PACIFIC JOURNAL OF MATHEMATICS
Vol. 81, No. 2, 1979

AN IMPLICIT FUNCTION THEOREM IN BANACH SPACES

IAIN RAEBURN

We prove the following theorem:

THEOREM: Suppose X, Y, and Z are complex Banach spaces, U and V are open sets in X and Y respectively, and $x \in U$, $y \in V$. Suppose $f: U \to V$ and $k: V \to Z$ are holomorphic maps with $f(x) = y$, $k \circ f$ constant and range $f'(x) = \ker k'(y) \neq \{0\}$. Let D be a domain in C^n, $z \in D$ and $g: D \to Y$ be a holomorphic map with $g(z) = y$ and $k \circ g$ constant. Then there is an open neighborhood W of z and a holomorphic map $h: W \to X$ such that $h(z) = x$ and $g|_W = f \circ h$.

We use this result to prove an Oka principle for sections of a class of holomorphic fibre bundles on Stein manifolds whose fibres are orbits of actions of a Banach Lie group on a Banach space.

Introduction. Suppose U is an open set in C^n, $x \in U$, and $f: U \to C^m$ is a holomorphic map such that $f'(x)$ is surjective. Then a form of the implicit function theorem tells us that there is a neighborhood V of $f(x)$ and a holomorphic map $\rho: V \to U$ such that $\rho(f(x)) = x$ and $f \circ \rho$ is the identity on V. This theorem remains true if f is a holomorphic map of an open set U in a Banach space X into a Banach space Y, provided that $\ker f'(x)$ is a complemented subspace of X. That this is also a necessary condition follows from the fact that $f'(x) \circ \rho'(f(x))$ is the identity operator on Y, so that $\rho'(f(x)) \circ f'(x)$ is a projection of X onto $\ker f'(x)$.

In general, implicit function theorems work well in a Banach space setting, provided that we impose suitable complementation conditions (see, for example [4]). In practice it can be very hard to find out whether a given subspace of a Banach space is complemented; our main theorem is an implicit function theorem which has no complementation hypothesis. Before we state our theorem, we shall reword the result mentioned above. Let X and Y be complex Banach spaces, U be open in X, $x \in U$, and $f: U \to Y$ be a holomorphic map such that $f'(x)$ is surjective and $\ker f'(x)$ is a complemented subspace of X. Then if V is an open set in a Banach space W, $w \in V$, and g is a holomorphic map of V into Y such that $g(w) = f(x)$, there is a neighborhood N of w and a holomorphic map $h(= \rho \circ g)$ of N into X such that $f \circ h = g$ on N. Our main theorem asserts that provided W is finite-dimensional, this theorem is still true without the hypothesis that $\ker f'(x)$ be complemented. More generally, suppose there is a third Banach space Z and a holomorphic map $k: Y \to Z$ such that $k \circ f$ is constant and range $f'(x) = \ker k'(f(x))$. Let D be an open set in $\boldsymbol{C}^n$, and let $z \in D$. Then our main theorem says that if g is a holomorphic map of D into $k^{-1}(k(f(x)))$ with $g(z) = f(x)$, then there is a holomorphic map h of a neighborhood N of z into X such that $f \circ h = g|_N$. We shall prove this theorem in § 2.

Grauert [2] has proved an Oka principle for sections of a holomorphic fibre bundle over a Stein manifold with fibre a complex Lie group. Ramspott [10] has generalized this result to allow homogeneous spaces as fibres, and Bungart [1] has extended it to the case where the fibres are infinite-dimensional Lie groups. In § 3, as an application of our implicit function theorem, we shall extend the theorems of Ramspott and Bungart to allow for infinite-dimensional fibres which are the orbits of suitable actions $(g, x) \rightarrow g \cdot x$ of an infinite-dimensional Lie group G on a Banach space X; more specifically, we demand that such an orbit M also be the level set of a holomorphic map k in such a way that the derivatives of the orbit map $g \rightarrow g \cdot x_0$ and k form an exact sequence at $x_0 \in M$.

