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BIHOLOMORPHIC MAPPINGS BETWEEN WEAKLY PSEUDOCONVEX DOMAINS

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Assume we have a biholomorphic mapping between weakly pseudoconvex domains. It is an old question whether this extends to a diffeomorphism between their closures. The well known theorem of Fefferman states that this is true for strongly pseudoconvex domains. We will show that if the map has a smooth extension to the boundary, then it cannot map an analytic disc in the boundary to a single point.

This is an immediate consequence of the following theorem.

THEOREM. *Assume Ω, W are bounded pseudoconvex sets with $\mathscr{C}^2$ boundary in $\boldsymbol{C}^n$, and assume $\Phi: \Omega \longrightarrow W$ is a biholomorphic map with a $\mathscr{C}^2$-extension $\Phi: \bar{\Omega} \longrightarrow \bar{W}$. Then Φ is a $\mathscr{C}^2$-diffeomorphism between $\bar{\Omega}$ and $\bar{W}$.*

This theorem generalizes a similar result for strongly pseudoconvex domains by the author [2], see also Pinchuk [3].

The theorem is false in general for $\mathscr{C}^1$-domains and maps Φ with $\mathscr{C}^1$-extensions. To see this, let $\Omega = \{z \in \boldsymbol{C};\ |z+1| < 1\}$ and let $\Phi(z) = z/\operatorname{Log} z$ and $W = \Phi(\Omega)$.

Also the theorem is false in general for proper holomorphic maps. For example, let $\Omega = \{(z, w) \in \boldsymbol{C}^2;\ |z|^2 + |w|^4 < 1\}$, $W = \{(z, w);\ |z|^2 + |w|^2 < 1\}$ and $\Phi(z, w) = (z, w^2)$.

Proof of the Theorem. Let us fix a point $p \in b\Omega$. We want to show that $\Phi'(p)$ is a nonsingular linear transformation. The proof is complete if this is true for all p in the boundary of Ω.

To simplify the computations, we will make affine complex changes of coordinates such that p becomes the origin in $\boldsymbol{C}^n$ with variables $(z_1, \cdots, z_n)$ and $\Phi(p)$ becomes the origin in $\boldsymbol{C}^n$ with variables $(w_1, \cdots, w_n)$. We may arrange this such that

$$\Omega = \{z = (z_1, \cdots, z_n);\ \rho(z) = \operatorname{Re} z_1 + R(z_1, \cdots, z_n) < 0\} \quad \text{and}$$
$$W = \{w = (w_1, \cdots, w_n);\ \sigma(w) = \operatorname{Re} w_1 + S(w_1, \cdots, w_n) < 0\}$$

where R, S are $\mathscr{C}^2$-functions which vanish to at least second order at the origin.

From Diederich, Fornaess [1] it follows that there exists a $\mathscr{C}^2$-defining function $\hat{\sigma}$ of W such that $-(-\hat{\sigma})^{2/3}$ is strictly plurisubharmonic in W near the origin. It follows that $-(-\hat{\sigma})^{2/3} \circ \Phi$ is strictly plurisubharmonic in Ω near the origin.

We apply the Hopf lemma to points in Ω of the form $(t, 0, \cdots, 0)$ with $-1 \ll t < 0$. There must exist a $K > 0$ such that

$$-(-\hat{\sigma})^{2/3}(\Phi(t, 0, \cdots, 0)) \leqq Kt .$$

Since σ is of the same order of magnitude as $\hat{\sigma}$, we obtain for a possibly different K, that

$$-\operatorname{Re}\varphi_1(t, 0, \cdots, 0) - S(\Phi(t, 0, \cdots, 0)) \geqq K|t|^{3/2}$$

where we have written $\Phi = (\varphi_1, \cdots, \varphi_n)$. Hence $(\partial\varphi_1/\partial z_1)(0, \cdots, 0) > 0$.

