PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 2, 1976

A GENERAL RATIO ERGODIC THEOREM FOR SEMIGROUPS

SHIGERU HASEGAWA AND RYOTARO SATO

The purpose of this note is to prove a ratio ergodic theorem, which is a continuous parameter version of Chacon's general ergodic theorem.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_1 = L_1(X, \mathscr{F}, \mu)$ the Banach space of equivalence classes of integrable complex-valued functions on X. Let $\Gamma = \{T_t;\ t > 0\}$ be a strongly continuous semigroup of linear contractions on L_1. It then follows (cf. [**6**, §4]) that for any $f \in L_1$ there exists a scalar function $T_tf(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathscr{F}$, such that $T_tf(x)$ belongs to the equivalence class of T_tf for each $t > 0$. Moreover there exists a set $N(f)$ with $\mu(N(f)) = 0$, dependent on f but independent of t, such that if $x \notin N(f)$, then $T_tf(x)$ is integrable on every finite interval (a, b) and the integral $\int_a^b T_tf(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_tf\,dt$.

THEOREM. *Let $p_t(x)$ be a nonnegative function on $(0, \infty) \times X$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathscr{F}$, such that $f \in L_1$ and $|f| \leqq p_s$ for some s imply $|T_tf| \leqq p_{s+t}$ for all $t > 0$. Then for any $f \in L_1$ the limit*

$$\lim_{b \to \infty} \int_0^b T_tf(x)\,dt \Big/ \int_0^b p_t(x)\,dt$$

exists and is finite a.e. *on* $\left\{x;\ \int_0^\infty p_t(x)dt > 0\right\}$.

LEMMA. *Let T be a linear contraction on L_1 and $\{p_n;\ n \geqq 0\}$ a sequence of nonnegative measurable functions on X such that $f \in L_1$ and $|f| \leqq p_n$ for some n imply $|Tf| \leqq p_{n+1}$. If $g \in L_1$, then*

$$\lim_n p_n(x) \Big/ \sum_{i=0}^{n-1} p_i(x) = 0$$

a.e. *on*

$$\left\{x;\ \sum_{i=0}^{\infty} p_i(x) > 0 \ \textit{and} \ \lim_n \left| \sum_{i=0}^{n} T^i g(x) \Big/ \sum_{i=0}^{n} p_i(x) \right| > 0\right\}.$$

Proof. By the Chacon theorem, $\lim_n \Sigma_{i=0}^n T^i g(x)/\Sigma_{i=0}^n p_i(x)$ exists and is finite a.e. on $\{x; \Sigma_{i=0}^\infty p_i(x) > 0\}$. Using this and the linear modulus **[3]** of T, the desired conclusion follows easily from the Chacon-Ornstein lemma **[5**, Theorem 2.4.2].

Proof of the Theorem. Write $q_n(x) = \int_n^{n+1} p_t(x)dt$. Then $\{q_n; n \geqq 0\}$ is a sequence of nonnegative measurable functions on X. We first prove that $f \in L_1$ and $|f| \leqq q_n$ for some n imply $|T_1 f| \leqq q_{n+1}$. For this purpose, let $\epsilon > 0$ be given, and choose a nonnegative function $h \in L_1$ such that if we set $p'_t(x) = \min(p_t(x), h(x))$, $q'_n(x) = \int_n^{n+1} p'_t(x)dt$ and $A = \{x; f(x) < q'_n(x)\}$, then $\|f1_{X-A}\| < \epsilon$. Define $s(x) = f(x)/q'_n(x)$ if $x \in A$, and $s(x) = 0$ if $x \notin A$. It follows that $\|f - sq'_n\| = \|f1_{X-A}\| < \epsilon$, and

$$T_1(sq'_n) = T_1\left(\int_n^{n+1} s(x)p'_t(x)\,dt\right) = \int_n^{n+1} T_1(sp'_t)\,dt.$$

(Here we note that $t \to sp'_t$ is strongly integrable over the interval $(n, n+1)$). Since $|sp'_t| \leqq p_t$, $|T_1(sp'_t)| \leqq p_{t+1}$ for all $t > 0$. Now let $r_t(x)$ be a scalar function on $(n, n+1) \times X$, measurable with respect to the product of the Lebesgue measurable subsets of $(n, n+1)$ and $\mathscr{F}$, such that for almost all t, $r_t(x)$ belongs to the equivalence class of $T_1(sp'_t)$ **[4**, Theorem III.11.17]. Then we have

$$\begin{aligned}\left|\int_n^{n+1} T_1(sp'_t)\,dt\right| &= \left|\int_n^{n+1} r_t(x)\,dt\right| \leqq \int_n^{n+1} \left|r_t(x)\right| dt \\ &\leqq \int_n^{n+1} p_{t+1}(x)dt = q_{n+1}(x) \quad \text{a.e.,}\end{aligned}$$

and hence $|T_1 f| \leqq q_{n+1}$.

Next, for any $f \in L_1$, put $f' = \int_0^1 T_t f\,dt$. If $b > 0$, write $b = n + a$, where $n = [b]$ and $0 \leqq a < 1$. Then, as in **[7]**,

$$\int_0^b T_t f(x)\,dt \Big/ \int_0^b p_t(x)\,dt$$

$$= \left(\frac{\sum_{i=0}^{n-1} T_1^i f'(x)}{\sum_{i=0}^{n-1} q_i(x)} + \frac{\int_n^b T_t f(x)\,dt}{\sum_{i=0}^{n-1} q_i(x)}\right) \Big/ \left(1 + \frac{\int_n^b p_t(x)\,dt}{\sum_{i-0}^{n-1} q_i(x)}\right),$$

and

$$\frac{\left|\int_n^b T_t f(x)dt\right|}{\sum_{i=0}^{n-1} q_i(x)} \leqq \frac{\tau^n\left(\int_0^1 |T_t f|\, dt\right)(x)}{\sum_{i=0}^{n-1} q_i(x)} \to 0$$

a.e. on $\{x; \Sigma_{i=0}^{\infty} q_i(x) > 0\}$ by the Chacon-Ornstein lemma, where τ denotes the linear modulus of T_1. Hence the Chacon theorem and our lemma complete the proof.

REFERENCES

1. R. V. Chacon, *Operator averages*, Bull. Amer. Math. Soc., **68** (1962), 351–353.
2. ———, *Convergence of operator averages*, Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), 89–120, Academic Press, New York, 1963.
3. R. V. Chacon and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc., **15** (1964), 553–559.
4. N. Dunford and J. T. Schwartz, *Linear Operators, Part* I, Interscience, New York, 1958.
5. A. M. Garsia, *Topics in Almost Everywhere Convergence*, Markham, Chicago, 1970.
6. D. S. Ornstein, *The sums of iterates of a positive operator*, Advances in Probability and Related Topics, **2** (1970), 85–115.
7. S. Tsurumi, *An ergodic theorem for a semigroup of linear contractions*, Proc. Japan Acad., **49** (1973), 306–309.

Received May 9, 1974.

SHIBAURA INSTITUTE OF TECHNOLOGY
AND
JOSAI UNIVERSITY