1. Preliminaries. Let X and Y be complex Banach spaces, let U be an open set in X and let f be a continuous map of U into Y. We say f is holomorphic in U if at each point of U f has a Fréchet derivative which is a complex linear map of X into Y. Equivalently, f is holomorphic in U if for each $x \in U$ and $h \in X$ the function $z \rightarrow f(x + zh)$ is holomorphic in a neighborhood of 0 in $\boldsymbol{C}$. If $f\colon U \rightarrow Y$ is holomorphic in U, then f has complex Fréchet derivatives of all orders; that is, for $x \in U$ and all n the nth derivative $f^{(n)}(x)$ exists as a complex multilinear map of X^n to Y. We give X^n the norm $\|(x_1, \cdots, x_n)\| = \sup\{\|x_i\|\}$ and put the corresponding operator norm on $L^n(X^n, Y)$, the space of complex n-linear maps of X^n into Y. If $f\colon U \subset X \rightarrow Y$ is holomorphic, it is well-known that $\limsup (\|f^{(n)}(x)\|/n!)^{1/n}$ is finite for each $x \in U$. For further details of this material, we refer to [7].

We shall use many times two differentiation techniques which are well-known in one variable; namely, the chain rule and Liebnitz' formula. Let U be open in X, V be open in Y, and let $f\colon U \rightarrow V$ and $g\colon V \rightarrow Z$ be differentiable. Then the chain rule [5, p. 99] says that $g \circ f$ is differentiable, and, for $x_0 \in U$, the derivative $(g \circ f)'(x_0) \in L(X, Z)$ is given by

$$(g \circ f)'(x_0)x = g'(f(x_0))[f'(x_0)x] \quad \text{for} \quad x \in X\ .$$

Let U be an open set in $\boldsymbol{C}$, and let $f\colon U \rightarrow L(Y, Z)$ and $g\colon U \rightarrow L(X, Y)$ be n times continuously differentiable maps. Then we can define $fg\colon U \rightarrow L(X, Z)$ by $fg(u) = f(u) \circ g(u)$ for $u \in U$, and a special case of the product formula [5, p. 97] gives that fg is differentiable and

$$(fg)'(u) = f(u) \circ g'(u) + f'(u) \circ g(u) .$$

Proceeding exactly as in the scalar case, an induction argument gives us our version of Liebnitz' formula: the function fg is n times continuously differentiable and

$$(fg)^{(n)}(u) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} f^{(r)}(u) \circ g^{(n-r)}(u) \quad \text{for} \quad u \in U .$$

2. The implicit function theorem.

THEOREM 2.1. *Suppose X, Y, and Z are complex Banach spaces, U and V are open sets in X and Y respectively, and $x \in U$, $y \in V$. Suppose f: $U \to V$ and k: $V \to Z$ are holomorphic maps with $f(x) = y$, $k \circ f$ constant and range $f'(x) = \ker k'(y) \neq \{0\}$. Let D be a domain in C^n, $z \in D$, and g: $D \to Y$ be a holomorphic map with $g(z) = y$ and $k \circ g$ constant. Then there is an open neighborhood W of z and a holomorphic map h: $W \to X$ such that $h(z) = x$ and $g|_W = f \circ h$.*

Proof. We shall assume for simplicity that x, y, and z are all 0. By shrinking D if necessary, we may assume that g has a power series representation

$$g(z) = \sum_{|I|=0}^{\infty} \frac{g^{(I)}(0)}{I!} z^I \quad \text{for} \quad z \in D ,$$

where I denotes the multiindex $(i_1, \cdots, i_n)$, $z^I = z_1^{i_1} \cdots z_n^{i_n}$, $I! = i_1! i_2! \cdots i_n!$, and

$$g^{(I)}(0) = \frac{\partial^{i_1}}{\partial z_1^{i_1}} \frac{\partial^{i_2}}{\partial z_2^{i_2}} \cdots \frac{\partial^{i_n}}{\partial z_n^{i_n}} g(0) .$$

We shall suppose first that such an h exists; then $f \circ h$ is a holomorphic map of D into Y. Let I be a nonzero multiindex, and assume without loss of generality that $i_1 > 0$. If $I' = (i_1 - 1, i_2, \cdots, i_n)$, then by the chain rule applied to the function $z_1 \to f \circ h(z_1, 0, \cdots, 0)$ we have

$$g^{(I)}(0) = (f \circ h)^{(I)}(0) = \left((f' \circ h) \frac{\partial h}{\partial z_1} \right)^{(I')}(0) .$$

