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ABSOLUTE SUMMABILITY OF WALSH-FOURIER SERIES

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We prove that for all $f \in \mathscr{H}^1$, $\sum_{k=1}^{\infty} (1/k)\,|\hat{f}(k)| \leq K\,||f||_{\mathscr{H}^1}$, where $\mathscr{H}^1$ is the Walsh function analogue of the classical Hardy-space and $\hat{f}(k)$ is the k^{th} Walsh-Fourier coefficient of f. We obtain this as a consequence of its dual result: given a sequence $\{a_k\}$ of numbers such that $a_k = O(1/k)$, there exists a function $h \in$ BMO with $\hat{h}(k) = a_k$.

We study the relation between our results and the theory of differentiation on the Walsh group, developed by Butzer and Wagner.

Introduction. We are interested in various properties of Walsh-Fourier series. $w_k(\cdot)$ will denote the k^{th} Walsh function in the Paley-enumeration and $\hat{f}(k)$ will be the corresponding Walsh-Fourier coefficient of $f \in L^1$. $\mathscr{H}^1$ and BMO will denote the Walsh function analogues of the classical Hardy space and the functions of bounded mean oscillation, respectively (see [3], pp. 3-4; also refer to the section on "Preliminaries", in this paper).

Our principal result is

THEOREM 1. *There exists a constant $K > 0$ such that*

$$\sum_{k=1}^{\infty} (1/k)\,|\hat{f}(k)| \leqq K\,||f||_{\mathscr{H}^1}\,,$$

for all $f \in \mathscr{H}^1$.

Our proof of Theorem 1 does not follow the lines of its trigonometric analogue (see [5], pp. 286-287). We use the fact that Theorem 1 is equivalent to

THEOREM 2. *Given a sequence $\{a_k\}$ of numbers such that $a_k = O(1/k)$, there exists a function h in* BMO *such that $\hat{h}(k) = a_k$ for all k.*

We give a direct proof of Theorem 2.

Theorem 2, combined with a result of Fine [2] gives Lip $(1, L^1) \subsetneqq$ BMO. However, Lip $(1, L^1) \not\subset L^\infty$, in contrast with the trigonometric case where Lip $(1, L^1) = BV \subsetneqq L^\infty$ (see [5], p. 180). Theorem 2 also has connections with the Butzer-Wagner theory of differentiation on the Walsh-group (see [1]). The antidifferentiation kernel $W(x) \sim 1 + \sum_{k=1}^{\infty} (1/k) w_k(x)$ was shown by Butzer-Wagner to be in Lip $(1, L^1)$.

W is thus a function in BMO, but W is not bounded. Hence we know that if both f and $D^{[1]}f$-the Butzer-Wagner derivative of f-are in L^1 then f must be in BMO. We give an example of an f in L^1 with $D^{[1]}f$ in L^1 but f not bounded.

We have that $W \in \mathrm{Lip}\,(1/p, L^p)$, for $1 \leqq p < \infty$, which gives: if both f and $D^{[1]}f$ are in L^p, $1 < p < \infty$, then f is in $\mathrm{Lip}\,(1/p', C(G))$ and the Walsh-Fourier series of f converges absolutely.

Theorem 1 can be restated in the context of the Butzer-Wagner theory as: if f and $D^{[1]}f$ are in $\mathscr{H}^1$ then $\sum_{k=1}^{\infty} |\hat{f}(k)| < \infty$. Equivalently, the Walsh-Fourier series of the 'indefinite integral' of any f in $\mathscr{H}^1$ converges absolutely.

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Preliminaries. $G = \Pi \boldsymbol{Z}_2$, the countable product of infinitely many copies of $\boldsymbol{Z}_2$, is called the Walsh-group. Addition in G, defined termwise modulo 2, is denoted by $\dot{+}$. For a fixed $x = (x_k) \in G$, the sets $V_0(x) = G$,

$$V_n(x) = \{(x_1, x_2, \cdots, x_n, z_{n+1}, z_{n+2}, \cdots) \in G\},\ n \geqq 1\ ,$$

define a neighbourhood system at x and the topology thus induced on G, makes it a compact, abelian group.

