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# ENUMERATION OF WEIGHTED $p$-LINE ARRAYS

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**Let $F_p(n,k;q_1,q_2,\cdots,q_p) = F_p(n,k)$ be defined by**

$$F_p(n,k) = \sum \prod_{i=1}^{p} q_i^{\sum_{j=1}^{n} a_{ij}},$$

**where the summation is over all $p$-line arrays of positive integers**

$$\boxed{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{matrix}}$$

**subject to the following conditions:**

$$\max\{a_{ij}: 1 \leqq i \leqq p\} \leqq \min\{a_{i,j+1}: 1 \leqq i \leqq p\}, \qquad 1 \leqq j \leqq n-1,$$

$$\max\{a_{ij}: 1 \leqq i \leqq p\} \leqq j, \qquad 1 \leqq j \leqq n,$$

**and**

$$a_{in} = k, \qquad 1 \leqq i \leqq p.$$

**Assuming $\prod_{t=1}^{p} q_t = 1$, formulas for $F_p(n,k)$ and two other enumerants, which are closely related to $F_p(n,k)$, are obtained in this paper. These three functions generalize enumerants which Carlitz has determined.**

**1. Introduction.** We consider the enumeration of $p$-line arrays of positive integers

$$(1.1) \qquad \boxed{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{matrix}}$$

satisfying certain conditions. We first require that

$$(1.2) \qquad \max\{a_{ij}: 1 \leqq i \leqq p\} \leqq \min\{a_{i,j+1}: 1 \leqq i \leqq p\}, \quad 1 \leqq j \leqq n-1,$$

and

(1.3) $$\max\{a_{ij}\colon 1 \leqq i \leqq p\} \leqq j, \qquad 1 \leqq j \leqq n.$$

We indicate further requirements by defining the enumerants we seek. Let $f_p(n, k; s_1, s_2, \cdots, s_p)$ represent the number of arrays (1.1) subject to the restrictions (1.2), (1.3),

(1.4) $$\sum_{j=1}^{n} a_{ij} = s_i, \qquad 1 \leqq i \leqq p,$$

and

(1.5) $$a_{in} = k, \qquad 1 \leqq i \leqq p,$$

and let

$$F_p(n, k; q_1, q_2, \cdots, q_p) = F_p(n, k) = F_p$$
$$= \sum{}^* f_p(n, k; s_1, s_2, \cdots, s_p) q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p},$$

where $\Sigma^*$ is the sum over the $p$-tuples $(s_1, s_2, \cdots, s_p)$. (We may view $(s_1, s_2, \cdots, s_p)$ as the weight of the array (1.1).) Let $g_p(n, k; s_1, s_2, \cdots, s_p)$ denote the number of arrays (1.1) satisfying (1.2), (1.3), (1.4) and

(1.6) $$\max\{a_{in}\colon 1 \leqq i \leqq p\} = k,$$

and let

$$G_p(n, k; q_1, q_2, \cdots, q_p) = G_p(n, k) = G_p$$
$$= \sum{}^* g_p(n, k; s_1, s_2, \cdots, s_p) q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p}.$$

Finally, we use $h_p(n, k; s_1, s_2, \cdots, s_p)$ to represent the number of arrays (1.1) subject to conditions (1.2), (1.3), (1.4), (1.6) and

(1.7) $$a_{i+1,j} \leqq a_{ij}, \qquad 1 \leqq i \leqq p-1, \ 1 \leqq j \leqq n,$$

and we let

$$H_p(n, k; q_1, q_2, \cdots, q_p) = H_p(n, k) = H_p$$
$$= \sum{}^* h_p(n, k; s_1, s_2, \cdots, s_p) q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p}.$$

The functions $F_1$, $G_1$ and $H_1$ coincide, and if $q_1 = 1$, they enumerate what MacMahon [7, p. 167] called two-element lattice permutations. Carlitz and Riordan [6] have studied these functions and a $q$-generalization. A related $q$-generalization, in fact $F_1(n,k;q_1)$, has also been investigated by Carlitz [5]. If $p=2$ and $q_1=q_2=1$, $F_2, G_2$ and $H_2$ are the enumerants $f, g$ and $h$ which Carlitz [4] has explicitly determined.

In this paper we first generalize some identities which Carlitz stated for $f, g$ and $h$. Then, by assuming

$$\prod_{i=1}^{p} q_i = 1, \tag{1.8}$$

we are able to use these results and Carlitz's technique for finding $f, g$ and $h$ to obtain formulas for $F_p$, $G_p$ and $H_p$. In general these formulas are in terms of functions $t(n,k)$ which are defined by

$$\Phi^n(x) = \sum_{k=0}^{\infty} t(n,k)x^k,$$

where $\Phi(x)$ is a known function. In some special cases the enumerants can be expressed in terms of binomial or $q$-binomial coefficients. For example, if $q_1=q_2=q$ and $q^2=1$,

$$F_2(n+1,k+1) = \frac{1}{n}\sum_{j=0}^{k}(n-j)b_2(n,j;q)$$

and

$$G_2(n,k+1) = \frac{n-k}{n}\, b_2(n,k;q)$$

where

$$\begin{aligned} b_2(n,k;q) &= \sum_{m=0}^{k}\binom{n}{k-m}\binom{2n+m-1}{m}q^m \\ &= \sum_{j=0}^{[k/2]}\binom{n}{k-2j}\binom{2n+2j-1}{2j} + \sum_{j=0}^{[(k-1)/2]}\binom{n}{k-2j-1}\binom{2n+2j}{2j+1}q. \end{aligned}$$

We also find that

$$H_2(n,k+1) = \frac{n-k}{n}\sum_{m=0}^{k}\binom{n+k-m-1}{k-m}\binom{n+m-1}{m}q_1^m$$

if $q_1 q_2 = 1$, and

$$H_p(n, k+1) = \frac{n-k}{n} \begin{bmatrix} np + k - 1 \\ k \end{bmatrix}_q$$

if $q_1 = q_2 = \cdots = q_p = q$ and $q^p = 1$. In §6 we interpret the formulas for $F_p, G_p$ and $H_p$ as partition theorems. It would be of interest to determine these enumerants without the restriction (1.8).

In a subsequent paper we shall consider

$$I_p(n, k; q_1, q_2, \cdots, q_p) = \sum{}^* i_p(n, k; s_1, s_2, \cdots, s_p) q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p},$$

where $i_p(n, k; s_1, s_2, \cdots, s_p)$ represents the number of arrays (1.1) satisfying (1.2), (1.3), (1.4) and (1.7), and having $k$ columns in which

$$a_{1j} = a_{2j} = \cdots = a_{pj}.$$

Carlitz **[3]** called such columns coincidences and has proved that

$$I_p(n, k; 1, 1, \cdots, 1) = \frac{1}{k} \binom{n-1}{k-1} \sum_{t=0}^{n-k} (-1)^{n-k-t} \binom{n-k}{t} \binom{2n + (p-1)t}{n-1}$$

for $q_1 = q_2 = \cdots = q_p = 1$.

