}{\|x_k - y_k\|} &= DF_r(u, s)\left(\frac{x_k - y_k}{\|x_k - y_k\|}\right) \\ &+ (DF_{r+\varepsilon_k}(y_k, t_k) - DF_r(y_k, t_k))\left(\frac{x_k - y_k}{\|x_k - y_k\|}\right) \\ &+ (DF_r(y_k, t_k) - DF_r(u, s))\left(\frac{x_k - y_k}{\|x_k - y_k\|}\right) + o(x_k - y_k) \end{aligned}$$

and by using appropriate subsequences we conclude that $DF_r(u, s)$ is singular. This is a contradiction since $DF_r(u, s) = DF_r(v_r(x, s, t), t)Dv_r(x, s, t)$ is the composition of two nonsingular maps in $\mathscr{L}(X)$, and the lemma is established.

LEMMA 3.3. *Let ε_0 be as determined in Lemma 3.2. Then for $0 \leqq \varepsilon < \varepsilon_0$ $F_{r+\varepsilon}$ (x, t) is a univalent subordination chain.*

Proof. We must show that for $0 \neq x \in B$ and $x^* \in T(x)$ we have

$$Re\, x^*([DF_{r+\varepsilon}(x, t)]^{-1}(g(x))) \geqq 0 ,$$

for then $\partial\|v_{r+\varepsilon}(x, s, t)\|/\partial t \leqq 0$, $s \leqq t$. It will follow that $v_{r+\varepsilon}(x, s, t) = F^{-1}_{r+\varepsilon}(F_{r+\varepsilon}(x, s), t)$ is a univalent Schwarz function.

Let $x^* \in T(x)$, $0 \neq x \in B$ and suppose

$$Re\, x^*\{[DF_{r+\varepsilon}(x, t)]^{-1}(g(x))\} < 0$$

for some t. Then since the reverse inequality holds for $t = 0$ and sufficiently large t, there exist s, t, u, v, $0 < s < t < \infty$, $u, v \in X$, $Re\, x^*(u) = Re\, x^*(v) = 0$ such that

$$e^s g(x) = Df(x)(u) + (e^s - 1)Dg(x)(u) \tag{3.1}$$

$$e^t g(x) = Df(x)(v) + (e^t - 1)Dg(x)(v) . \tag{3.2}$$

Let

$$L = \{y \in X\colon Re\, x^*(y) = 0\}$$

and

$$L_1 = L \cap (Df(x))^{-1}(Dg(x)(L)) = L \cap (Dg(x))^{-1}(Df(x)(L))$$

and view L and L_1 as linear spaces over the *real* numbers. If $L = L_1$, then $g(x)$ is in the space $Df(x)(L) = Dg(x)(L)$ which is impossible since $Re\, x^*\{[Df(x)]^{-1}(g(x))\} > 0$ by (2.1). Thus L and L_1 have real dimension $2n - 1$ and $2n - 2$ respectively where n is the complex dimension of X.

We wish to show that $u = v$ and $s = t$. Let $y_0 \in L - L_1$ and observe that we may write u and v uniquely in the form $u = ay_0 +$

u_1, $v = by_0 + v_1$ where a and b are real, $u_1, v_1 \in L_1$. Then (3.1) and (3.2) yield that

$$
\begin{aligned}
g(x) &= a[e^{-s}Df(x)(y_0) + (1 - e^{-s})Dg(x)(y_0)] + w_1 \\
&= b[e^{-t}Df(x)(y_0) + (1 - e^{-s})Dg(x)(y_0)] + w_2
\end{aligned}
\tag{3.3}
$$

where $w_1, w_2 \in Dg(x)(L_1) = Df(x)(L_1)$. We shall show that $g(x)$ has a unique representation of the form $\alpha Df(x)(y_0) + \beta Dg(y_0) + w$ where $w \in Dg(x)(L_1)$ and α, β are real. To this end, we assume that

$$\alpha Df(x)(y_0) + \beta Dg(x)(y_0) \in Dg(x)(L_1)$$

for some real α, β. Then $Df(x)(\alpha y_0) = Dg(x)(w_3 - \beta y_0)$ for some $w_3 \in L_1$ and consequently $\alpha y_0 \in L_1$. This implies that $\alpha = 0$ and then $\beta y_0 = w_3 \in L_1$ and $\beta = 0$. Thus from (3.3) we conclude $ae^{-s} = be^{-t}$, $a(1 - e^{-s}) = b(1 - e^{-t})$ and therefore $a = b$ and $s = t$. This contradicts our assumption that $s < t$ and completes the proof of the lemma.