The Haar-measure 'dx' on G is normalized so that $\int_G dx = 1$. The character group $\hat{G}$ of G is the set of all continuous, nonzero functions χ on G, satisfying

$$\chi(x \dot{+} y) = \chi(x)\chi(y),\ \forall x, y \in G\ ,$$

endowed with the compact-open topology. Fine [2] has shown that these functions are given by

$$w_n(x) = \prod_{k=0}^{\infty} [r_k(x)]^{\varepsilon_k}\ ,$$

where $r_k(x) = (-1)^{x_{k+1}}$ and $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$ is the unique binary expansion of the integer $n \geqq 0$. r_k's are called the Rademacher functions and w_j's the Walsh-functions (in Paley's enumeration). The system $\{w_j\}$ is closed under pointwise multiplication; more precisely, $w_n \cdot w_m = w_{n \dot{+} m}$, where for $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$, and $m = \sum_{j=0}^{\infty} \eta_j 2^j$, $\varepsilon_j, \eta_j \in \{0, 1\}$, we have $n \dot{+} m = \sum_{j=0}^{\infty} |\varepsilon_j - \eta_j|\, 2^j$.

For $m \geqq 1$, the m^{th} Dirichlet kernel is defined as:

$$D_m(x) = \sum_{k=0}^{m-1} w_k(x)\ .$$

For $m = 2^n$,

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in V_n(0) , \\ 0 & \text{if } x \notin V_n(0) . \end{cases}$$

For $f,\ g \in L^1$,

$$(f*g)(x) = \int_G f(y)g(x \dotplus y)dy .$$

$\hat{f}(k) = \int_G f(x)w_k(x)dx$ denotes the k^{th} Walsh-Fourier coefficient and

$$(S_n f)(x) = \sum_{k=0}^{n-1} \hat{f}(k)w_k(x) = (f*D_n)(x)$$

is the n^{th} partial sum of the Walsh-Fourier series of f. Thus,

$$(S_{2^m} f)(x) = 2^m \int_{V_m(x)} f(t)dt .$$

Moreover, $(f*g)\hat{}(k) = \hat{f}(k)\hat{g}(k)$, $\forall k \geqq 0$ and $f,\ g \in L^1$. Henceforth, all functions f are assumed to satisfy $\int_G f(x)dx = \hat{f}(0) = 0$.

L^p, $1 \leqq p \leqq \infty$ denote the usual Lebesgue spaces on G; $C(G)$ is the space of continuous functions on G.

BMO is defined to be the space of all functions f such that $\sup_{n \geqq 1} \| S_{2^n}[f - S_{2^{n-1}} f]^2 \|_\infty < \infty$. $\mathscr{H}^1$ is the space of those functions f for which $Sf = (\sum_{n=1}^\infty [S_{2^n} f - S_{2^{n-1}} f]^2)^{1/2} \in L^1$. Moreover, $\| f \|_{\mathscr{H}^1} = \| Sf \|_{L^1}$ (see [3]).

For $h = (h_n) \in G$, let $\lambda(h) = \sum_{n=1}^\infty h_n \cdot 2^{-n}$, and $\mathrm{Lip}\,(\alpha, L^p) = \{f \in L^p \colon \| f(\cdot) - f(\cdot \dotplus h) \|_{L^p} = O[\lambda(h)^\alpha]\}$, $1 \leqq p < \infty$, $\alpha > 0$.

For $p = \infty$, we replace L^∞ by $C(G)$. If

$$\omega_p(f; \delta) = \sup_{\lambda(h) \leqq \delta} \| f(\cdot) - f(\cdot \dotplus h) \|_{L^p} ,$$

then $f \in \mathrm{Lip}\,(\alpha, L^p) \Leftrightarrow \omega_p(f; \delta) = O(\delta^\alpha) \Leftrightarrow \omega_p(f; 2^{-n}) = O(2^{-n\alpha})$.

Let X denote L^p, $1 \leqq p < \infty$, or $C(G)$.

Define $e_j = \{x_s^j\}$ where $x_s^j = \delta_{js}$. For an $f \in X$, if there exists a $g \in X$ such that $\lim_{m \to \infty} \| 1/2 \sum_{j=0}^m 2^j [f(\cdot) - f(\cdot \dotplus e_{j+1})] - g(\cdot) \|_X = 0$, then f is said to be differentiable in X (see [1]). g is called the derivative of f and we write $D^{[1]} f = g$. Differentiable functions in X are completely characterized by the Theorem (see [1]):

For $f \in X$, the following are equivalent: (*)

(1) $D^{[1]} f = g$ exists.

(2) There is a $g \in X$ such that $k\hat{f}(k) = \hat{g}(k)$, $\forall k$.

(3) There is a $g \in X$ such that $f = W*g$ where $W(x) \sim 1 + \sum_{k=1}^\infty (1/k) w_k(x)$.