**2. Preliminary results.** Generalizing (2.1)–(2.4), (2.7) and (2.9) of **[4]**, we have

$$F_p(n+1, k) \tag{2.1}$$

$$= \left[\prod_{i=1}^{p} q_i\right]^k \sum_{m=1}^{k} \left[\prod_{i=1}^{p} [k-m+1]_{q_i} - \prod_{i=1}^{p} q_i [k-m]_{q_i}\right] F_p(n, m), \quad k \leqq n+1,$$

where

$$[k]_{q_i} = \sum_{j=0}^{k-1} q_i^j,$$

$$F_p(n+1, k) = \left[\prod_{i=1}^{p} q_i\right]^k \sum_{m=1}^{k} G_p(n, m), \qquad k \leqq n+1, \tag{2.2}$$

$$G_p(n+1, k) = \sum_{m=1}^{k} \left[\prod_{i=1}^{p} q_i\right]^m \left[\prod_{i=1}^{p} [k-m+1]_{q_i} - \prod_{i=1}^{p} [k-m]_{q_i}\right] G_p(n, m), \qquad k \leqq n+1, \tag{2.3}$$

$$G_p(n,k) = \sum_{m=1}^{k} \left[ \prod_{i=1}^{p} [k-m+1]_{q_i} - \left(1 + \prod_{i=1}^{p} q_i\right) \prod_{i=1}^{p} [k-m]_{q_i} + \prod_{i=1}^{p} q_i \prod_{i=1}^{p} [k-m-1]_{q_i} \right] F_p(n,m), \qquad k \leqq n, \tag{2.4}$$

$$G_p(n+k,k) = \sum_{m=1}^{k} \left[ \prod_{i=1}^{p} q_i \right]^{(n+k-m)(m-1)} G_p(m,m) G_p(n+k-m, k-m+1), \qquad n \geqq 1, \tag{2.5}$$

and

$$F_p(n+k,k) = \sum_{j=1}^{k} \left[ \prod_{i=1}^{p} q_i \right]^{(n+k-m)(m-1)} G_p(m,m) F_p(n+k-m, k-m+1), \qquad n \geqq 1. \tag{2.6}$$

Let

$$\theta_p(k; q_2, q_3, \cdots, q_p) = \sum \prod_{i=2}^{p} q_i^{a_{in}}, \qquad p \geqq 2, \tag{2.7}$$

where the summation is over all $(p-1)$-tuples $(a_{2n}, a_{3n}, \cdots, a_{pn})$ with the $a_{in}$ satisfying $0 \leqq a_{pn} \leqq \cdots \leqq a_{3n} \leqq a_{2n} \leqq k$. Then corresponding to (6.3) and (6.8) of **[4]** we have

$$H_p(n,k) = \sum_{m=1}^{k} \left[ \prod_{i=1}^{p} q_i \right]^{m} q_1^{k-m} \theta_p(k-m; q_2, q_3, \cdots, q_p) H_p(n-1, m), \qquad 1 \leqq k \leqq n, \tag{2.8}$$

where it is understood that $H_p(n-1, n) = 0$, and

$$H_p(n+k,k) = \sum_{j=1}^{k} \left[ \prod_{i=1}^{p} q_i \right]^{(n+k-m)(m-1)} H_p(m,m) H_p(n+k-m, k-m+1), \qquad n \geqq 1. \tag{2.9}$$

The proofs of (2.1)–(2.6), (2.8) and (2.9) are simply generalizations of the proofs of their analogues in **[4]**. To prove (2.1) it suffices to assume $k \leqq n$ since $F_p(n+1, n+1) = F_p(n+1, n)$. For $k \leqq n$ we consider the array

$$\boxed{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & k \\ a_{21} & a_{22} & \cdots & a_{2n} & k \\ \vdots & \vdots & & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} & k \end{matrix}}$$

satisfying (1.2), (1.3) and

$$\max\{a_{in} : 1 \leqq i \leqq p\} \leqq k. \tag{2.10}$$

Let

$$\min\{a_{in} : 1 \leqq i \leqq p\} = m. \tag{2.11}$$

Using (2.10), (2.11) and the definition of $F_p(n, k)$, we have

$$F_p(n+1, k) = \left[\prod_{i=1}^{p} q_i\right]^k \sum_{m=1}^{k} \sum \prod_{i=1}^{p} q_i^{\sum_{j=1}^{n} a_{ij}}, \tag{2.12}$$

where the inner summation is over all arrays (1.1) satisfying (1.2), (1.3), (2.10) and (2.11). From (2.12) we get

$$F_p(n+1, k) = \left[\prod_{i=1}^{p} q_i\right]^k \sum_{m=1}^{k} \sum{}' \prod_{i=1}^{p} q_i^{a_{in}-m} F_p(n, m), \tag{2.13}$$

where $\Sigma'$ is the sum over all $p$-tuples $(a_{1n}, a_{2n}, \cdots, a_{pn})$ subject to conditions (2.10) and (2.11). Since

$$\begin{aligned} \sum{}' \prod_{i=1}^{p} q_i^{a_{in}-m} &= \sum_{\substack{m \leqq a_{in} \leqq k \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q_i^{a_{in}-m} - \sum_{\substack{m+1 \leqq a_{in} \leqq k \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q_i^{a_{in}-m} \\ &= \sum_{\substack{0 \leqq a_{in} \leqq k-m \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q_i^{a_{in}} - \sum_{\substack{0 \leqq a_{in} \leqq k-m-1 \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q_i^{a_{in}+1} \\ &= \prod_{i=1}^{p} [k-m+1]_{q_i} - \prod_{i=1}^{p} q_i [k-m]_{q_i}, \end{aligned}$$

(2.1) follows from (2.13).

Equation (2.2) follows immediately from the definitions. The proof of (2.3) is similar to the proof of (2.1), but to obtain (2.3) we consider the array

$$\begin{array}{|ccccc|} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} & a_{p,n+1} \end{array}$$

where conditions (1.2) and (1.3) (with $n$ replaced by $n + 1$) are satisfied and where

$$\max\{a_{in}: 1 \leqq i \leqq p\} = m \tag{2.14}$$

and

$$\max\{a_{i,n+1}: 1 \leqq i \leqq p\} = k. \tag{2.15}$$

As (2.14) and (1.2) imply that

$$\min\{a_{i,n+1}: 1 \leqq i \leqq p\} \geqq m, \tag{2.16}$$

we find that

$$G_p(n + 1, k) = \sum_{m=1}^{k} \sum \prod_{i=1}^{p} q_i^{a_{i,n+1}} G_p(n, m), \tag{2.17}$$

where the inner sum is over all $p$-tuples $(a_{1,n+1}; a_{2,n+1}; \cdots; a_{p,n+1})$ satisfying (2.15) and (2.16). From (2.17) we get (2.3).

