

CHARACTERIZATIONS OF SOME C^* -EMBEDDED SUBSPACES OF $\beta\mathbb{N}$

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Let K be a compact F -space such that $|C^*(K)| = 2^{\omega}$. Using the continuum hypothesis we characterize those subspaces of K that are C^* -embedded in K . We also characterize the class of extremally disconnected Tychonoff spaces of countable cellularity. As corollaries of these theorems, using various set-theoretic hypotheses we characterize the C^* -embedded, and the extremally disconnected C^* -embedded, subspaces of $\beta\mathbb{N}$.

1. Introduction. Our notation and terminology follows that of the Gillman-Jerison text [4]. All hypothesized topological spaces are assumed to be completely regular and Hausdorff (i.e., Tychonoff). As usual βX denotes the Stone-Ćech compactification of the Tychonoff space X , and \mathbb{N} denotes the countable discrete space. $C^*(X)$ denotes the family of bounded real-valued continuous functions on X . A subspace S of X is C^* -embedded in X if given $f \in C^*(S)$ there exists $g \in C^*(X)$ such that $g|_S = f$. A cozero-set of X is a set of the form $X - f^{-1}(0)$ where $f \in C^*(X)$. The collection of cozero-sets of X is denoted by $\text{coz}(X)$. A space X is zero-dimensional if its open-and-closed (clopen) sets form a base for its open sets. X is strongly zero-dimensional if βX is zero-dimensional.

A space X is weakly Lindelöf if given an open cover \mathcal{V} of X , there is a countable subfamily \mathcal{C} of \mathcal{V} such that $\bigcup \mathcal{C}$ is dense in X (if \mathcal{C} is a collection of subsets of a set we denote $\bigcup \{C: C \in \mathcal{C}\}$ by $\bigcup \mathcal{C}$). A space X has the countable chain condition, or countable cellularity, if each family of pairwise disjoint nonempty open subsets of X is countable. We abbreviate this by writing “ X has c.c.c.” The following lemma, which came to the attention of the author through a letter from W.W. Comfort, is easily proved.

LEMMA 1.1. *A space has c.c.c. iff each of its open subsets is weakly Lindelöf.*

A space X is extremally disconnected if disjoint open subsets have disjoint closures. It is an F -space if its cozero-sets are C^* -embedded. It is an F' -space if disjoint cozero-sets have disjoint closures. Each extremally disconnected space is an F -space, and each F -space is an F' -space. Proofs of these facts, plus other information on these classes of spaces, may be found in [1] and [4]. We shall

need the following facts.

THEOREM 1.2 (1H and 6M of [4]). *The following are equivalent for a space X .*

- (1) X is extremally disconnected.
- (2) Each dense subspace of X is extremally disconnected.
- (3) Each open subspace of X is extremally disconnected.
- (4) Each dense subspace of X is C^* -embedded in X .
- (5) Each open subspace of X is C^* -embedded in X .

THEOREM 1.3 (14.25 and 14.26 of [4]).

- (1) Each C^* -embedded subspace of an F -space is an F -space.
- (2) X is an F -space iff βX is an F -space.

The following lemma appears as the “note added on September 16, 1968” on page 494 of [1].

LEMMA 1.4. *If X is an F' -space and if each open subset of X is weakly Lindelöf, then X is extremally disconnected.*

LEMMA 1.5 (Corollary 1.7 of [1]). *Each weakly Lindelöf subspace of an F' -space is C^* -embedded in its own closure.*

The symbol [CH] preceding the statement of a theorem indicates that the continuum hypothesis ($2^\omega = \omega_1$) is used in the proof of the theorem. The cardinality of a set S is denoted by $|S|$. The *weight* of a topological space X , denoted by $w(X)$, is the least cardinal of a base for the open subsets of X . If α is a cardinal number then $D(\alpha)$ is the discrete space of cardinality α and $\log \alpha = \min \{\gamma: 2^\gamma \geq \alpha\}$.

Finally, we shall use the following theorem, which appears as Remark 8, page 274 of [2].

THEOREM 1.6. *Each compact extremally disconnected space K such that $w(K) \leq 2^\alpha$ can be topologically embedded in $\beta D(\alpha)$.*

2. C^* -embedded subsets of $\beta \underline{N}$. The proof of the implication in Theorem 2.2 that requires the continuum hypothesis—namely (3) \rightarrow (1)—relies heavily on a theorem, and a technique of proof, due to Fine and Gillman [3]. We first state the theorem.

