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ON BOUNDED AND SUBCONTINUOUS MULTIFUNCTIONS

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In this paper we extend Kocela's conditions of boundedness of real valued functions to the case of multifunctions. Moreover the concept of subcontinuity (introduced by R. E. Smithson) is considered with application to the following generalization of a result of Ka-Sing Lau:

Let $F: X \to Y$ be a point closed and convex multifunction taking values in a locally convex space Y and suppose F is subcontinuous. Then it is f-continuous if and only if for every functional $y' \in Y'$ the function $x \to \sup\{y'(y): y \in F(x)\}$ is continuous.

1. Let X, Y denote topological spaces. $\mathscr{A}(Y)$ is the family of all nonempty subsets of Y; 2^Y and $\mathscr{C}(Y)$ are the families of closed and compact sets in $\mathscr{A}(Y)$, respectively.

A multifunction $F: X \to \mathscr{A}(Y)$ is called *f-upper semicontinuous at $x_0 \in X$ (f-usc at x_0)* if for each open set G, containing $F(x_0)$ the set $\{x \in X: F(x) \subset G\}$ is open. F is *f-lower semicontinuous at $x_0 \in X$ (f-lsc at x_0)* if for every open G such that $F(x_0) \cap G \neq \varnothing$ the set $\{x \in X: F(x) \cap G \neq \varnothing\}$ is open. F is *f-continuous at $x_0 \in X$* if it is f-usc and f-lsc at $x_0 \in X$.

Now let $(Y, \mathfrak{A})$ be an uniform space. For $V \in \mathfrak{A}$ and $A \subset Y$ we denote $V[A] = \{y \in Y: (z, y) \in V \textit{ for some } z \in A\}$. Let $\mathscr{U}(x_0)$ denote a base of neighbourhoods of $x_0 \in X$. We say that a multifunction $F: X \to \mathscr{A}(Y)$ is *u-continuous at $x_0 \in X$* if for every $V \in \mathfrak{A}$ there is $U(x_0) \in \mathscr{U}(x_0)$ such that $x \in U(x_0)$ implies

$$F(x_0) \subset V[F(x)] \quad \text{and} \quad F(x) \subset V[F(x_0)].$$

The grill of $\mathscr{U}(x_0)$, denoted by $\mathscr{U}''(x_0)$, consists of all sets $U''(x_0)$ contained in X such that $U''(x_0) \cap U(x_0) \neq \varnothing$ for every $U(x_0) \in \mathscr{U}(x_0)$. We say that a multifunction $F: X \to \mathscr{A}(Y)$ is *p-upper semicontinuous at $x_0 \in X$ (p-usc at x_0)* if

$$p\text{-}\limsup_{x \to x_0} F(x) = \bigcap_{U(x_0) \in \mathscr{U}(x_0)} \left[\overline{\bigcup_{x \in U(x_0)} F(x)}\right] \subset F(x_0),$$

and that F is *p-lower semicontinuous at $x_0 \in X$ (p-lsc at x_0)* if

$$F(x_0) \subset p\text{-}\liminf_{x \to x_0} F(x) = \bigcap_{U''(x_0) \in \mathcal{U}''(x_0)} \left[\overline{\bigcup_{x \in U''(x_0)} F(x)} \right].$$

A multifunction $F: X \to \mathcal{A}(Y)$ is called *p-continuous at $x_0 \in X$* if it is p-lsc and p-usc at x_0. F is p-usc (i.e. p-usc at each $x \in X$) iff it has a closed graph $G(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ (Choquet [**1**]). For more information about these notions of continuity see for example [**6**].

2. Let Y denote a real Hausdorff topological vector space and $\mathcal{B}(Y)$ [$\mathcal{K}(Y)$] the collection of nonempty bounded [nonempty closed and convex] subsets of Y. A multifunction $F: X \to \mathcal{A}(Y)$ is called *bounded* if the set $F(X) = \bigcup_{x \in X} F(x) \subset Y$ is bounded.

A topological space X is called *σ-absolutely closed* [**5**] if for each open countable cover $\{G_n\}$ there is a finite subset $\{G_1, \cdots, G_k\}$ of $\{G_n\}$ such that

$$X = \overline{\bigcup_{i=1}^{k} G_i}.$$

The following three propositions generalize the results of E. Kocela [**5**] established for real valued functions.