Consider the $\mathscr{C}^2$-map $\Lambda(z_1, \cdots, z_n) = (\varphi_1(z_1, \cdots, z_n), z_2, \cdots, z_n)$. Then $\Lambda'(0)$ is invertible. Therefore Λ maps the germ of Ω at the origin to the germ of a pseudoconvex set U with $\mathscr{C}^2$ boundary at the origin. We can describe U by

$$U = \{\eta = (\eta_1, \cdots, \eta_n); \tau(\eta) = \operatorname{Re}\eta_1 + T(\eta_1, \cdots, \eta_n) < 0\}$$

for some $\mathscr{C}^2$-function T vanishing to at least second order at the origin.

We will study the map $\Psi = \Phi\circ\Lambda^{-1}$, $\Psi = (\psi_1, \cdots, \psi_n)$. It suffices to show that $\Psi'(0)$ is a nonsingular linear map. We can describe Ψ by

$$\Psi(\eta_1, \cdots, \eta_n) = (\eta_1, \psi_2(\eta_1, \cdots, \eta_n), \cdots, \psi_n(\eta_1, \cdots, \eta_n)) .$$

If $\Psi'(0)$ is singular, then we may assume, after a linear change in the $(w_2, \cdots, w_n)$ — and $(\eta_2, \cdots, \eta_n)$ — direction, that

$$\Psi'(0) = \begin{bmatrix} 1 & 0 & 0 \cdots & 0 \\ \dfrac{\partial\psi_2}{\partial\eta_1}(0) & 0 & \dfrac{\partial\psi_2}{\partial\eta_3}(0) \cdots & \dfrac{\partial\psi_2}{\partial\eta_n}(0) \\ \vdots & \vdots & \vdots & \\ \dfrac{\partial\psi_n}{\partial\eta_1}(0) & 0 & \dfrac{\partial\psi_n}{\partial\eta_3}(0) \cdots & \dfrac{\partial\psi_n}{\partial\eta_n}(0) \end{bmatrix} .$$

Next we consider points $t_1 = (t, 0, \cdots, 0)$ and $t_2 = (t, t, 0, \cdots, 0)$ in U with $-1 \ll t < 0$. We then have the estimates, for $\tau_i = \Psi(t_i)$, $i = 1, 2$:

$$\tau_i = \left(t, \frac{\partial\psi_2}{\partial\eta_1}(0)\cdot t + 0(t^2), \cdots, \frac{\partial\psi_n}{\partial\eta_1}(0)\cdot t + 0(t^2)\right) ,$$

$i = 1, 2.$

Define W_t to be the set

$$W_t = \{(w_1, \cdots, w_n) \in W; w_1 = t\} .$$

There exists some $\delta > 0$ independent of t such that

$$W_t \supset \{(t, w_2, \cdots, w_n);\ \|(w_2, \cdots, w_n)\| < \delta|t|^{1/2}\} = \tilde{W}_t\ .$$

Let us write $\Phi^{-1}: W \longrightarrow \Omega$ as $\Phi^{-1} = (\mu_1, \cdots, \mu_n)$. We then have that for some constant $K > 0$, independent of t, that

$$|\mu_2(t, w_2, \cdots, w_n)| \leqq K$$

for all points in $\tilde{W}_t$, since Ω is bounded. The points τ_1, τ_2 are in $\tilde{W}_t$ and satisfy the estimates $\|\tau_1\| \leqq K_1|t|$, $\|\tau_2\| \leqq K_1|t|$ and $\|\tau_1 - \tau_2\| \leqq K_1|t|^2$ for some constant K_1 independent of t. It follows from Schwarz's lemma that for some constant K_2, independent of t, we have

$$|\mu_2(\tau_1) - \mu_2(\tau_2)| \leqq K_2|t|^{3/2}\ .$$

However, by construction we know that $|\mu_2(\tau_1) - \mu_2(\tau_2)| = |t|$ which is a contradiction for all small enough $|t|$.

REFERENCES

1. K. Diederich and J. E. Fornaess, *Exhaustion functions and Stein neighborhoods for smooth pseudoconvex domains*, Proc. Nat. Acad. Sci. USA, **72** (1975), 3279–3280.

2. J. E. Fornaess, *Embedding strictly pseudoconvex domains in convex domains*, Amer. J. Math., **98** (1976), 529–569.

3. S. I. Pinchuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J., (1975), 644–649 (translated from Sibirskii Mat. Zhurnal, **15** (1974), 909–917).

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