THEOREM 2.1 (4.1(c) of [3]). *Let X be an F -space, let $\{S_\alpha: \alpha < \omega_1\}$ be a family of ω_1 cozero-sets of X , and put $S = \bigcup_{\alpha < \omega_1} S_\alpha$. If $G \subset S$ and $G \cap S_\alpha \in \text{coz}(S_\alpha)$ for each $\alpha < \omega_1$, then G is C^* -embedded in S .*

We now state and prove the main theorem of this section.

THEOREM 2.2 [CH]. *Let K be a compact F -space such that $|C^*(K)| = 2^\omega$. Let X be a subspace of K . The following are equivalent:*

- (1) X is weakly Lindelöf.
- (2) X is C^* -embedded in K .
- (3) $|C^*(X)| = 2^\omega$.

Proof.

(1) \rightarrow (2): By 1.5 X is C^* -embedded in $cl_K X$, which in turn is C^* -embedded in K by the Urysohn extension theorem (see 3.11(c) of [4]).

(2) \rightarrow (3): Since $|C^*(K)| = 2^\omega$ this is obvious.

(3) \rightarrow (1): Assume (1) fails; we shall prove that (3) fails also. Let X be a subspace of K that is not weakly Lindelöf. Let \mathcal{V} be an open cover of X which has no countable subcollection whose union is dense in X . By writing each member of \mathcal{V} as the intersection of X with a union of cozero sets of $cl_K X$, and noting that $cl_K X$ has only $2^\omega (= \omega_1)$ cozero subsets, we see that without loss of generality we may assume that $\mathcal{V} = \{A_\alpha \cap X : \alpha < \omega_1\}$, where each A_α is a cozero subset of $cl_K X$. Put $U = \bigcup \{A_\alpha : \alpha < \omega_1\}$. Fix $\alpha_0 < \omega_1$, and inductively assume that for each $\alpha < \alpha_0$, we have found a nonempty cozero-set B_α of $cl_K X$ such that $B_\alpha \subset U$ and $\gamma < \alpha < \alpha_0$ implies that $B_\alpha \cap (A_\gamma \cup B_\gamma) = \emptyset$. Now $\bigcup_{\alpha < \alpha_0} A_\alpha \cup B_\alpha$ is a cozero-set of $cl_K X$ contained in U . If it were dense in U , then as cozero-sets of compact spaces are Lindelöf there would be a countable subcollection \mathcal{C} of $\{A_\alpha : \alpha < \omega_1\}$ whose union covers $\bigcup_{\alpha < \alpha_0} A_\alpha \cup B_\alpha$. Thus $\bigcup \mathcal{C}$ would be dense in U , and so $\{C \cap X : C \in \mathcal{C}\}$ would be a countable subfamily of \mathcal{V} whose union is dense in X , contradicting hypothesis. Thus assume that $U - cl_K (\bigcup_{\alpha < \alpha_0} A_\alpha \cup B_\alpha) \neq \emptyset$. Hence a nonempty cozero-set B_{α_0} of $cl_K X$ can be found such that $B_{\alpha_0} \cap (\bigcup_{\alpha < \alpha_0} A_\alpha \cup B_\alpha) = \emptyset$ and $B_{\alpha_0} \subset U$. Now let $B = \bigcup_{\alpha < \omega_1} B_\alpha$. As $\gamma > \alpha$ implies $B_\gamma \cap A_\alpha = \emptyset$, evidently $B \cap A_\alpha = \bigcup_{\gamma \leq \alpha} B_\gamma \cap A_\alpha \in \text{coz}(U)$. Thus by 2.1 B is C^* -embedded in U . But B is the union of ω_1 pairwise disjoint nonempty open subsets of $cl_K X$, so $|C^*(B)| \geq 2^{\omega_1}$. Thus $|C^*(U)| \geq 2^{\omega_1}$ and as X is dense in U , $|C^*(X)| \geq 2^{\omega_1} > 2^\omega$. Thus (3) fails, and the proof is complete.