2.1. *If X is σ-absolutely closed then every f-continuous multifunction $F: X \to \mathcal{B}(Y)$ is bounded.*

Proof. Let V be a neighbourhood of 0 in Y and W a neighbourhood of 0 such that $\bar{W} \subset V$. Define $G_n = \{x \in X : F(x) \subset nW\}$, $n \in N$. By assumption $X = \overline{\bigcup_{i=1}^{k} G_i}$, for some $k \in N$. Hence by f-lower semicontinuity of F we have

$$\begin{aligned} X = \bar{G}_k &= \overline{\{x \in X : F(x) \subset kW\}} \subset \{x \in X : F(x) \subset k\bar{W}\} \\ &\subset \{x \in X : F(x) \subset kV\}, \quad \text{i.e.,} \ F(X) \subset kV. \end{aligned}$$

A pair of families of sets $\{A_\lambda\}$, $\{B_\lambda\}$, $\lambda_0 \leqq \lambda < \infty$, ($\lambda_0$ is a fixed real number) is said to satisfy condition $(*)$ if

(1) for every $\lambda \geqq \lambda_0$ the set A_λ is closed while the set B_λ is open,

(2) for every pair $\lambda_1 < \lambda_2$ holds $A_{\lambda_1} \supset B_{\lambda_1} \supset A_{\lambda_2} \supset B_{\lambda_2}$,

(3) $A_\lambda \neq \varnothing \neq B_\lambda$.

A topological space X is called *$*$-compact* if for every pair of families satisfying condition $(*)$ $\bigcap A_\lambda = \bigcap B_\lambda \neq \varnothing$ holds [**5**].

2.2. *If X is $*$-compact and Y is a locally convex space then every f-continuous multifunction $F: X \to \mathcal{B}(Y)$ is bounded.*

Proof. Let $(p_t)_{t \in T}$ be a family of seminorms determining the topology of the space Y. Suppose that there exists a nonbounded f-continuous multifunction $F: X \to \mathcal{B}(Y)$. Then there is $t \in T$ such that the set $F(X)$ is unbounded in the space (Y, p_t). For every $\lambda \geqq \lambda_0 > 0$ we define

$$A_\lambda = \{x \in X: F(x) \cap (Y - \lambda V) \neq \emptyset\},$$
$$B_\lambda = \{x \in X: F(x) \cap (Y - \lambda \bar{V}) \neq \emptyset\},$$

where $V = \{y \in Y: p_t(y) < 1\}$. Then the pair of families $\{A_\lambda\}$, $\{B_\lambda\}$ satisfies the condition $(*)$, but $\cap A_\lambda = \cap B_\lambda = \emptyset$ holds, because $F(x) \in \mathcal{B}(Y)$ for $x \in X$.

2.3. *Let (Y, τ) be a locally convex space, (X, ρ) a metrizable compact space and ρ_1 a topology on X stronger than ρ. If (X, ρ_1) is $*$-compact then every $(\rho_1)u$-continuous multifunction $F: X \to \mathcal{K}(Y) \cap \mathcal{B}(Y)$ is $(\rho)u$-continuous.*

Proof. Suppose that $F: X \to \mathcal{K}(Y) \cap \mathcal{B}(Y)$ is $(\rho_1)u$-continuous but it is not $(\rho)u$-continuous at $x_0 \in X$. The multifunction F may be considered as a (ρ_1)continuous function taking values in the locally convex topological vector space $(R(Y), r(\tau))$ from Minkowski–Rådström–Hörmander theorem (see e.g. [**2**; Th.2.1]). Let $(p_t)_{t \in T}$ be a family of seminorms determining the topology $r(\tau)$ of $R(Y)$. Then there is a seminorm p_t such that the function $p_t \circ F: X \to R$ is (ρ_1)continuous but it is not (ρ)continuous at x_0. Thus applying [**5**; Th. 6] we obtain that (X, ρ_1) is not $*$-compact.

Now let Y be an arbitrary topological space. Following Smithson [**8**] a multifunction $F: X \to \mathcal{A}(Y)$ is said to be *subcontinuous* if and only if whenever $(x_t)_{t \in T}$ is a convergent net in X and $(y_t)_{t \in T}$ is a net in $F(X) = \bigcup_{x \in X} F(x)$ with $y_t \in F(x_t)$ then $(y_t)_{t \in T}$ has a convergent subnet.