To prove (2.4) consider the array (1.1) subject to conditions (1.2), (1.3), (1.6) and (2.11). Corresponding to (2.13) and (2.17) in the previous proofs, we have

$$G_p(n, k) = \sum_{m=1}^{k} \sum \prod_{i=1}^{p} q_i^{a_{in}-m} F_p(n, m), \tag{2.18}$$

where the inner summation is over the $p$-tuples $(a_{1n}, a_{2n}, \cdots, a_{pn})$ satisfying (1.6) and (2.11). From (2.18) we get

$$G_p(n, k) = \sum_{m=1}^{k} \left[ \sum_{\substack{m \leqq a_{in} \leqq k \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q_i^{a_{in}-m} - \sum_{\substack{m \leqq a_{in} \leqq k-1 \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q_i^{a_{in}-m} \right.$$
$$\left. - \sum_{\substack{m+1 \leqq a_{in} \leqq k \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q \quad + \sum_{\substack{m+1 \leqq a_{in} \leqq k-1 \\ 1 \leqq i \leqq p}} \prod_{i=1}^{p} q \quad \right] F_p(n, m),$$

and (2.4) follows.

Since the proofs of (2.5), (2.6) and (2.9) are similar, we shall only establish (2.5). To this end we observe that

$$\max\{a_{i1}\colon 1 \leqq i \leqq p\} = 1$$

implies that there exists a greatest $m$ such that

$$\max\{a_{im}\colon 1 \leqq i \leqq p\} = m.$$

Therefore

$$\max\{a_{i,m+1}\colon 1 \leqq i \leqq p\} = m,$$

$$a_{i,m+1} = m, \qquad 1 \leqq i \leqq p,$$

and we can divide our original array into two sub-arrays as follows:

$$\text{(2.19)} \qquad \begin{array}{|ccc|ccc|} \hline 1 & \cdots & a_{1m} & m & \cdots & a_{1,n+k} \\ 1 & \cdots & a_{2m} & m & \cdots & a_{2,n+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & a_{pm} & m & \cdots & a_{p,n+k} \\ \hline \end{array}$$

By subtracting $m-1$ from each entry in the right sub-array of (2.19), we get

$$g_p(n+k, k; s_1, s_2, \cdots, s_p)$$
$$= \sum_{m=1}^{k} g_p(m, m; u_1, u_2, \cdots, u_p) g_p(n+k-m, k-m+1; v_1, v_2, \cdots, v_p),$$

where $u_i + v_i = s_i - (n+k-m)(m-1)$, $1 \leqq i \leqq p$. Now (2.5) follows immediately.

We obtain (2.8) by considering the array

$$\begin{array}{|ccccc|} \hline a_{11} & a_{12} & \cdots & m & k \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{p,n-1} & a_{pn} \\ \hline \end{array}$$

where (1.2)–(1.4), (1.6), (1.7) and the condition

$$a_{1,n-1} = m$$

are satisfied. Clearly

$$h_p(n,k;s_1,s_2,\cdots,s_p)$$
$$= \sum_{m=1}^{k} \sum{}'' h_p(n-1,m;s_1-k,s_2-a_{2n},\cdots,s_p-a_{pn}),$$

where $\Sigma''$ is the sum over all $(p-1)$-tuples such that

$$m \leqq a_{pn} \leqq a_{p-1,n} \leqq \cdots \leqq a_{2n} \leqq k.$$

Thus

$$H_p(n,k) = \sum_{m=1}^{k} q_1^k \sum{}'' \prod_{i=2}^{p} q_i^{a_{in}} H_p(n-1,m)$$

and we have (2.8).

**3. Techniques for determining the enumerants.** To obtain the following results we use Carlitz's method **[4]** for finding $f$, $g$ and $h$ in a somewhat more general setting. Assuming (1.8), Theorem 1 provides formulas for $G_p$ and $H_p$ while Corollary 1 yields an expression for $F_p$.

Before stating the theorem it is convenient to define some functions. Using $\mathbf{N}$ to denote the nonnegative integers and $\mathbf{N}^*$ to represent the positive integers, let $r(n,k)$ be a function from $\mathbf{N}^* \times \mathbf{N}^*$ into a field $F$ and $\phi(n)$ be a function from $\mathbf{N}$ into $F$. Let

$$R_n(x) = \sum_{k=1}^{\infty} r(n+k-1,k)x^{n+k-1}, \qquad n \geqq 1,$$

and

$$\Phi(x) = \sum_{n=0}^{\infty} \phi(n)x^n.$$

Furthermore we define $t(n,k)$, a function from $\mathbf{N} \times \mathbf{N}$ into $F$, by

$$\Phi^n(x) = \sum_{k=0}^{\infty} t(n,k)x^k.$$

THEOREM 1. *If $r(n,k)$ and $\phi(n)$ satisfy*

$$r(1,1) = 1, \tag{3.1}$$

(3.2) $$r(n,k)=0, \qquad n<k,$$

(3.3) $$\phi(0)=1,$$

(3.4) $$r(n+1,k)=\sum_{m=1}^{k}\phi(k-m)r(n,m), \qquad 1\leqq k\leqq n+1,$$

*and*

(3.5) $$r(n+k,k)=\sum_{m=1}^{k} r(m,m)r(n+k-m,k-m+1), \qquad n\geqq 1,$$

*and if $\Phi(z)$ is analytic about $z=0$, then*

(3.6) $$r(n,k+1)=\frac{n-k}{n}\,t(n,k).$$

*Proof.* By (3.5)

$$\begin{aligned} R_{n+1}(x) &= \sum_{k=1}^{\infty} x^{n+k}\sum_{m=1}^{k} r(m,m)r(n+k-m,k-m+1) \\ &= \sum_{m=1}^{\infty} r(m,m)x^{m}\sum_{k=1}^{\infty} r(n+k-1,k)x^{n+k-1} \\ &= R_1(x)\,R_n(x). \end{aligned}$$

Thus

$$R_n(x)=R_1^n(x) \qquad n\geqq 1.$$

Using (3.1)–(3.4), we find that

$$\begin{aligned} \Phi(R_1(x)) &= 1+\sum_{n=1}^{\infty}\phi(n)\,R_n(x) \\ &= 1+\sum_{n=1}^{\infty}\phi(n)\sum_{k=1}^{\infty} r(n+k-1,k)x^{n+k-1} \\ &= 1+\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\phi(n-k)\,r(n-1,k)x^{n-1} \\ &= 1+\sum_{n=2}^{\infty} r(n,n)x^{n-1} \\ &= x^{-1}R_1(x). \end{aligned}$$

It follows that

$$x = \frac{z}{\Phi(z)},$$

where $z = R_1(x)$.