The proof of Theorem 1 is now complete for we have shown that R is a nonempty subset of $[0, 1]$ that is both open and closed. Hence $R = [0, 1]$ and $F(x, t) = F_1(x, t)$ is a univalent subordination chain by Lemma 3.1.

4. *Proof of Theorem* 2. By hypothesis there are univalent Schwarz functions $v(x, s, t)$ such that $F(x, s) = F(v(x, s, t), t)(0 \leqq s \leqq t)$ for the chain $F(x, t)$ defined in (2.3). It is clear from the form of (2.3) that the derivative

$$\frac{\partial F}{\partial t}(x, t) = \lim_{s \to t} \frac{F(x, s) - F(x, t)}{s - t} \tag{4.1}$$

exists and the convergence is uniform on compact subsets of B.

We fix $t > 0$, let $s < t$ and write

$$
\begin{aligned}
F(x, s) - F(x, t) &= F(x, s) - F(v(x, s, t), t) \\
&= DF(x, t)(v(x, s, t) - x) + o(v - x)
\end{aligned}
$$

where $o(v - x)/\|v - x\|$ tends to zero uniformly for x in a compact subset of B as $v(x, s, t) - x$ tends to zero. Thus

$$\frac{F(x, s) - F(x, t)}{s - t} = DF(x, t)\Big(\frac{x - v(x, s, t)}{t - s}\Big) + \frac{o(v - x)}{s - t} \tag{4.2}$$

and since $DF(x, t)$ is nonsingular we can argue (as in [8] Lemma 2) that $(x - v(x, s, t))/(t - s)$ is bounded and tends to a limit, and that $o(v(x, s, t) - x)/(s - t)$ tends to zero as s tends to t (the univalence of the chain insures that $v(x, s, t)$ tends to x as $s \to t$). Since $t - s > 0$ and

$$\begin{aligned} Re\, x^*(x - v(x, s, t)) &= \|x\| - Re\, x^*(v(x, s, t)) \\ &\geqq \|x\| - \|v(x, s, t)\| \geqq 0 \end{aligned}$$

for each $x^* \in T(x)$ it follows that the function

$$h(x, t) = \lim_{s \to t} \frac{x - v(x, s, t)}{t - s}, \ t > 0 , \tag{4.3}$$

is in the class $\mathscr{M}$.

From (4.1) – (4.3) we conclude that $F(x, t)$ satisfies the generalized Loewner differential equation [5]

$$\partial F(x, t)/\partial t = DF(x, t)(h(x, t)), \ x \in B , \tag{4.4}$$

for each $t > 0$. For the specific subordination chain (2.3) it is clear that we may let t tend to zero in (4.4) to obtain

$$g(x) = Df(x)(h(x, 0)) ,$$

and $h(x, 0) \in \mathscr{M}$ since the properties of $\mathscr{M}$ are preserved by local uniform convergence. This completes the proof of Theorem 2.

5. EXAMPLES. (1) Let $f(z) = z + \cdots$ be close to the starlike function $g(z) = z + \cdots$ where f and g are complex valued analytic functions of z in the open unit disk, $|z| < 1$. Let X be a complex finite dimensional inner product space with inner product $\langle , \rangle$ and let $x_0 \in X$, $\|x_0\| = 1$. Define the vector valued holomorphic maps

$$F(x) = \frac{f(\langle x, x_0 \rangle)}{\langle x, x_0 \rangle} x, \ G(x) = \frac{g(\langle x, x_0 \rangle)}{\langle x, x_0 \rangle} x$$

for x in B, the unit ball in X. Then

$$DG(x) = \frac{-\langle \cdot, x_0 \rangle}{\langle x, x_0 \rangle^2} g(\langle x, x_0 \rangle) x + \frac{\langle \cdot, x_0 \rangle}{\langle x, x_0 \rangle} g'(< x, x_0 \rangle) x + \frac{g(\langle x, x_0 \rangle)}{\langle x, x_0 \rangle} I$$