Proof of Theorem 2. Since $a_k = O(1/k)$, say $|a_k| \leqq M_1(1/k)$,

$\forall k \geqq 1$, $\sum_{k=1}^{\infty} a_k w_k(x)$ defines a function $h \in L^2$ such that $\hat{h}(k) = a_k$ for $k \geqq 1$ and $\hat{h}(0) = 0$. Let us put, $D_n(h, \nu)(t) = S_{2^n}\{S_{2^\nu}h(\cdot) S_{2^{\nu-1}}h(\cdot)\}^2(t)$.

Then to prove $h \in \mathrm{BMO}$, it suffices to show that (see [3]), $\exists M > 0 \ni \|\sum_{\nu=n}^{\infty} D_n(h, \nu)(t)\|_{(L^\infty, dt)} \leqq M$, $\forall n \geqq 1$.

Now

$$\begin{aligned}\{S_{2^\nu} h(\cdot) - S_{2^{\nu-1}} h(\cdot)\}^2 &= \left\{\sum_{k=2^{\nu-1}}^{2^\nu-1} a_k w_k(\cdot)\right\}^2 \\ &= \sum_{k=2^{\nu-1}}^{2^\nu-1} a_k^2 + 2 \sum_{k=2^{\nu-1}+1}^{2^\nu-1} \sum_{l=2^{\nu-1}}^{k-1} a_k \cdot a_l \cdot w_{k \dotplus l}(\cdot)\ .\end{aligned}$$

Also, for any $t \in G$ and n fixed,

$$S_{2^n}(w_j)(t) = \pm \chi_n(j) = \begin{cases} \pm 1 & \text{if } \ 0 \leqq j < 2^n\ , \\ 0 & \text{otherwise.} \end{cases}$$

Since $S_{2^n}(w_{k \dotplus l})(t) = \pm \chi_n(k \dotplus l)$, we have for $\nu \geqq n$

$$\begin{aligned}&D_n(h, \nu)(t) \\ &\quad = \sum_{k=2^{\nu-1}}^{2^\nu-1} a_k^2 + 2 \sum_{k=2^{\nu-1}+1}^{2^\nu-1} \sum_{l=2^{\nu-1}}^{k-1} \pm a_k \cdot a_l \cdot \chi_n(k \dotplus l)\ .\end{aligned}$$

Thus

$$\begin{aligned}&\left|\sum_{\nu=n}^{\infty} D_n(h, \nu)(t)\right| \\ &\quad \leqq M_1^2 \left[\sum_{k=1}^{\infty} \frac{1}{k^2} + 2 \sum_{\nu=n}^{\infty} \sum_{k=2^{\nu-1}+1}^{2^\nu-1} \sum_{l=2^{\nu-1}}^{k-1} \frac{1}{k} \cdot \frac{1}{l} \cdot |\chi_n(k \dotplus l)|\right].\end{aligned}$$

Note that, $|\chi_n(k \dotplus l)| = 1$ for $0 \leqq k \dotplus l < 2^n$ and 0 otherwise. For a fixed k, $k \dotplus l < 2^n$ iff the dyadic expansions of k and l agree at and after the n^{th} stage. Thus, there are exactly 2^n values of l for which $|\chi_n(k \dotplus l)| = 1$, if k is fixed. Therefore

$$\begin{aligned}&\sum_{\nu=n}^{\infty}\left\{\sum_{k=2^{\nu-1}+1}^{2^\nu-1} \frac{1}{k} \sum_{l=2^{\nu-1}}^{k-1} \frac{1}{l} \cdot |\chi_n(k \dotplus l)|\right\} \\ &\qquad < \sum_{\nu=n}^{\infty}\left\{\sum_{k=2^{\nu-1}+1}^{2^\nu-1} \frac{1}{k} \cdot \frac{2^n}{2^{\nu-1}}\right\} < \sum_{\nu=n}^{\infty} \frac{2^n}{2^{\nu-1}} = 4\ .\end{aligned}$$

Since $\sum_{k=1}^{\infty} (1/k^2) < \infty$, $\exists M < \infty$ such that $|\sum_{\nu=n}^{\infty} D_n(h, \nu)(t)| < M$, i.e., $\|\sum_{\nu=n}^{\infty} D_n(h, \nu)(t)\|_{(L^\infty, dt)} \leqq M < \infty$, $\forall n \geqq 1$.

COROLLARY 1. $\mathrm{Lip}\,(1, L^1) \subseteqq \mathrm{BMO}$, *but* $\mathrm{Lip}\,(1, L^1) \not\subseteqq L^\infty$.

Proof. Fine [2] had proved that, for each f in $\mathrm{Lip}\,(1, L^1)$, $\hat{f}(k) = O(1/k)$. So $f \in \mathrm{BMO}$ by Theorem 2.