By the Lagrange expansion formula [**9**, p. 125] the equation

$$x = \frac{z}{\Phi(z)} \qquad (\Phi(0) = 1),$$

where $\Phi(z)$ is analytic in a neighborhood of $z = 0$, implies

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} f'(z)\Phi^n(z) \right]_{z=0}. \tag{3.7}$$

Since

$$R_m(x) = R_1^m(x) = z^m,$$

we can determine $R_m(x)$ by letting $f(z) = z^m$ in (3.7). Thus we have

$$\begin{aligned} R_m(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} (mz^{m-1}) \sum_{k=0}^{\infty} t(n,k)z^k \right]_{z=0} \\ &= \sum_{n=1}^{\infty} x^n \frac{m}{n} t(n, n-m), \end{aligned}$$

and

$$\sum_{n=m-1}^{\infty} r(n, n-m+1)x^n = \sum_{n=1}^{\infty} x^n \frac{m}{n} t(n, n-m). \tag{3.8}$$

We obtain (3.6) by equating coefficients of $x^n$ in (3.8).

To state the corollary we must introduce two more functions. Let $s(n,k)$ be a function from $\mathbf{N}^* \times \mathbf{N}^*$ into $F$ and define $S_n(x)$ by

$$S_n(x) = \sum_{k=1}^{\infty} s(n+k-1, k)x^{n+k-1}, \qquad n \geqq 1.$$

COROLLARY 1. *If $r(n,k)$, $s(n,k)$ and $\phi(n)$ satisfy* (3.1)–(3.5),

$$s(1,1) = 1, \tag{3.9}$$

$$s(n,n) = s(n, n-1), \qquad n \geqq 2, \tag{3.10}$$

*and*

$$(3.11)\; s(n+k,k)=\sum_{m=1}^{k} r(m,m)s(n+k-m,k-m+1), \qquad n \geqq 1,$$

*then*

$$s(n+1,k+1)=\frac{1}{n}\sum_{j=0}^{k}(n-j)\,t(n,j). \tag{3.12}$$

*Proof.* Using (3.9)–(3.11), we find that

$$\begin{aligned} S_1(x) &= x+\sum_{k=1}^{\infty} x^{k+1}\sum_{j=1}^{k} r(m,m)\,s(k-m+1,k-m+1)\\ &= x+R_1(x)\,S_1(x), \end{aligned}$$

and from (3.11) we get

$$\begin{aligned} S_{n+1}(x) &= \sum_{k=1}^{\infty} x^{n+k}\sum_{m=1}^{k} r(m,m)\,s(n+k-m,k-m+1)\\ &= S_n(x)\,R_1(x), \qquad n \geqq 1. \end{aligned}$$

Thus

$$\begin{aligned} S_{n+1}(x) &= S_1(x)\,R_1^n(x)\\ &= \frac{xR_1^n(x)}{1-R_1(x)} \end{aligned}$$

or

$$\frac{S_{m+1}(x)}{x}=\frac{z^m}{1-z},$$

where $z=R_1(x)$. Again making use of (3.7), this time with

$$f(z)=\frac{z^m}{1-z},$$

we see that

$$x^{-1}S_{m+1}(x)=\delta_{m,0}+\sum_{n=1}^{\infty}\frac{x^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left[\frac{mz^{m-1}}{1-z}+\frac{z^m}{(1-z)^2}\right]\sum_{k=0}^{\infty} t(n,k)z^k\right]_{z=0}.$$

Hence we have

$$\sum_{n=m-1}^{\infty} s(n+1, n-m+1)x^n = \delta_{m,0} + \sum_{n=1}^{\infty} x^n \frac{1}{n} \sum_{j=0}^{n-m} (m+j)\, t(n, n-m-j)$$

and (3.12) follows.

**4. The functions $F_p$ and $G_p$.** Throughout the rest of this paper we assume condition (1.8) holds. With this assumption we can use the results of the two previous sections to determine $F_p$, $G_p$ and $H_p$. In this section we consider $F_p$ and $G_p$ and in §5 we find $H_p$.

THEOREM 2. *If $\prod_{i=1}^{p} q_i = 1$, then*

$$G_p(n, k+1) = \frac{n-k}{n} b_p(n, k) \tag{4.1}$$

*and*

$$F_p(n+1, k+1) = \frac{1}{n} \sum_{j=0}^{k} (n-j)\, b_p(n, j), \tag{4.2}$$

*where $b_p(n, k; q_1, \cdots, q_p) = b_p(n, k)$ is defined by*

$$\left[ \sum_{k=0}^{\infty} \left[ \prod_{i=1}^{p} [k+1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i} \right] x^k \right]^n = \sum_{k=0}^{\infty} b_p(n, k) x^k.$$

*Proof.* This theorem is an immediate consequence of Theorem 1 and Corollary 1, for (1.8), (2.3), (2.5), (2.6) and the definitions of $F_p$ and $G_p$ imply that the hypotheses of these results are satisfied if $r(n, k) = G_p(n, k)$, $s(n, k) = F_p(n, k)$ and

$$\phi(k) = \prod_{i=1}^{p} [k+1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i}.$$

Instead of Corollary 1 we can use (2.2) to obtain (4.2). We remark that by virtue of (1.8) it is possible to reduce

$$\prod_{i=1}^{p} [k+1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i},$$

and thus $b_p(n, k)$, to a function of $p-1$ $q_i$'s.

Now

$$\sum_{k=0}^{\infty}\left[\prod_{i=1}^{p}[k+1]_{q_i}-\prod_{i=1}^{p}[k]_{q_i}\right]x^k = x^{-1}H_p(x^{-1}\mid q_1,q_2,\cdots,q_p),$$

where $H_p(x \mid q_1, q_2, \cdots, q_p) = H_p(x)$ is the generalized Eulerian function defined and studied by Roselle **[12]**. Because $H_p(x)$ is quite complicated for $p \geqq 3$, in general it is not feasible to find a simple formula for $b_p(n, k)$ in terms of more familiar coefficients. However, we can find $b_2(n, k)$ without difficulty.