Now $f' \circ h$ is a holomorphic map of D into $L(X, Y)$ and we can regard $\partial h / \partial z_1$ as a holomorphic map of D into $L(C, X) \cong X$, so our Liebnitz formula applies; we obtain

$$g^{(I)}(0) = \sum_{r=0}^{i_1 - 1} \begin{bmatrix} i_1 - 1 \\ r \end{bmatrix} \left[\frac{\partial^r}{\partial z_1^r} (f' \circ h) \frac{\partial^{i_1 - r}}{\partial z_1^{i_1 - r}} h \right]^{(I-(i_1, 0, \cdots, 0))}(0) .$$

By successively applying the Liebnitz formula to the different variables, we obtain

$$g^{(I)}(0) = \sum_{J \leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} (f' \circ h)^{(J)}(0) \circ h^{(I-J)}(0)$$

where

$$\begin{bmatrix} I' \\ J \end{bmatrix} = \begin{bmatrix} i_1 - 1 \\ j_1 \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} \cdots \begin{bmatrix} i_n \\ j_n \end{bmatrix} .$$

Hence if such an h exists, for all multiindices I its derivatives satisfy

$$(1) \qquad (f' \circ h)(0) h^{(I)}(0) = g^{(I)}(0) - \sum_{0 < J \leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] .$$

We observe that by repeating this process on the term $(f' \circ h)^{(J)}(0)$, we find that each $(f' \circ h)^{(J)}(0) h^{(I-J)}(0)$ can be written as a linear combination of points of Y of the form

$$(f^{(j)} \circ h)(0)[h^{(L_1)}(0), \cdots, h^{(L_j)}(0)]$$

for some $j \geqq 2$ and multiindices $L_1, \cdots, L_j$ with $L_i > 0$ for all i and $\sum_{i=1}^{j} L_i = I$.

We first define $h(0) = 0$. Then $(f' \circ h)(0) = f'(0): X \to Y$, and range $f'(0) = \ker k'(0)$, a closed linear subspace of Y. Then by the open mapping theorem there is a constant C such that for each $y \in$ range $f'(0)$ there exists $x \in X$ with $f'(0)x = y$ and $\|x\| \leqq C\|y\|$. We shall assume that $C\|f'(0)\| \geqq 1$. We shall define $h^{(I)}(0)$ inductively so that (1) holds and

$$(2) \qquad \|h^{(I)}(0)\| \leqq C \left\| g^{(I)}(0) - \sum_{0 < J \leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] \right\|$$

where by $(f' \circ h)^{(I)}(0) h^{(I-J)}(0)$ we mean the linear combination described above. We observe that for $|I| = 1$, $(k \circ g)^{(I)}(0) = k'(0) g^{(I)}(0)$, and since $k \circ g$ is constant we have $g^{(I)}(0) \in \ker k'(0) = \text{range} f'(0)$, so that we can choose $h^{(I)}(0)$ as required. Suppose now that for all J with $|J| < |I|$ the right hand side of (1) is in the range of $f'(0)$ and we have chosen $h^{(J)}(0)$ satisfying (1) and (2). For notational convenience we shall regard h as the polynomial

$$h(z) = \sum_{0 \leqq J < I} \frac{h^{(J)}(0)}{J!} z^J \quad \text{for} \quad z \in D ,$$

so that for $J < I$ the terms $(f' \circ h)^{(J)}(0)$, $(k' \circ f \circ h)^{(J)}(0)$ and so on all make sense, and all such terms agree with those given by expanding and using (1). To show that we can define $h^{(I)}(0)$ as required it is enough to show that

$$(3) \qquad k'(0)\left[g^{(I)}(0) - \sum_{0<J\leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)]\right] = 0 .$$

Since $k \circ g = 0$, $(k \circ g)^{(I)}(0) = 0$, and so as before

$$k'(0)g^{(I)}(0) = - \sum_{0<K\leqq I'} \begin{bmatrix} I' \\ K \end{bmatrix} (k' \circ g)^{(K)}(0)[g^{(I-K)}(0)] .$$