REMARKS 2.3. (1) The hypotheses on K in Theorem 2.2 are satisfied by a large class of spaces. One such class is the class of extremally disconnected spaces of weight no greater than 2^ω , such as βN , or the absolute of a compact separable space (see [2] for details concerning absolutes of compact spaces). Another such class

is the class of spaces of the form $\beta X - X$, where X is a locally compact σ -compact non-compact space with $|C^*(X)| = 2^\omega$ (see 14.27 of [4]); $\beta R - R$ is such a space, where R denotes the space of real numbers. Under assumption of the continuum hypothesis Theorem 2.2 gives a characterization of the C^* -embedded subspaces of each of these spaces.

(2) Let K satisfy the hypotheses imposed in 2.2. One consequence of 2.2 is that the question of whether a subspace X of K is C^* -embedded in K depends only on the topology of X and not on "how X is placed" in K . In the general case, by contrast, a space T can contain two homeomorphic subspaces, one C^* -embedded in T and the other not. For example the space Q of rational numbers is C^* -embedded in βQ , but its homeomorphs $Q - \{0\}$ and $Q \cap (0, \infty)$, for example, are not C^* -embedded in βQ .

(3) If $2^{\omega_1} = 2^\omega$ then Theorem 2.2 fails; for by 1.6 $\beta D(\omega_1)$ could be topologically embedded in $\beta \underline{N}$. Hence $\beta \underline{N}$ would contain a C^* -embedded copy of $D(\omega_1)$, which certainly is not weakly Lindelöf. I do not know whether Theorem 2.2 holds only if the continuum hypothesis holds; neither do I know whether the (possibly weaker) implication (3) \rightarrow (2) can hold in the absence of the continuum hypothesis.

Theorem 2.2 tells us when a subspace of $\beta \underline{N}$ will be C^* -embedded in $\beta \underline{N}$. A slight generalization of a theorem of Louveau (stated below) allows us to characterize (assuming the continuum hypothesis) those Tychonoff spaces that are homeomorphic to some C^* -embedded subspace of $\beta \underline{N}$. The following theorem appears in [5].

THEOREM 2.4 [CH]. *A compact space K is homeomorphic to a subspace of $\beta \underline{N}$ iff K is a zero-dimensional F -space and $w(K) \leq 2^\omega$.*

THEOREM 2.5 [CH]. *The following are equivalent for a space X :*

- (1) X is a strongly zero-dimensional F -space and $|C^*(X)| = 2^\omega$.
- (2) X is homeomorphic to a C^* -embedded subspace of $\beta \underline{N}$.

Proof.

(1) \rightarrow (2): By 1.3 βX is a compact zero-dimensional F -space and $|C^*(\beta X)| = 2^\omega$. Thus $w(\beta X) \leq 2^\omega$ so by 2.4 there is a compact subspace K of $\beta \underline{N}$ and a homeomorphism $h: \beta X \rightarrow K$. Evidently $h[X]$ is homeomorphic to X and C^* -embedded in $\beta \underline{N}$.

(2) \rightarrow (1): By hypothesis $\text{cl}_{\beta \underline{N}} X = \beta X$. Thus βX is zero-dimensional so X is strongly zero-dimensional. As βX is C^* -embedded in $\beta \underline{N}$, by 1.3 βX is an F -space and $|C^*(\beta X)| = 2^\omega$. Hence $|C^*(X)| = 2^\omega$ and by 1.3 X is an F -space.

One interesting consequence of 2.2 and 2.5 is that if the con-

tinuum hypothesis is assumed, if $X \subset \beta N$ and $|C^*(X)| = 2^\omega$ then X is a strongly zero-dimensional F -space.

3. **Extremally disconnected spaces of countable cellularity.** By combining 1.1, 1.5, and 1.6 we obtain the following.

THEOREM 3.1. *Let X be a Tychonoff space of countable cellularity. The following are equivalent:*

- (1) X is extremally disconnected.
- (2) X is homeomorphic to a subspace of $\beta D(\log w(\beta X))$.

Further, if X is homeomorphic to a subspace Y of $\beta D(\alpha)$ for some α , then Y is C^ -embedded in $\beta D(\alpha)$.*

Proof. Let $\beta D(\log w(\beta X)) = K$.

(1) \rightarrow (2): βX is extremally disconnected (see 6M of [4]), so by 1.6 βX can be embedded in K .