COROLLARY 2. *Let* $p = 2$. *If* $q_1q_2 = 1$, *then*

$$G_2(n, k+1) = \frac{n-k}{n}\, b_2(n, k)$$

*and*

$$F_2(n+1, k+1) = \frac{1}{n}\sum_{j=0}^{k}(n-j)\, b_2(n, j),$$

*where*

$$(4.3)\quad b_2(n,k) = \sum_{m=0}^{k}\binom{n}{k-m}\sum_{j=0}^{m}\binom{n+j-1}{j}\binom{n+m-j-1}{n-1} q_1^{m-2j}.$$

*Proof.* Since

$$H_2(x) = \frac{x+q_1q_2}{(x-q_1)(x-q_2)}$$

and $q_1q_2 = 1$, we find that

$$x^{-1}H_2(x^{-1}) = \frac{1+x}{(1-q_1x)(1-q_1^{-1}x)}.$$

Thus

$$\begin{aligned}(x^{-1}H_2(x^{-1}))^n &= \sum_{m=0}^{n}\binom{n}{m}x^m\sum_{k=0}^{\infty}\binom{n+k-1}{k}q_1^kx^k\ \sum_{j=0}^{\infty}\binom{n+j-1}{j}q_1^{-j}x^j \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{n}{k-m}\sum_{j=0}^{m}\binom{n+j-1}{j}\binom{n+m-j-1}{m-j}q_1^{m-2j}x^k\end{aligned}$$

and (4.3) follows.

Another special case of interest is that in which $q_1 = q_2 = \cdots = q_p = q$. In this case condition (1.8) becomes $q^p = 1$ and the possible values for $q$ are the $p$th roots of unity. Let $H_p(x \mid q_1, q_2, \cdots, q_p) = H_p(x \mid q)$ and $b_p(n, k; q_1, q_2, \cdots, q_p) = b_p(n, k; q)$ when $q_1 = \cdots = q_p = q$. We also require the notation

$$[x] = [x]_q = \frac{q^x - 1}{q - 1}$$

and

$$\begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} x \\ m \end{bmatrix}_q = \prod_{j=1}^{m} \frac{q^{x-j+1} - 1}{q^j - 1}.$$

Carlitz [2] has proved that

$$H_p(x \mid q) = \sum_{j=1}^{p} A_{p,j}(q) x^{j-1} \Big/ \prod_{n=1}^{p} (x - q^n), \tag{4.4}$$

where $A_{p,j}(q)$ are the $q$-Eulerian numbers defined by

$$[x]^m = \sum_{s=1}^{m} A_{m,s}(q) \begin{bmatrix} x + s - 1 \\ m \end{bmatrix}, \qquad m \geqq 1.$$

Now (4.4) implies

$$x^{-1} H_p(x^{-1} \mid q) = \sum_{j=1}^{p} A_{p,j}(q) x^{j-1} \Big/ \prod_{n=1}^{p} (1 - q^n x).$$

If we define $a_p(n, k; q)$ by

$$\left[ \sum_{j=1}^{p} A_{p,j}(q) x^{j-1} \right]^n = \sum_{k=0}^{\infty} a_p(n, k; q) x^k, \tag{4.5}$$

then

$$\begin{aligned}(x^{-1} H_p(x^{-1} \mid q))^n &= \sum_{k=0}^{\infty} a_p(n, k; q) x^k \sum_{m=0}^{\infty} \begin{bmatrix} np + m - 1 \\ m \end{bmatrix} (qx)^m \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{k} a_p(n, k - m; q) \begin{bmatrix} np + m - 1 \\ m \end{bmatrix} q^m x^k.\end{aligned}$$

Thus we have proved

COROLLARY 3. *If* $q_1 = q_2 = \cdots = q_p = q$ *and* $q^p = 1$, *then*

$$G_p(n, k+1) = \frac{n-k}{n} b_p(n, k; q)$$

*and*

$$F_p(n+1, k+1) = \frac{1}{n} \sum_{j=0}^{k} (n-j)\, b_p(n, k; q),$$

*where*

$$b_p(n, k; q) = \sum_{m=0}^{k} a_p(n, k-m; q) \begin{bmatrix} np+m-1 \\ m \end{bmatrix} q^m \tag{4.6}$$

*and* $a_p(n, k; q)$ *is defined by* (4.5).

The following result follows immediately from Corollaries 2 and 3.

COROLLARY 4. *Let* $p = 2$. *If* $q_1 = q_2 = q$ *and* $q^2 = 1$, *then*

$$G_2(n, k+1) = \frac{n-k}{n} b_2(n, k; q)$$

*and*

$$F_2(n+1, k+1) = \frac{1}{n} \sum_{j=0}^{k} (n-j)\, b_2(n, j; q),$$

*where*

$$b_2(n, k; q) = \sum_{m=0}^{k} \binom{n}{k-m} \binom{2n+m-1}{m} q^m \tag{4.7}$$

*or*

$$b_2(n, k; q) = \sum_{j=0}^{[k/2]} \binom{n}{k-2j} \binom{2n+2j-1}{2j} + \sum_{j=0}^{[(k-1)/2]} \binom{n}{k-2j-1} \binom{2n+2j}{2j+1} q \tag{4.8}$$

*or*

$$b_2(n,k;q)=\sum_{m=0}^{k}\binom{n}{k-m}\begin{bmatrix}2n+m-1\\ m\end{bmatrix}q^{n-k}. \tag{4.9}$$

*Proof.* We deduce (4.7) from Corollary 2 by using the hypothesis $q^2=1$ and a binomial identity found in Riordan [**10**, p. 9]. Because $q^2=1$, (4.8) follows from (4.7). We get (4.9) from Corollary 3 by observing that

$$a_2(n,k;q)=\binom{n}{k}q^{n-k}.$$

If $q=1$ in Corollary 4 we have

COROLLARY 5. *Let $p=2$. If $q_1=q_2=1$, then*

$$G_2(n,k+1)=\frac{n-k}{n}\sum_{m=0}^{k}\binom{n}{k-m}\binom{2n+m-1}{m} \tag{4.10}$$

*and*

$$F_2(n+1,k+1)=\frac{1}{n}\sum_{j=0}^{k}(n-j)\sum_{m=0}^{j}\binom{n}{j-m}\binom{2n+m-1}{m}. \tag{4.11}$$

We note that (4.10) is precisely Carlitz's formula for $g(n,k+1)$, while (4.11) is equivalent to his formula for $f(n+1,k+1)$.