By the inductive hypothesis

$$g^{(I-K)}(0) = \sum_{0\leqq L\leqq (I'-K)} \begin{bmatrix} I' - K \\ L \end{bmatrix} (f' \circ h)^{(L)}(0)[h^{(I-K-L)}(0)]$$

for all $K \leqq I'$. Hence

$$(4) \qquad k'(0)g^{(I)}(0) = - \sum_{0<K\leqq I'} \sum_{0\leqq L\leqq (I'-K)} \begin{bmatrix} I' \\ K \end{bmatrix} \begin{bmatrix} I' - K \\ L \end{bmatrix} \times (k' \circ f \circ h)^{(K)}(0) \circ (f' \circ h)^{(L)}(0)[h^{(I-K-L)}(0)]$$

since all derivatives of $f \circ h$ of less than Ith order are those of g. Now $(k \circ f)' = 0$, and so

$$0 = (k \circ f)' \circ h = (k' \circ f \circ h)(f' \circ h) .$$

Thus for every $J < I$

$$\begin{aligned} 0 &= ((k' \circ f \circ h)(f' \circ h))^{(J)}(0) \\ &= \sum_{0<M\leqq J} \begin{bmatrix} J \\ M \end{bmatrix} (k' \circ f \circ h)^{(M)}(0)(f' \circ h)^{(J-M)}(0) , \end{aligned}$$

and so as elements of $L(X, Z)$,

$$k'(0)(f' \circ h)^{(J)}(0) = - \sum_{0<M\leqq J} \begin{bmatrix} J \\ M \end{bmatrix} (k' \circ f \circ h)^{(M)}(0)/(f' \circ h)^{(J-M)}(0) .$$

Thus

$$\begin{aligned} &- \sum_{0<J\leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} k'(0) \circ (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] \\ &= \sum_{0<J\leqq I'} \sum_{0\leqq M\leqq J} \begin{bmatrix} I' \\ J \end{bmatrix} \begin{bmatrix} J \\ M \end{bmatrix} (k' \circ f \circ h)^{(M)}(0) \circ (f' \circ h)^{(J-M)}(0)[h^{(I-J)}(0)] \\ &= \sum_{0<M\leqq I'} \sum_{M\leqq J\leqq I'} \begin{bmatrix} I' \\ M \end{bmatrix} \begin{bmatrix} I' - M \\ J - M \end{bmatrix} (k' \circ f \circ h)^{(M)}(0) \circ (f' \circ h)^{(J-M)}(0)[h^{(I-J)}(0)] \\ &= - k'(0)g^{(I)}(0) \qquad \text{by (4).} \end{aligned}$$

Thus we have proved (3) and we can define $h^{(I)}(0)$ to satisfy (1) and (2).

Now define

$$h(z) = \sum_{|I|=0}^{\infty} \frac{h^{(I)}(0)}{I!} z^I \quad \text{for} \quad z \in \boldsymbol{C}^n . \tag{5}$$

If we can show that this series converges absolutely in some neighborhood of 0, then we shall be done. Now, by (2), if I is a multiindex

$$\| h^{(I)}(0) \| \leqq k \left\{ \| g^{(I)}(0) \| + \sum_{0<J\leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} \chi_J^{I'} \right\} ,$$

where $\chi_J^{I'}$ is a linear combination of terms of the form

$$\chi = \| f^{(j)}(0) \| \, \| h^{(L_1)}(0) \| \cdots \| h^{(L_j)}(0) \| ,$$

for some $j \geqq 2$, $L_1, \cdots, L_j > 0$ with $\sum_{i=1}^{j} L_i = I$. Define $F(0) = \| f(0) \| = 0$, $F^{(1)}(0) = \| f'(0) \|$, and $F^{(n)}(0) = - \| f^{(n)}(0) \|$ for $n > 1$. Since f: $U \to Y$ is holomorphic, we have that $\limsup (\| f^{(n)}(0) \| / n!)^{1/n}$ is finite, and so there is an open neighborhood V of 0 such that

$$F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} t^n \qquad \text{for } t \in V ,$$

defines a holomorphic function of V into $\boldsymbol{C}$. Similarly we define a holomorphic map G of D into $\boldsymbol{C}$ by $G^{(I)}(0) = \| g^{(I)}(0) \|$ for all multiindices I and

$$G(z) = \sum_{|I|=0}^{\infty} \frac{G^{(I)}(0)}{I!} z^I \qquad \text{for } z \in D .$$