THEOREM 2.4. (Smithson [**8**]). *Let $F: X \to \mathcal{A}(Y)$ be a subcontinuous multifunction which has a closed graph (i.e. F is p-usc). Then F is f-usc and $F: X \to \mathcal{C}(Y) \cap 2^Y$.*

In the case Y is regular every f-usc multifunction $F: X \to 2^Y$ is p-usc ([**6**], [**8**]). Moreover

THEOREM 2.5. *Every f-usc multifunction $F: X \to \mathscr{C}(Y)$ is subcontinuous.*

Proof. Let $(x_t)_{t\in T}$ be a net in X which is convergent to x_0, and let $(y_t)_{t\in T}$ be a net with $y_t \in F(x_t)$ for each $t \in T$. Next let $\mathscr{F}$ be the filter associated with $(y_t)_{t\in T}$ and $\mathscr{F}_0$ be an ultrafilter $\mathscr{F}_0 \supset \mathscr{F}$. Then $\mathscr{F}_0$ converges to some point of $F(x_0)$. To see this, suppose that each $y \in F(x_0)$ is contained in an open set $V(y) \notin \mathscr{F}_0$. By compactness of $F(x_0)$ there are points $y_1, \cdots, y_n \in F(x_0)$ such that $F(x_0) \subset \bigcup_{i=1}^n V(y_i)$. Then, since F is f-usc, there is a neighbourhood U of x_0 with property $F(U) \subset \bigcup_{i=1}^n V(y_i)$. There is $t_0 \in T$ such that $\{y_t : t \geqq t_0\} \subset \bigcup_{i=1}^n V(y_i)$. Thus $\bigcup_{i=1}^n V(y_i) \in \mathscr{F}_0$. Since $\mathscr{F}_0$ is an ultrafilter, it has to contain at least one $V(y_i)$, $1 \leqq i \leqq n$, but this is impossible.

From Theorem 2.4, the remark following it and [**6**; 1.2 and 1.18] (see also [**7**; Th. 3.4]) we get immediately

COROLLARY 2.6. *Let Y be a Hausdorff uniform space. If a multifunction $F: X \to 2^Y$ is subcontinuous then the following statements are equivalent:*

(a) *F is p-continuous.*
(b) *F is u-continuous.*
(c) *F is f-continuous.*

REMARK. Theorem 2.4 improves a result of G. Choquet [**1**] obtained in the case of compact Y. (See also [**9**; Prop. 8] and [**6**; Prop. 1.7].)

Henceforth Y denotes a real Hausdorff locally convex vector space under topology τ, Y' its topological dual space. If $A \subset Y$ then the *support function of A* is defined by

$$y' \to \varphi(y', A) = \sup\{y'(y) : y \in A\}.$$

The following two propositions are easy to prove.

2.7. *For every multifunction $F: X \to \mathscr{A}(Y)$ the following statements are equivalent:*

(a) *For every $y' \in Y'$ and $r \in R$ the set $\{x \in X : F(x) \cap H \neq \varnothing\}$, where $H = \{y \in Y : y'(y) > r\}$, is open.*

(b) *For every $y' \in Y'$ the function $x \to \varphi(y', F(x))$ is lower semicontinuous.*

2.8. *Let $F: X \to \mathscr{A}(Y)$ and consider the following statements:*

(a) *For every* $y' \in Y'$ *and* $r \in R$ *the set* $\{x \in X : F(x) \subset H\}$, *where* $H = \{y \in Y : y'(y) > r\}$, *is open.*

(b) *For every* $y' \in Y'$ *the function* $x \to \varphi(y', F(x))$ *is upper semicontinuous.*

We have

(i) (a) $\Rightarrow$ (b).

(ii) *If* $F(x)$ *is weakly compact for every* $x \in X$, *then* (b) $\Rightarrow$ (a).

It is easy to show that in general the condition (b) does not imply the following one:

(c) F *is* f-*semicontinuous in the topology* $\sigma(Y, Y')$. The implication (b) $\Rightarrow$ (c) holds under additional assumptions on a multifunction F. More exactly

THEOREM 2.9. (Valadier **[10, 11]** and Godet-Thobie **[3]**.) *Let* [*F be a locally bounded multifunction* **[3]** *and*] *for each* $x \in X$ $F(x)$ *be a convex and weakly compact* [*bounded*] *subset of* Y. *Then* F *is* f-*usc* [f-*lsc*], *in the topology* $\sigma(Y, Y')$, *iff for every* $y' \in Y'$ *the function* $x \to \varphi(y', F(x))$ *is upper semicontinuous* [*lower semicontinuous*].