where $I \in \mathscr{L}(X)$ is the identity. A similar formula holds for $DF(x)$. Setting $H(x) = g(\langle x, x_0 \rangle)x/(\langle x, x_0 \rangle g'(\langle x, x_0 \rangle))$ we see that $H \in \mathscr{M}$ and $DG(x)(H(x)) = G(x)$ so G is starlike [7]. Similarly if $K(x) = g(\langle x, x_0 \rangle)x/(\langle x, x_0 \rangle f'(\langle x, x_0 \rangle))$ then $K \in \mathscr{M}$ and $DF(x)(K(x)) = G(x)$ so F is close-to-starlike. Note that F and G both reduce to the identity map on the subspace orthogonal to x_0. An interesting choice of f and g is $f(z) = (1/2) \log [(1 + z)/(1 - z)]$, $g(z) = z/(1 + z)^2$. Then $f + (e^t - 1)g$ maps the unit disk onto the entire plane slit along two parallel rays when $0 < t < \infty$. Also $F(x) + (e^t - 1)G(x)$ has similar behavior on the one dimensional slice $\{\alpha x_0 : \alpha \in C, |\alpha| < 1\}$.

(2) Let $X = C^2$ with the usual inner product and Euclidean norm

$$\langle x, y\rangle = \sum_{j=1}^{2} x_j \bar{y}_j,\ \|x\| = \langle x, x\rangle^{1/2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in C^2. We define the functions

$$f(x) = (2^{-1}[(1 - x_1)^{-2} - 1],\ x_2 + ax_1x_2), \tag{5.1}$$

$$g(x) = (x_1/(1 - x_1)^2,\ x_2[1 + 2bx_1 + bx_1^2]), \tag{5.2}$$

$$h(x) = (x_1(1 - x_1),\ x_2[1 + 4ax_1/(2a - 1)]) \tag{5.3}$$

where $\|x\| < 1$, $b = a(2a + 1)/(2a - 1)$ and a is a complex number with small modulus. We claim that if $|a|$ is sufficiently small then: (I) $h(x)$ belongs to the class $\mathscr{M}$, (II) $g(x) = x + \cdots \in \mathscr{H}(B)$ is starlike, (III) $f(x) = x + \cdots \in \mathscr{H}(B)$ is close-to-$g(x)$, and (IV) f is not starlike.

(I) Clearly (5.3) is holomorphic in B and has the required normalization $h(x) = x + \cdots$. Furthermore, if $|a|$ is sufficiently small then

$$Re < h\langle x\rangle, x\rangle = |x_1|^2\, Re\,(1 - x_1) + |x_2|^2\, Re\left(1 + \frac{4ax_1}{2a - 1}\right) > 0, \tag{5.4}$$

for all $x \in B$ and $h \in \mathscr{M}$ [8, p. 577].

(II) The holomorphy and normalization of (5.2) are clear. We must show that $(Dg(x))^{-1}(g(x))$ belongs to $\mathscr{M}$ if $|a|$ is small. Elementary computations with (5.2) yield that

$$(Dg(x))^{-1}(g(x)) = \left(x_1\left(\frac{1 - x_1}{1 + x_1}\right),\ x_2\left[1 - \frac{2bx_1(1 - x_1)}{1 + 2bx_1 + bx_1^2}\right]\right)$$

and therefore $Re\,\langle (Dg(x))^{-1}(g(x)), x\rangle \geqq 0$ for all $x \in B$ and small $|a|$.

(III) It is easy to verify that (5.1), (5.2), and (5.3) satisfy the equation $Df(x)(h(x)) = g(x)$ and hence that f is close-to-g.

(IV) We must show that $(Df(x))^{-1}(f(x))$ does not belong to $\mathscr{M}$. This follows when $|a|$ is small since

$$(Df(x))^{-1}(f(x)) = \left(\frac{x_1(2 - x_1)(1 - x_1)}{2},\ x_2\left[1 - \frac{ax_1(2 - x_1)(1 - x_1)}{1 + ax_1}\right]\right),$$

and $Re\,(2 - x_1)(1 - x_1) < 0$ at some points in the unit disk $|x_1| < 1$.

Finally we mention that the functions (5.1), (5.2), and (5.3) provide an example similar to the preceding one when we consider $X = C^2$ with the sup norm, $\|x\|_\infty = \max(|x_1|, |x_2|)$. In this setting the condition (5.4) for membership in $\mathscr{M}$ is replaced by the condition $Re\,(h_j(x)/x_j) > 0$ when $\|x\|_\infty = |x_j| > 0$ [8].

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Received March 21, 1974. This research was performed while the authors were visitors in the Department of Mathematics at the University of Maryland. We express our sincere thanks to the chairman and members of that department for their hospitality and support during our visit. The research of the second author was partially supported by NSF Grant GP-39053.

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