If $p=1$, condition (1.8) implies $q_1=1$. In this case we get

COROLLARY 6. *Let $p=1$. If $q_1=1$, then*

$$F_1(n,k+1)=G_1(n,k+1)=\frac{n-k}{n}\binom{n+k-1}{k}. \tag{4.12}$$

*Proof.* Letting $p=1$ and $q=1$ in Corollary 3 we obtain

$$G_1(n,k+1)=\frac{n-k}{n}\sum_{m=0}^{k}a_1(n,k-m;1)\binom{n+m-1}{m}. \tag{4.13}$$

Since

$$a_1(n,k;1)=\delta_{k,0},$$

(4.13) reduces to

$$G_1(n, k+1) = \frac{n-k}{n}\binom{n+k-1}{k}.$$

Then (4.12) follows because $F_1(n, k+1) = G_1(n, k+1)$ by definition. Formula (4.12) is the result given by Bertrand [1] as the number of two-element lattice permutations.

**5. The function $H_p$.** By virtue of (1.8), (2.8) and (2.9), the hypotheses of Theorem 1 are satisfied if $r(n, k) = H_p(n, k)$ and

$$\phi(n) = q_1^n \theta_p(n; q_2, q_3, \cdots, q_p),$$

where $\theta_p(n; q_2, q_3, \cdots, q_p)$, or more briefly $\theta_p(n)$, is defined by (2.7) for $p \geqq 2$ and by $\theta_p(n) = 1$ for $p = 1$. Thus we can express $H_p(n, k)$ in terms of the coefficients $c_p(n, k) = c_p(n, k; q_1, q_2, \cdots, q_p)$ defined by

(5.1) $$\Theta_p^n(x) = \sum_{k=0}^{\infty} c_p(n, k) x^k,$$

where

$$\Theta_p(x) = \Theta_p(x \mid q_1, q_2, \cdots, q_p) = \sum_{k=0}^{\infty} \theta_p(k)(q_1 x)^k.$$

In fact we have

THEOREM 3. *If* $\Pi_{i=1}^{p} q_i = 1$, *then*

(5.2) $$H_p(n, k+1) = \frac{n-k}{n} c_p(n, k),$$

*where* $c_p(n, k)$ *is defined by* (5.1).

(In view of condition (1.8) it is possible to express $\Theta_p(x)$ and $c_p(n, k)$ as a function of $p - 1$ $q_i$'s.)

Since the coefficients $c_p(n, k)$ are so closely related to $\theta_p(k)$ and $\Theta_p(x)$, we reduce $\theta_p(k)$ from a $(p-1)$-tuple summation to a single sum. Using this simplification of $\theta_p(k)$, we can write $\Theta_p(x)$ as a single, finite sum.

THEOREM 4. *If $p \geqq 2$, then*

$$\theta_p(k) = \frac{1}{\gamma_1(q_2, \cdots, q_p)} + \sum_{j=2}^{p} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}(q_2 q_3 \cdots q_j)^k}{\gamma_j(q_2, \cdots, q_p)}, \tag{5.3}$$

*where*

$$\gamma_1(q_2, \cdots, q_p) = (1-q_2)(1-q_2q_3)\cdots(1-q_2\cdots q_p),$$

*and*

$$\gamma_j(q_2, \cdots, q_p) = (1-q_2\cdots q_j)(1-q_3\cdots q_j)\cdots(1-q_j)(1-q_{j+1}) \cdot (1-q_{j+1}q_{j+2})\cdots(1-q_{j+1}\cdots q_p), \quad j \geqq 2, \tag{5.4}$$

*with the understanding that*

$$(1-q_i\cdots q_j) = 1 \quad \text{if} \quad i > j. \tag{5.5}$$

*Proof.* We use induction on $p$. From definition (2.7) we get

$$\theta_2(k) = \sum_{m=0}^{k} q_2^m.$$

Thus

$$\theta_2(k) = \frac{1-q_2^{k+1}}{1-q_2}$$

and we have verified (5.3) for $p = 2$. Using first the definition of $\theta_{p+1}(k)$ and then (5.3) (as the induction hypothesis), we find that

$$\theta_{p+1}(k; q_2, \cdots, q_{p+1}) = \sum_{a_2=0}^{k} q_2^{a_2} \sum_{a_3=0}^{a_2} q_3^{a_3} \cdots \sum_{a_{p+1}=0}^{a_p} q_{p+1}^{a_{p+1}}$$

$$= \sum_{a_2=0}^{k} q_2^{a_2} \theta_p(a_2; q_3, \cdots, q_{p+1})$$

$$= \sum_{m=0}^{k} q_2^m \left[\frac{1}{\gamma_1(q_3, \cdots, q_{p+1})} + \sum_{j=2}^{p} \frac{(-1)^{j+1} q_3 q_4^2 \cdots q_{j+1}^{j-1}(q_3 \cdots q_{j+1})^m}{\gamma_{j+1}(q_3, \cdots, q_{p+1})}\right]$$

$$= \frac{1-q_2^{k+1}}{(1-q_2)\gamma_1(q_3,\cdots,q_{p+1})} + \sum_{j=3}^{p+1} \frac{(-1)^j q_3 q_4^2 \cdots q_j^{j-2}(1-(q_2\cdots q_j)^{k+1})}{(1-q_2\cdots q_j)\gamma_j(q_3,\cdots,q_{p+1})}.$$

$$= \sum_{j=2}^{p+1} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}(q_2\cdots q_j)^k}{\gamma_j(q_2,\cdots,q_{p+1})} + \sum_{j=2}^{p+1} \frac{(-1)^j q_3 q_4^2 \cdots q_j^{j-2}}{\gamma_j(q_2,\cdots,q_{p+1})}.$$

To complete the induction it suffices to prove

LEMMA 1. *We have*

$$\frac{1}{(1-q_1)(1-q_2)(1-q_2q_3)\cdots(1-q_2\cdots q_p)} + \sum_{j=2}^{p} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}}{\gamma_j(q_1,\cdots,q_p)}$$

(5.6)
$$= \frac{1}{(1-q_1)(1-q_1q_2)\cdots(1-q_1\cdots q_p)},$$

*where* $\gamma_j$ *is defined by* (5.4) *and* (5.5).

