PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

A GEOMETRIC INEQUALITY WITH APPLICATIONS TO LINEAR FORMS

JEFFREY D. VAALER

Let C_N be a cube of volume one centered at the origin in $\boldsymbol{R}^N$ and let P_K be a K-dimensional subspace of $\boldsymbol{R}^N$. We prove that $C_N \cap P_K$ has K-dimensional volume greater than or equal to one. As an application of this inequality we obtain a precise version of Minkowski's linear forms theorem. We also state a conjecture which would allow our method to be generalized.

1. Introduction. Let $C_N = [-1/2, 1/2]^N$ be the N-dimensional cube of volume one centered at the origin in $\boldsymbol{R}^N$ and suppose that P_K is a K-dimensional linear subspace of $\boldsymbol{R}^N$. Dr. Anton Good has conjectured that the K-dimensional volume of $P_K \cap C_N$ is always greater than or equal to one. In case $K = N - 1$ this has recently been proved by Hensley [6], who also obtained upper bounds for this volume. Our purpose in this paper is to prove the conjecture for arbitrary K and to give some applications to Minkowski's theorem on linear forms. In fact we prove a more general inequality for the product of spheres of various dimensions which contains the conjecture as a special case.

We write $\bar{x}$ for the column vector $\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ in $\boldsymbol{R}^n$ and

$$|\bar{x}| = \left(\sum_{j=1}^{n} (x_j)^2\right)^{1/2}$$

for its length. We define the sphere S_n by

$$S_n = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| \leqq \rho_n\}$$

where $\rho_n = \pi^{-1/2}\{\Gamma(n/2 + 1)\}^{1/n}$. It follows that $\mu_n(S_n) = 1$ where μ_n is Lebesgue measure on $\boldsymbol{R}^n$. Also we let $\chi_U(\bar{x})$ denote the characteristic function of a subset U in $\boldsymbol{R}^n$.

Our first main result is contained in the following theorem.

THEOREM 1. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers, $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$, and A is a real $N \times K$ matrix,* rank$(A) = K$. *Then*

$$|\det A^T A|^{-1/2} \leqq \int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x})\, d\mu_K(\bar{x}) \,, \tag{1.1}$$

where A^T is the transpose of A.

We note that if $\operatorname{rank}(A) < K$ then each side of (1.1) is infinite. From Theorem 1 we easily deduce a lower bound for $\mu_K(Q_N \cap P_K)$.

COROLLARY. *Let Q_N be as in Theorem 1 and let P_K be a K-dimensional subspace of $\boldsymbol{R}^N$. Then $\mu_K(Q_N \cap P_K) \geqq 1$.*

Proof. Choose A in Theorem 1 so that the columns of A form an orthonormal basis for P_K in $\boldsymbol{R}^N$. Then the left hand side of (1.1) is 1 while the right hand side is $\mu_K(Q_N \cap P_K)$.

The corollary clearly contains Good's conjecture since $Q_N = C_N$ if $n_j = 1$ and $J = N$.

Next we suppose that $L_j(\bar{x})$, $j = 1, 2, \cdots, N$ are N linear forms in K variables,

$$L_j(\bar{x}) = \sum_{k=1}^{K} a_{jk} x_k ,$$

so that $A = (a_{jk})$ is an $N \times K$ matrix. We assume that the forms L_j are real for $j = 1, 2, \cdots, r$ and that the remaining forms consist of s pairs of complex conjugate forms arranged so that $L_{r+2j-1} = \bar{L}_{r+2j}$ for $j = 1, 2, \cdots, s$. Thus $N = r + 2s$. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N$ be positive with $\varepsilon_{r+2j-1} = \varepsilon_{r+2j}$ for $j = 1, 2, \cdots, s$. We define the $N \times N$ diagonal matrix E by $E = (c_j \delta_{jk})$ where $c_j = \varepsilon_j^{-1}$ if $j = 1, 2, \cdots, r$, $c_j = (2/\pi)^{1/2} \varepsilon_j^{-1}$ if $j = r + 1, r + 2, \cdots, N$ and δ_{jk} is the Kronecker delta. Theorem 1 allows us to prove the following precise version of Minkowski's classical result on linear forms.

THEOREM 2. *Let M be a positive integer and suppose that*

$$M |\det A^* E^2 A|^{1/2} \leqq 1 , \tag{1.2}$$

where A^* is the complex conjugate transpose of the matrix A. Then there exist at least M distinct pairs of nonzero lattice points $\pm \bar{v}_m$, $m = 1, 2, \cdots, M$, such that

$$|L_j(\pm \bar{v}_m)| \leqq \varepsilon_j \tag{1.3}$$

for each j and each m. In particular if $|\det A^* A| > 0$ then there exists a pair of nonzero lattice points $\pm \bar{v}$ such that

$$|L_j(\pm \bar{v})| \leqq |\det A^* A|^{1/2K} \tag{1.4}$$

for $j = 1, 2, \cdots, r$, and

$$|L_j(\pm \bar{v}) \leqq \left(\frac{2}{\pi}\right)^{1/2} |\det A^* A|^{1/2K} \tag{1.5}$$

for $j = r + 1, r + 2, \cdots, N$.