Since $F'(0) = \| f'(0) \| \neq 0$, by the inverse function theorem for one variable there is a neighborhood W of 0 in D, and a holomorphic map H of W into $\boldsymbol{C}$ with $F \circ H = G|_W$ and $H(0) = 0$. By differentiating $F \circ H$ using the chain rule and Liebnitz' formula, we obtain

$$F'(0) H^{(I)}(0) = G^{(I)}(0) - \sum_{0<J\leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} (F' \circ H)^{(J)}(0) H^{(I-J)}(0) .$$

Again, we expand each $(F' \circ H)^{(J)}(0)$ in the same way and obtain

$$F'(0) H^{(I)}(0) = G^{(I)}(0) + \sum_{0<J\leqq I'} \begin{bmatrix} I' \\ J \end{bmatrix} \xi_J^{I'} ,$$

where $\xi_J^{I'}$ is identical to $\chi_J^{I'}$ with each $\| h^{(L_i)}(0) \|$ replaced by $H^{(L_i)}(0)$. We shall now prove that there is a constant M such that for all multiindices I

$$\| h^{(I)}(0) \| \leqq M^{2|I|-1} H^{(I)}(0) . \tag{6}$$

In fact, take $M = C\,||f'(0)||$ which was chosen to be $\geqq 1$. The inequality is trivially true for $I = 0$. Suppose (6) holds for all J with $|J| < |I|$. Then for each term χ of $\chi_J^{I'}$

$$\begin{aligned}\chi &\leqq ||f^{(j)}(0)||\, M^{\sum_{i=1}^{j}(2|L_j|-1)} H^{(L_1)}(0) \cdots H^{(L_j)}(0) \\ &\leqq M^{2|I|-2}\, ||f^{(j)}(0)||\, H^{(L_1)}(0) \cdots H^{(L_j)}(0)\,,\end{aligned} \tag{7}$$

since $j \geqq 2$ and $\sum L_i = I$. The right hand side of (7) is $M^{2|I|-2}$ times the term of $\xi_J^{I'}$ corresponding to χ, and so we have

$$\begin{aligned}||h^{(I)}(0)|| &\leqq k\left\{||g^{(I)}(0)|| + \sum_{0<J\leqq I'}\begin{bmatrix} I' \\ J \end{bmatrix} M^{2|I|-2}\xi_J^{I'}\right\} \\ &\leqq kM^{2|I|-2}F'(0)H^{(I)}(0) \\ &= kM^{2|I|-2}\,||f'(0)||\, H^{(I)}(0)\end{aligned}$$

as required. Since H is holomorphic in a polydisc, from (6) it follows that the power series (5) converges in a polydisc about 0, and the proof is complete.

3. Sections of holomorphic fibre bundles.

We shall start this section with a couple of technical results which we shall need later. The first is an application of the mean value theorem [5, p. 103].

LEMMA 3.1. *Let X and Y be Banach spaces, let U be open in X, and let f: $U \to Y$ be continuously differentiable. Let K be a compact Hausdorff space and define $\tilde{f}$: $C(K, U) \to C(K, Y)$ by $(\tilde{f}\phi)(k) = f(\phi(k))$ for $\phi \in C(K, U)$ and $k \in K$. Then $\tilde{f}$ is continuously differentiable and for $\phi \in C(K, U)$*

$$(\tilde{f}'(\phi)\psi)(k) = [\tilde{f}'(\phi(k))]\psi(k) \quad \textit{for all} \quad \psi \in C(K, X),\ k \in K\,.$$

Let X and Y be Banach spaces, $T \in L(X, Y)$ and suppose T has closed range. Then by the open mapping theorem T: $X \to$ range T has a bounded inverse T^{-1}. Call $||T^{-1}||$ the inversion constant of T. Let K be a compact Hausdorff space, and let T: $K \to L(X, Y)$ be a continuous map. Then T induces a bounded linear map $\tilde{T}$: $C(K, X) \to C(K, Y)$, where

$$(\tilde{T}f)(k) = T(k)f(k) \quad \text{for} \quad f \in C(K, X),\ k \in K\,.$$

LEMMA 3.2. *Suppose that $T(k)$ has closed range for each $k \in K$ and suppose that the inversion constant of $T(k)$ is less than M for each $k \in K$. Then*

(1) *If $g \in C(K, Y)$ satisfies $g(k) \in$ range $T(k)$ for all $k \in K$, then for each $\varepsilon > 0$ there is $f \in C(K, X)$ with $||f|| \leqq M\,||g||$ and $||\tilde{T}f - g|| < \varepsilon$.*