*Proof.* This proof was suggested by Carlitz. In the expression

(5.7)
$$\frac{(1-q_1q_2)\cdots(1-q_1\cdots q_p)}{(1-q_2)(1-q_2q_3)\cdots(1-q_2\cdots q_p)}$$
$$+ \sum_{j=2}^{p} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}(1-q_1)\cdots(1-q_1\cdots q_p)}{\gamma_j(q_1,\cdots,q_p)}$$

fix $q_2,\cdots,q_p$. Then (5.7) is a polynomial in $q_1$ of degree $p-1$. Since this polynomial is 1 for $p$ values of $q_1$, namely

$$q_1 = 1, \frac{1}{q_2}, \cdots, \frac{1}{q_2q_3\cdots q_p},$$

it is identically 1 and (5.6) follows. Using the definition of $\Theta_p(x)$ and (5.3), we find that

$$\Theta_p(x) = \frac{1}{(1-q_1x)\gamma_1(q_2,\cdots,q_p)} + \sum_{j=2}^{p} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}}{(1-q_1\cdots q_jx)\gamma_j(q_2,\cdots,q_p)}.$$

In general $\theta_p(x)$ is quite complicated and it is not feasible to determine the coefficients $c_p(n,k)$ explicitly. However, for $p=2$ and $q_1q_2=1$, we have

$$\Theta_2(x) = \frac{1}{(1-q_1x)(1-q_2)} - \frac{q_2}{(1-x)(1-q_2)}$$
$$= \frac{1}{(1-q_1x)(1-x)}.$$

It follows that

$$\theta_2^n(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{n+k-m-1}{k-m}\binom{n+m-1}{m}q_1^m x^k$$

and

$$c_2(n,k) = \sum_{m=0}^{k}\binom{n+k-m-1}{k-m}\binom{n+m-1}{m}q_1^m.$$

Hence, as a corollary of Theorem 3, we have

COROLLARY 7. *Let $p=2$. If $q_1q_2=1$, then*

$$(5.8)\qquad H_2(n,k+1) = \frac{n-k}{n}\sum_{m=0}^{k}\binom{n+k-m-1}{k-m}\binom{n+m-1}{m}q_1^m.$$

If, in addition to (1.8), we assume $q_1 = q_2 = \cdots = q_p = q$, $\Theta_p(x)$ is considerably simpler. In this case let $\Theta_p(x) = \Theta_p(x \mid q)$, $\theta_p(k) = \theta_p(k;q)$ and $c_p(n,k) = c_p(n,k;q)$. From the definition of $\theta_p(k;q)$ it follows that

$$\theta_p(k;q) = \sum q^m,$$

where the summation is over all $(p-1)$-tuples $(a_2,\cdots,a_p)$ such that $1 \leqq a_p \leqq \cdots \leqq a_2 \leqq k+1$, and

$$m+p-1 = \sum_{i=2}^{p} a_i.$$

Hence it is evident that $\theta_p(k;q)$ generates the number of partitions of $m+p-1$ into $p-1$ parts with each part at most $k+1$. It is well-known (see, for example, [8, p. 5]) that such a function is $\begin{bmatrix} k+p-1 \\ k \end{bmatrix}$. Thus

$$\Theta_p(x \mid q) = \sum_{k=0}^{\infty} \begin{bmatrix} k+p-1 \\ k \end{bmatrix} (qx)^k$$

$$= \prod_{k=1}^{p} (1-q^k x)^{-1}.$$

Since $q^p = 1$,

$$\Theta_p(x \mid q) = \prod_{k=0}^{p-1} (1-q^k x)^{-1}, \tag{5.9}$$

and

$$\Theta_p^n(x \mid q) = \prod_{k=0}^{p-1} (1-q^k x)^{-n}$$

$$= \prod_{k=0}^{np-1} (1-q^k x)^{-1}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} np+k-1 \\ k \end{bmatrix} x^k.$$

Therefore

$$c_p(n, k; q) = \begin{bmatrix} np+k-1 \\ k \end{bmatrix},$$

and we have proved

COROLLARY 8. *If $q_1 = q_2 = \cdots = q_p = q$ and $q^p = 1$, then*

$$H_p(n, k+1) = \frac{n-k}{n} \begin{bmatrix} np+k-1 \\ k \end{bmatrix}. \tag{5.10}$$

We observe that if $q$ is a primitive $p$th root of unity, say $\xi$, (5.9) reduces to

$$\Theta_p(x \mid \xi) = (1-x^p)^{-1}.$$

Then

$$\Theta_p^n(x \mid \xi) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{pk}. \tag{5.11}$$

From (5.11) it follows that

$$c_p(n,k;\xi)=\begin{cases}\binom{n+k/p-1}{n-1} & \text{if } p/k\\ 0 & \text{otherwise}\end{cases} \tag{5.12}$$

As an immediate consequence of Corollary 8 we have

COROLLARY 9. *If* $q_i = 1$, $1 \leqq i \leqq p$, *then*

$$H_p(n,k+1)=\frac{n-k}{n}\binom{np+k-1}{k}. \tag{5.13}$$

*We can also obtain* (5.13) *by viewing* $\theta_p(k;1)$ *as the number of* $(p-1)$*-combinations with repetition of* $k+1$ *distinct objects. Then we know that* (*see Riordan* [**11,** *p.* 7])

$$\theta_p(k;1)=\binom{k+p-1}{p-1}. \tag{5.14}$$

*From* (5.14) *we can deduce* (5.13).

If $p=2$, $q_1=q_2=q$ and $q^2=1$, the formulas for $H_2(n,k+1)$, given in the following corollary, are particularly simple.

COROLLARY 10. *Let* $p=2$. *If* $q_1=q_2=1$, *then*

$$H_2(n,k+1)=\frac{n-k}{n}\binom{2n+k-1}{k}; \tag{5.15}$$

if $q_1=q_2=-1$,

$$H_2(n,k+1)=\begin{cases}\frac{n-k}{n}\binom{n+k/2-1}{n-1} & \text{if } k \text{ is even}\\ 0 & \text{if } k \text{ is odd}\ .\end{cases} \tag{5.16}$$

*Proof.* From either (5.8) or (5.13) we get (5.15), while (5.16) follows from (5.12). Carlitz's result for $h(n,k+1)$ coincides with (5.15).

Since (5.13) is valid for $p=1$, we have the expected result

COROLLARY 11. *Let* $p=1$. *If* $q_1=1$, *then*

$$H_1(n,k+1)=\frac{n-k}{n}\binom{n+k-1}{k}.$$

**6. Partitions.** From the definition of $g_p(n, k; s_1, s_2, \cdots, s_p)$ it is clear that this enumerant is the number of partitions of the $p$-partite $(s_1, s_2, \cdots, s_p)$ of the form

$$\sum_{j=1}^{n} a_{ij} = s_i, \qquad 1 \leqq i \leqq p,$$

where the $a_{ij}$ are positive integers subject to conditions (1.2), (1.4) and (1.6). Thus $G_p(n, k)$ generates these partitions. If we replace (1.6) by (1.5), these statements are true for $f_p(n, k)$ and $F_p(n, k)$. Adding (1.7) to the conditions for $g_p(n, k)$, we have a partition interpretation for $h_p(n, k)$ and $H_p(n, k)$.