(2) \rightarrow (1): We may assume $X \subset K$. As K is extremally disconnected and hence an F -space, its C^* -embedded subspace $cl_K X$ is an F -space. But $cl_K X$ has c.c.c. as X has; hence by 1.1 and 1.4 $cl_K X$ is extremally disconnected. Thus by 1.2 X is extremally disconnected. The final statement of the theorem follows from 1.1 and 1.5.

COROLLARY 3.2. *A separable Tychonoff space is extremally disconnected iff it is homeomorphic to a subspace of βN .*

Proof. If X is separable then $w(\beta X) \leq 2^\omega$ (as βX will have no more than 2^ω regular open subsets), so $\log(w(\beta X)) = \omega$.

We now consider extremally disconnected C^* -embedded subspaces of βN . Note that 3.1 says that a subspace βN having c.c.c. will be extremally disconnected and C^* -embedded in βN . The following theorem describes when the converse holds.

THEOREM 3.3. *The following are equivalent:*

- (1) $2^{\omega_1} > 2^\omega$.
- (2) *Each extremally disconnected C^* -embedded subspace of βN has c.c.c.*

Proof.

(1) \rightarrow (2): Suppose X were an extremally disconnected C^* -embedded subspace of βN but that X does not have c.c.c. Let \mathcal{M} be a family of ω_1 pairwise disjoint open subsets of X . By 1.2 $\bigcup \mathcal{M}$ is C^* -embedded in X and hence in βN . But evidently $|C^*(\bigcup \mathcal{M})| \geq 2^{\omega_1}$; thus $|C^*(\beta N)| \geq 2^{\omega_1}$. But $|C^*(\beta N)| = 2^\omega$ so $2^\omega = 2^{\omega_1}$. Hence if (2)

fails, so does (1).

(2) \rightarrow (1): If $2^{\omega_1} = 2^\omega$ then $w(\beta D(\omega_1)) = 2^\omega$. Thus by 1.6 $\beta D(\omega_1)$ can be embedded in βN . Hence there will be a C^* -embedded copy of $D(\omega_1)$ in βN , and $D(\omega_1)$ is extremally disconnected but does not have c.c.c. Hence if (1) fails, so does (2).

REMARKS 3.4. (1) Part of 3.3 appears as Corollary 10 of [2], where it is shown that $2^\omega < 2^{\omega_1}$ iff each compact extremally disconnected space of weight 2^ω has c.c.c.

(2) Not every compact subspace of βN with c.c.c. is separable. Let B denote the Boolean algebra of Lebesgue measurable subsets of the unit interval, modulo sets of measure zero, and let X denote the Stone space of B . Then X is compact, extremally disconnected, has c.c.c., is not separable, and $w(X) = 2^\omega$. Hence X can be embedded in βN . A discussion of X , together with references to further sources of information about it, may be found in Example 7.5 of [7].

(3) In Remark 2.3 (2) we have seen that if $2^{\omega_1} = 2^\omega$ then βN has some discrete C^* -embedded subspaces of cardinality ω_1 . It would be interesting to know whether it is consistent that every discrete subspace of βN of cardinality ω_1 is C^* -embedded in βN . More generally, if one assumes, say, Martin's axiom [MA] but not CH, is it true that each discrete subspace of βN of cardinality less than 2^ω is C^* -embedded in βN ? (It is known that MA plus not CH implies that if $\kappa < 2^\omega$ then $2^\kappa = 2^\omega$; see, for example, page 21 of [6].)

(4) There is an interesting parallel between Theorems 2.2 and 3.3 as follows. Lemma 1.2 of [1] asserts that each cozero-set of a weakly Lindelöf space is weakly Lindelöf. Hence assuming the continuum hypothesis, a subspace of βN is C^* -embedded in βN iff all its cozero-sets are weakly Lindelöf; it is extremally disconnected and C^* -embedded in βN iff all its open subsets are weakly Lindelöf.

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Received January 7, 1976 and in revised form April 12, 1976. The research discussed in this paper was conducted while the author was on sabbatical leave at the University of Wisconsin (Madison). The author would like to thank the Department of Mathematics at the University of Wisconsin, and particularly Mrs. Rudin, for their kind hospitality. The author also wishes to thank the referee for a number of useful suggestions. This research was partially supported by Grant No. A7592 from the National Research Council of Canada.

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