(2) *$\tilde{T}$ has closed range and the inversion constant of $\tilde{T}$ is less than $2M$.*

Proof. Part (1) follows by a standard partition of unity argument. To prove part (2) it is enough to show that for each $g \in C(K, Y)$ with $g(k) \in \operatorname{range} T(k)$ for all $k \in K$, there is some $f \in C(K, X)$ with $\tilde{T}f = g$ and $\|f\| \leqq 2M\|g\|$. Let such a g be given. Then by (1) we can choose f_1 such that $\|f_1\| \leqq M\|g\|$ and $\|Tf_1 - g\| \leqq 1/2\|g\|$. Then $(g - \tilde{T}f_1)(k) \in \operatorname{range} T(k)$ for each $k \in K$, and so by (1) we can find $f_2 \in C(K, X)$ such that $\|f_2\| \leqq M\|g - \tilde{T}f_1\| \leqq M(1/2)\|g\|$ and $\|\tilde{T}f_2 + \tilde{T}f_1 - g\| \leqq 1/4\|g\|$. In this way we can find a sequence $\{f_n\} \subset C(K, X)$ such that $\|f_n\| \leqq M\|g\|/2^{n-1}$ and $\|\tilde{T}(\sum_{i=1}^{n} f_i) - g\| \leqq \|g\|/2^n$. Then $f = \sum_{i=1}^{\infty} f_i$ is the required function.

Let G be a Banach Lie group, and suppose that G is acting holomorphically on a Banach space X. Let $x_0 \in X$, write $\pi(g) = g \cdot x_0$ for $g \in G$, and set $F = \pi(G)$. We shall say F is a homogeneous space under the action of G if there is a Banach space Y and a holomorphic map k: $X \to Y$ satisfying

(1) $k(x) = y_0$ for all $x \in F$ and some $y_0 \in Y$;

(2) $\operatorname{range} \pi'(1) = \ker k'(x_0)$;

(3) there is a neighborhood N of 1 in G such that $k'(g \cdot x_0)$ has closed range for $g \in N$ and inversion constant uniformly bounded over N;

(4) $H = \{g \in G : g \cdot x_0 = x_0\}$ is a Banach Lie group.

EXAMPLES. (1) Let A and B be Banach algebras with identity, and let Hom (A, B) be set of continuous homomorphisms of A into B. If $\phi \in \operatorname{Hom}(A, B)$ we set

$$F_\phi = \{\psi \in \operatorname{Hom}(A, B) : \exists b \in B^{-1} \text{ with } \psi(a) = b\phi(a)b^{-1} \text{ for } a \in A\} .$$

Denote by B the two sided Banach A-module consisting of B with the products

$$a \cdot b = \phi(a)b , \quad b \cdot a = b\phi(a) \quad \text{for} \quad a \in A,\ b \in B.$$

Then if the Hochschild cohomology groups $H^1(A, B_\phi)$ and $H^2(A, B_\phi)$ vanish (for the definitions, see [3]), F_ϕ is a homogeneous space under the action of B^{-1}. That conditions (1), (2), and (3) hold is checked in [9, § 3]; (4) follows from the observation that $\{b \in B^{-1} : b\phi(a)b^{-1} = \phi(a) \text{ for } a \in A\}$ is the set of invertible elements in $\phi(A)'$, the commutant of $\phi(A)$, which is a closed subalgebra of B.

(2) Let F_1 be the set of continuous algebra multiplications on A which give algebras isomorphic to A. Then if the Hochschild groups

$H^2(A, A)$ and $H^3(A, A)$ vanish, F_1 is a homogeneous space under the action of $L(A)^{-1}$ given by

$$\phi \cdot m(a, b) = \phi^{-1}(m(\phi(a)\phi(b))) \quad \text{for} \quad a \in A,\ b \in B,$$

where $\phi \in L(A)^{-1}$ and m is a multiplication on A. Again, (1), (2), and (3) are checked in [9, §4]; (4) follows since the isotropy group of the usual multiplication is the set of algebra automorphisms of A, which is a Banach Lie group with Lie algebra the set of bounded derivations of A.