Since we have only obtained $F_p$, $G_p$ and $H_p$ under the assumption (1.8), we describe how this restriction affects the partitions. If $\Pi_{i=1}^{p} q_i = 1$ and $p \geqq 2$,

$$G_p(n, k) = \sum{}^{*} g_p(n, k; s_1, s_2, \cdots, s_p) q_1^{s_1 - s_p} q_2^{s_2 - s_p} \cdots q_{p-1}^{s_{p-1} - s_p},$$

where $\Sigma^*$ is as defined in §1. Thus we have

THEOREM 5. *If* $\Pi_{i=1}^{p} q_i = 1$, *the function* $G_p(n, k)$ *generates the number of partitions of the* $(p-1)$*-partite* $(m_1, m_2, \cdots, m_{p-1})$ *of the form*

$$m_i = \sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} a_{pj},$$

*where the* $a_{ij}$ *are positive integers satisfying* (1.2), (1.4) *and* (1.6).

We have corresponding interpretations for $F_p$ and $H_p$.
For example, by Corollaries 2 and 7 we have

$$G_2(n, k+1) = \frac{n-k}{n} \sum_{m=-k}^{k} \sum_{j=\max\{0,-m\}}^{[(k-m)/2]} \binom{n}{k-m-2j} \binom{n+j-1}{j} \binom{n+m+j-1}{m+j} q_1^m,$$

$$F_2(n+1, k+1) = \frac{1}{n} \sum_{m=-k}^{k} \sum_{j=\max\{m,-m\}}^{k} (n-j) \sum_{s=\max\{0,-m\}}^{[(j-m)/2]} \binom{n}{j-m-2s} \binom{n+s-1}{s} \binom{n+m+s-1}{m+s} q_1^m,$$

and

$$H_2(n, k+1) = \frac{n-k}{n} \sum_{m=0}^{k} \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q_1^m.$$

Thus

(6.1) $$G_2(3, 2) = 2q_1^{-1} + 2 + 2q_1,$$

(6.2) $$F_2(3, 2) = q_1^{-1} + 2 + q_1$$

and

(6.3) $$H_2(3, 2) = 2 + 2q_1.$$

The following arrays are enumerated by $G_2(3, 2)$:

$$(1)\ \boxed{\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}} \qquad (2)\ \boxed{\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 2 & 2 \end{array}} \qquad (3)\ \boxed{\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 2 \end{array}}$$

$$(4)\ \boxed{\begin{array}{ccc} 1 & 2 & 2 \\ 1 & 2 & 2 \end{array}} \qquad (5)\ \boxed{\begin{array}{ccc} 1 & 2 & 2 \\ 1 & 1 & 2 \end{array}} \qquad (6)\ \boxed{\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}}$$

Now arrays (1) and (2) satisfy

$$-1 = \sum_{j=1}^{3} a_{1j} - \sum_{j=1}^{3} a_{2j};$$

for (3) and (4) we have

$$0 = \sum_{j=1}^{3} a_{1j} - \sum_{j=1}^{3} a_{2j},$$

while (5) and (6) are subject to the condition

$$1 = \sum_{j=1}^{3} a_{1j} - \sum_{j=1}^{3} a_{2j}.$$

The arrays counted by $F_2(3, 2)$ are (2)–(5). Array (2) accounts for the coefficient of $q_1^{-1}$, (3) and (4) are enumerated by the constant term, and (5) is counted by the coefficient of $q_1$. Finally, $H_2(3, 2)$ counts arrays (3)–(6). The first two of these are enumerated by the constant term; (5) and (6) account for the coefficient of $q_1$.

If we assume $q_1 = \cdots = q_p = q$,

$$G_p(n, k) = \sum{}^* g_p(n, k; s_1, \cdots, s_p) q^{\sum_{i=1}^p s_i}.$$

Then condition (1.8) implies $q^p = 1$ and we have

$$G_p(n, k) = \sum_{m=0}^{p-1} \sum g_p(n, k; s_1, \cdots, s_p) q^m$$

where the inner sum is over all $p$-tuples satisfying

$$\sum_{i=1}^{p} s_i \equiv m \pmod{p}.$$

Therefore, we have

THEOREM 6. *If $q_1 = q_2 = \cdots = q_p = q$, $q^p = 1$, and $\Pi_{p,n,k}(r)$ is the number of partitions of the form*

$$r = \sum_{i=1}^{p} s_i = \sum_{i=1}^{p} \sum_{j=1}^{n} a_{ij},$$

*where the $a_{ij}$ are positive integers subject to conditions* (1.2), (1.3), (1.4) *and* (1.6), *then $G_p(n, k)$ generates*

$$\sum_t \Pi_{p,n,k}(m + tp), \qquad 0 \leqq m \leqq p - 1.$$

For example, from Corollary 4 we see that

$$\sum_t \Pi_{2,n,k+1}(2t) = \frac{n-k}{n} \sum_{j=0}^{[k/2]} \binom{n}{k-2j} \binom{2n+2j-1}{2j}$$

and

$$\sum_t \Pi_{2,n,k+1}(2t+1) = \frac{n-k}{n} \sum_{j=0}^{[(k-1)/2]} \binom{n}{k-2j-1} \binom{2n+2j}{2j+1}.$$

We have corresponding results for $F_p$ and $H_p$.

Returning to the illustration used above, if $q_1 = q_2 = q$, we may write (6.1), (6.2) and (6.3) as

$$G_2(3, 2) = 2 + 4q, \tag{6.4}$$

(6.5) $$F_2(3, 2) = 2 + 2q,$$

and

(6.6) $$H_2(3, 2) = 2 + 2q.$$

Since the sums of arrays (3) and (4) are even, these arrays are counted by the constant term in (6.4)–(6.6). In each case the coefficient of $q$ enumerates the arrays having an odd sum.

## REFERENCES


1. J. Bertrand, *Solution d'un probleme*, Comptes Rendus Académie des Sciences. Paris, **105** (1887), 369.
2. L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76** (1954), 332–350.
3. ———, *Enumeration of rectangular arrays by length and coincidences*, Annali di Matematica pura ed applicata, **99** (1974), 155–182.
4. ———, *Enumeration of two-line arrays*, The Fibonacci Quarterly, **2** (1973), 113–130.
5. ———, *Sequences, paths, ballot numbers*, The Fibonacci Quarterly, **10** (1972), 531–550.
6. L. Carlitz and J. Riordan, *Two element lattice permutation numbers and their q-generalization*, Duke Math. J., **31** (1964), 371–388.
7. P. A. MacMahon, *Combinatorial Analysis*, Vol. 1, Cambridge, 1915.
8. ———, *Combinatorial Analysis*, vol. 2, Cambridge, 1916.
9. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Springer, Berlin, **1** (1925).
10. J. Riordan, *Combinatorial Identities*, Wiley, New York, 1968.
11. ———, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
12. D. P. Roselle, *Generalized Eulerian functions and a problem in partitions*, Duke Math. J., **33** (1966), 293–304.


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