Butzer and Wagner [1] have shown that $W(x) \sim 1 + \sum_{k=1}^{\infty'} (1/k) w_k(x)$ is in $\mathrm{Lip}(1, L^1)$. But $W \notin L^\infty$ because $\{S_{2^n} g\}$ is uniformly bounded whenever $g \in L^\infty$; $S_{2^m} W(x) = 1 + \sum_{k=1}^{2^m - 1} (1/k)$, $\forall x \in V_m(0)$.

REMARK. The above corollary is in contrast with the trigonometric case. We know that $\mathrm{Lip}(1, L^1) = BV \subsetneqq L^\infty$ in the latter context [5, p. 180].

Proof of Theorem 1. Recall that $f \in \mathscr{H}^1$. We want to show that $\sum_{k=1}^{\infty} (1/k) |\hat{f}(k)| \leqq K \cdot \|f\|_{\mathscr{H}^1}$, with K independent of f.

Let us put $b_k = (\operatorname{sgn} \hat{f}(k))/k$, $k \geqq 1$, $b_0 = 0$. Then by Theorem 2, $\exists g \in \mathrm{BMO}$ such that $\hat{g}(k) = b_k$. Therefore

$$\sum_{k=1}^{2^N - 1} (1/k) |\hat{f}(k)| = (S_{2^N} g * S_{2^N} f)(0)$$
$$= \int_G (S_{2^N} g)(y) \cdot (S_{2^N} f)(y) dy \,.$$

But (see [3], p. 8) the last integral is majorized by

$$\sqrt{2} \|g\|_{\mathrm{BMO}} \|f\|_{\mathscr{H}^1} \,.$$

Thus $\sum_{k=1}^{\infty} (1/k) |\hat{f}(k)| \leqq \sqrt{2} \|g\|_{\mathrm{BMO}} \|f\|_{\mathscr{H}^1}$. By the proof of Theorem 2, $\|g\|_{\mathrm{BMO}} \leqq \pi^2/6 + 8$. Hence, there exists a constant $K > 0$, independent of f, such that

$$\sum_{k=1}^{\infty} |\hat{f}(k)| (1/k) \leqq K \|f\|_{\mathscr{H}^1} \,.$$

REMARK. It can be easily shown that Theorem 1 implies Theorem 2.

Butzer and Wagner ([1]) introduced the notion of differentiation on the Walsh-group. They showed that $W(x) \sim 1 + \sum_{k=1}^{\infty} (1/k) w_k(x)$ is the 'antidifferentiation' kernel and W belongs to $\mathrm{Lip}(1, L^1)$. In the proof of Corollary 1, we have shown that $W \in \mathrm{BMO}$ but $W \notin L^\infty$. Since convolution of an L^1 function and a BMO function is again a BMO function, we obtain f and $D^{[1]} f$ are in $L^1 \Rightarrow f = W * D^{[1]} f$ is in BMO. Rubinshtein [4] has shown that $\sum_{n=1}^{\infty} (1/n \log n) w_n(x)$ defines an unbounded L^1-function g, and that $\sum_{n=2}^{\infty} (1/\log n) w_n(x) \sim h(x)$ is in L^1. Thus, we have g and $h = D^{[1]} g$ both in L^1 but g is not bounded.

It is easy to prove that $W \in \mathrm{Lip}(1/2, L^2)$; then using interpolation and duality, we get $W \in \mathrm{Lip}(1/p, L^p)$, $1 \leqq p < \infty$. By the characterization of differentiable functions in L^p (see [1]), we then have that if f and $D^{[1]} f$ are in L^p for some $1 < p < \infty$, then $f \in \mathrm{Lip}(1/q, C(G))$, where $1/p + 1/q = 1$. This leads to the fact that the Walsh-Fourier series of such an f converges absolutely. Theorem

1 actually strengthens this result, as we see below.

The definition of derivative can be given for $\mathscr{H}^1$ as in [1]. A characterization similar to (*) for differentiability in $\mathscr{H}^1$ remains true: $f \in \mathscr{H}^1$ is differentiable iff $\exists g \in \mathscr{H}^1$ such that $k\hat{f}(k) = \hat{g}(k)$, $\forall k$.

Now, if f is differentiable in $\mathscr{H}^1$, then

$$\sum_{k=1}^{\infty} |\hat{f}(k)| = \sum_{k=1}^{\infty} (1/k)\,|\hat{g}(k)| < \infty$$

by Theorem 1, because $g \in \mathscr{H}^1$; thus f has an absolutely convergent Walsh-Fourier series. The same fact can be stated as: The Walsh-Fourier series of the "indefinite integral" $W*g$ of any $g \in \mathscr{H}^1$, is absolutely convergent.

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