THEOREM 3.3. *Let F be a homogeneous space under the action of a Banach Lie group G. Let M be a Stein manifold, N be a closed submanifold of M and suppose E is a holomorphic fibre bundle over M with fibre F and structure group G. Then*

(I) *If s: $M \to E$ is a continuous section such that $s|_N$ is holomorphic, then s is homotopic in the space of sections which extend $s|_N$ to a holomorphic section $\tilde{s}$: $M \to E$.*

(II) *If two holomorphic sections f_1 and f_0 of E over M are homotopic in the space of continuous sections, then they are homotopic in the space of holomorphic sections.*

Proof. Let s: $M \to E$ be a continuous section whose restriction to N is holomorphic, and let p: $E \to M$ denote the bundle projection. We shall show that there is an open cover $\{U_j\}_{j \in J}$ of M by holomorphically convex sets such that $E|_{U_j}$ is trivial for each j, and satisfying:

(*) Let Φ_j: $U_j \times F \to p^{-1}(U_j)$ be a trivialization of $E|_{U_j}$, and for $m \in U_j$ define $\Phi_{j,m}$: $F \to p^{-1}(m)$ by $\Phi_{j,m}(e) = \Phi_j(m, e)$ for $e \in F$. Then $p_j(e) = \Phi^{-1}_{j,p(e)}(e)$ for $e \in p^{-1}(U_j)$ defines a holomorphic map p_j of $p^{-1}(U_j)$ into F. There exist continuous maps s_j: $U_j \to G$ such that $\pi \circ s_j = p_j \circ s|_{u_j}$ for all j and such that $s_j|_{U_j \cap N}$ is holomorphic.

Let $m \in M$; it is enough to show that m has a neighborhood U satisfying (*). Choose a neighborhood V of m such that

(a) V is relatively compact;

(b) $E|_V$ is trivial via Φ: $V \times F \to p^{-1}(V)$;

(c) $V \cap N$ is a co-ordinate neighborhood in N.

Since G acts transitively on the fibre F, there is some $g \in G$ with $\pi(g) = \Phi_m^{-1}(s(m))$. Define a continuous map t: $V \to F$ by

$$t(m') = g^{-1} \cdot (\Phi_{m'}^{-1} s(m')) \quad \text{for} \quad m' \in V.$$

Then $t|_{V \cap N}$ is a holomorphic map of $V \cap N$ into $F \subset X$. By Theorem 2.1, there is a neighborhood $W \subset V$ of m in M and a holomorphic map f of $W \cap N$ into G such that $\pi \circ f = t|_{W \cap N}$.

Let $K = \bar{W}$. Then if G has Lie algebra $\mathfrak{G}$, $C(K, G)$ is a Banach Lie group with Lie algebra $C(K, \mathfrak{G})$. As in Lemma 3.1, the sequence $G \xrightarrow{\pi} X \xrightarrow{k} Y$ induces a sequence

$$(1) \qquad C(K, G) \xrightarrow{\tilde{\pi}} C(K, X) \xrightarrow{\tilde{k}} C(K, Y)$$

of holomorphic maps. Since $(\tilde{k} \circ \tilde{\pi})(g) = \underline{y}_0$ for $g \in C(K, G)$, where $\underline{y}_0$ denotes the constant function value y_0, the derivatives form a complex

$$(2) \qquad C(K, \mathfrak{G}) \xrightarrow{\tilde{\pi}'(1)} C(K, X) \xrightarrow{\tilde{k}'(\underline{x}_0)} C(K, Y) .$$

Now, since, near 1, $C(K, G)$ can be identified with $C(K, \mathfrak{G})$, we can apply Lemma 3.1 to deduce that

$$(\tilde{\pi}'(1)\psi)(k) = \pi'(1)\psi(k) \quad \text{for} \quad \psi \in C(K, \mathfrak{G}),\ k \in K$$

$$(\tilde{k}(\underline{x}_0)\alpha)(k) = k'(x_0)\alpha(k) \quad \text{for} \quad \alpha \in C(K, X),\ k \in K .$$

Now range $\pi'(1) = \ker k'(x_0)$, and so in particular range $\pi'(1)$ is closed. Thus (see, for example, [6]) there is a continuous map η: range $\pi'(1) \to X$ such that $\pi'(1) \circ \eta$ is the identity on range $\pi'(1)$. Now let $\alpha \in \ker \tilde{k}'(\underline{x}_0)$. Then $\alpha(k) \in \ker k'(x_0)$ for every k in K, and so $\eta \circ \alpha$ is a continuous map of K into X such that $\tilde{\pi}'(1)(\eta \circ \alpha) = \alpha$, proving that the complex (2) is exact. For $\alpha \in C(K, X)$ close to $\underline{x}_0$, Lemma 3.1 gives

$$(\tilde{k}'(\alpha)\beta)k = k'(\alpha(k))\beta(k) \quad \text{for} \quad \beta \in C(K, X),\ k \in K .$$