An immediate consequence of the above theorem is the following result, due to Ka-Sing Lau:

THEOREM 2.10. (Ka-Sing Lau **[4]**.) *Suppose* K *is a compact subset of* Y. *Then a multifunction* $F: X \to \mathcal{K}(K)$ *is* f-*lsc if and only if the set* $\{x \in X : F(x) \cap H \neq \emptyset\}$ *is open for any half space* $H = \{y \in Y : y'(y) > r\}$, *where* $y' \in Y'$, $r \in R$.

By using subcontinuity we can get a generalization of this.

THEOREM 2.11. *Let* $F: X \to \mathcal{K}(Y)$ *be a subcontinuous and locally bounded multifunction. Then it is* f-*lsc if and only if for every* $y' \in Y'$ *the function* $x \to \varphi(y', F(x))$ *is lower semicontinuous.*

Proof. The necessity is clear. Suppose now that F is not f-lsc at the point $x_0 \in X$. Let $\mathcal{U}(x_0) = (U_t)_{t \in T}$ be a base of neighbourhoods of x_0 and partial order $\mathcal{U}(x_0)$ by inclusion downward. Then there is a point $y_0 \in F(x_0)$ and a τ-open set $G \ni y_0$ such that, for each $t \in T$, the equality $F(x_t) \cap G = \emptyset$ holds for some $x_t \in U_t$. Without loss of generality we can consider that the net (x_t) converges to x_0. Let $\mathcal{V}(y_0)$ be a base of neighbourhoods of y_0 for the topology $\sigma(Y, Y')$ and partial order $\mathcal{V}(y_0)$ by inclusion downward. Then give $T \times \mathcal{V}(y_0)$ the product partial order. By Theorem 2.9 the mapping F is f-lsc with respect to the topology $\sigma(Y, Y')$. It follows that for each $(t, V) \in T \times \mathcal{V}(y_0)$ there is an $s = s(t, V) \in T$ such that $s \geqq t$ and $F(x_s) \cap V \neq \emptyset$. Choose such x_s for every element (t, V). Then the net (x_s) is a subnet of (x_t) and converges to x_0. Let $y_s \in F(x_s) \cap V \neq \emptyset$. The net (y_s) has a subnet $(y_{s'})$ σ-convergent to y_0. Now since F is subcontinuous, some subnet $(y_{s''})$ of $(y_{s'})$ τ-converges to some $y_1 \in Y$, and consequently, σ-converges to y_1. Thus $y_0 = y_1$. Since $y_{s''} \in F(x_{s''}) \subset Y - G$ we have $y_0 = y_1 \in \overline{Y - G}^{\tau} = Y - G$, but this is impossible because $y_0 \in G$.

2.12. *If $F: X \to \mathcal{K}(Y)$ and for every $y' \in Y'$ the function $x \to \varphi(y', F(x))$ is upper semicontinuous then a graph $G(F)$ of F is a closed subset of $X \times (Y, \sigma(Y, Y'))$ (i.e. F is p-usc in the topology $\sigma(Y, Y')$).*

Proof. Let $(x_0, y_0) \notin G(F)$, i.e. $y_0 \notin F(x_0)$. Then by the separation theorem there is $y' \in Y'$ such that $\varphi(y', F(x_0)) < \lambda < y'(y_0)$ for some λ. As the mapping $x \to \varphi(y', F(x))$ is upper semicontinuous there is a neighbourhood $U(x_0)$ of x_0 such that $\varphi(y', F(x)) < \lambda$ provided $x \in U(x_0)$. Then $U(x_0) \times V(y_0) \subset X \times Y - G(F)$, where $V(y_0) = \{y \in Y: y'(y) > \lambda\} \in \sigma(Y, Y')$.

THEOREM 2.13. *Let $F: X \to \mathcal{K}(Y)$ be a subcontinuous multifunction. Then it is f-usc if and only if for every $y' \in Y'$ the function $x \to \varphi(y', F(x))$ is upper semicontinuous.*

COROLLARY 2.14. *Let X be a locally compact space and $F: X \to \mathcal{K}(Y)$ a subcontinuous multifunction. Then it is f-continuous if and only if for every $y' \in Y'$ the function $x \to \varphi(y', F(x))$ is continuous.*

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