Thus, by Lemma 3.2, for α sufficiently close to $\underline{x}_0$, $k'(\alpha)$ has closed range and bounded inversion constant. Hence we can apply [9, Theorem 1] to the complex (1) and deduce that there is $\varepsilon > 0$ such that if $\psi \in C(K, X)$ satisfies $\tilde{k}(\psi) = \underline{y}_0$ and $\| \psi - \underline{x}_0 \| < \varepsilon$, ψ has a preimage in $C(K, G)$.

Now choose a neighborhood $W' \subset W$ of m such that $\| t(m') - t(m) \| < \varepsilon$ for $m' \in W'$, and choose a neighborhood U of m such that $\bar{U} \subset \operatorname{int} W'$ and U is holomorphically convex. Since K is a compact Hausdorff space, by Urysohn's lemma there is a continuous function ϕ: $K \to [0, 1]$ with $\phi = 0$ off $\bar{W}'$ and $\phi = 1$ on $\bar{U}$. Then $\phi t + (1 - \phi)\underline{x}_0$ is within ε of $\underline{x}_0$ on K, and so there is a continuous map $\tilde{t}$: $K \to G$ such that $\pi \circ \tilde{t}\,|_{\bar{U}} = t\,|_{\bar{U}}$. Now $\tilde{t}^{-1}f$ is a continuous map of $\bar{U} \cap N$ into H, and so by shrinking U if necessary, we can assume $\tilde{t}^{-1}f$ is a continuous map of $\bar{U} \cap N$ into a Banach space. Thus by Dugundji's extension theorem $\tilde{t}^{-1}f$ extends to a continuous map u of $U \cap N$ into H. Then $v = \tilde{t}u$ is a continuous map of U into G with $\pi \circ v = t$ and $v\,|_{U \cap N} = f$ holomorphic. The map $\tilde{s}$ defined by $\tilde{s}(m') = gv(m')$ for $m' \in U$ is the required lift of s.

We are now in the situation that Ramspott is in after the first paragraph of § 4 of [10]. We can use the rest of his proof, using Theorem 8.4 of [1] and Theorems A and B of [8, § 3] in place of the corresponding finite-dimensional theorems of Grauert. We note that the hypothesis—which has not been used so far—that the isotropy group of x_0 is a Banach Lie group is required to apply the lemma in [10, § 5].

Note. The results of Grauert, Ramspott, and Bungart apply to bundles over Stein spaces; since our basic technique involves lifting of power series it does not immediately apply in this more general setting.

REFERENCES

1. L. Bungart, *On analytic fibre bundles—I, holomorphic fibre bundles with infinite dimensional fibres*, Topology, **7** (1968), 55-68.

2. H. Grauert, *Analytische Faserungen uber holomorphvollstandigen Raumen*, Math. Ann., **135** (1958), 263-276.

3. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., **127** (1972).

4. S. Lang, *Differential Manifolds*, Addison-Wesley, Reading, Mass., 1972.

5. ———, *Analysis II*, Addison-Wesley, Reading, Mass., 1969.

6. E. Michael, *Continuous selections I*, Ann. Math., **63** (1956), 361-382.

7. L. Nachbin, *Topology on spaces of holomorphic mappings*, Springer-Verlag, Berlin, 1969.

8. I. Raeburn, *The relationship between a commutative Banach algebra and its maximal ideal space*, J. Functional Analysis, **25** (1977), 366-390.

9. I. Raeburn and J. L. Taylor, *Hochschild cohomology and perturbations of Banach algebras*, J. Functional Analysis, **25** (1977), 258-266.

10. K. J. Ramspott, *Stetige und holomorphe Schnitte in Bündeln mit homogener Faser*, Math. Zeit., **89** (1965), 234-246.

Received January 10, 1977.

UNIVERSITY OF NEW SOUTH WALES, KENSINGTON
NEW SOUTH WALES, AUSTRALIA 2033