Theorem 2 was first proved in the case $N \leqq K$ and $M = 1$ by Minkowski [8, p. 104]. Subsequently the extension of Minkowski's convex body theorem by van der Corput [5] allowed Theorem 2 to be proved for $N \leqq K$ and arbitrary M. Of course if $N = K$ then (1.2) becomes the more familiar condition

$$M\left(\frac{2}{\pi}\right)^s |\det A| \leqq \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N ,$$

and if $N < K$ then (1.2) is trivially satisfied since the left hand side is zero. The novelty in our result is that Theorem 2 now holds for $1 \leqq K < N$. Previously in the case $1 \leqq K < N$ we knew only that (1.3) held if

$$2^K M \leqq \mu_K(\{\bar{x} \in \boldsymbol{R}^K \colon |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) . \tag{1.6}$$

We prove Theorem 2 by showing that the right hand side of (1.6) is bounded from below by $2^K |\det A^* E^2 A|^{-1/2}$. As will be clear from the proof, Theorem 2 could be generalized to include linear forms with values in $\boldsymbol{R}^n$ for various n.

In §5 we state a conjecture which would allow us to obtain a significant improvement in Theorem 1. Specifically, we deduce from this conjecture an analogue of Theorem 1 in which Q_N is replaced by an arbitrary closed, convex, symmetric subset of $\boldsymbol{R}^N$ having N-dimensional volume equal to one.

The author wishes to thank Professors Patrick Brockett, Douglas Hensley, and Bruce Palka for several helpful discussions on the subject of this paper.

2. Preliminary results. In this section we briefly summerize some facts about logarithmically concave measures and functions. A more detailed discription can be found in the papers of Kanter [7] and Prékopa [9].

A function $f \colon \boldsymbol{R}^n \to [0, \infty)$ is said to be *log-concave* if for every pair of vectors $\bar{x}_1, \bar{x}_2$ in $\boldsymbol{R}^n$ and every λ, $0 < \lambda < 1$, we have

$$f(\lambda \bar{x}_1 + (1 - \lambda)\bar{x}_2) \geqq (f(\bar{x}_1))^\lambda (f(\bar{x}_2))^{1-\lambda} .$$

A probability measure ν defined on the measurable subsets of $\boldsymbol{R}^n$ is *log-concave* if for every pair of open convex sets U_1 and U_2 in $\boldsymbol{R}^n$ and every λ, $0 < \lambda < 1$, we have

$$\nu(\lambda U_1 + (1 - \lambda) U_2) \geqq (\nu(U_1))^\lambda (\nu(U_2))^{1-\lambda} , \tag{2.1}$$

where $+$ on the left hand side of (2.1) indicates Minkowski addition of sets. Clearly (2.1) holds for all open convex sets U_1 and U_2 if and only if it holds for all closed convex sets U_1 and U_2. The relationship between log-concave measures and log-concave functions is contained in the following lemma.

LEMMA 3. *Let ν be a log-concave probability measure on $\boldsymbol{R}^n$ and suppose that the support of ν spans the k-dimensional subspace P_k in $\boldsymbol{R}^n$. Then there is a log-concave probability density function f defined on P_k such that $d\nu = fd\mu_k$, where μ_k is k-dimensional Lebesgue measure on P_k. Conversely for any log-concave probability density function f defined on a k-dimensional subspace P_k in $\boldsymbol{R}^n$, the probability measure defined by $d\nu = fd\mu_k$ is log-concave, where μ_k is Lebesgue measure on P_k.*

The first part of Lemma 3 is a result of Borell [2, p. 123] while the converse was proved by Prékopa [9], (see also Kanter [7, Lemma 2.1]).

Let ν_1 and ν_2 be probability measures on $\boldsymbol{R}^n$. We say that ν_2 is *more peaked* than ν_1 if

$$\nu_1(U) \leqq \nu_2(U)$$

for all closed, convex, symmetric subsets U in $\boldsymbol{R}^n$. (We recall that $U \subseteqq \boldsymbol{R}^n$ is symmetric if $U = -U$.) If f_1 and f_2 are probability density functions on $\boldsymbol{R}^n$ we say that f_2 is *more peaked* than f_1 if the measure $f_2d\mu_n$ is more peaked than the measure $f_1d\mu_n$. The notion of peakedness was introduced by Birnbaum [1] and Sherman [10]. A complementary relation is that of symmetric dominance in the sense of Kanter [7]. If ν_3 and ν_4 are measures on $\boldsymbol{R}^n$ then ν_3 symmetrically dominates ν_4 if

$$\nu_3(\boldsymbol{R}^n \backslash U) \geqq \nu_4(\boldsymbol{R}^n \backslash U)$$

for all closed, convex, symmetric subsets U in $\boldsymbol{R}^n$. It is clear that if ν_3 and ν_4 are both probability measures then ν_3 symmetrically dominates ν_4 if and only if ν_4 is more peaked than ν_3. For our purposes it is more convenient to work with the relation of peakedness.

If ν_1 and ν_2 are log-concave probability measures on $\boldsymbol{R}^n$ then the convolution $\nu_1 * \nu_2$ is also log-concave on $\boldsymbol{R}^n$ (Kanter [7, Lemma 2.3]). It follows that if ν_1 and ν_2 are log-concave probability measures on $\boldsymbol{R}^{n_1}$ and $\boldsymbol{R}^{n_2}$ respectively then the product measure $\nu_1 \times \nu_2$ is log-concave on $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$. Forming product measures also preserves the peakedness relation.

LEMMA 4. *Suppose that ν_1, ν_2, ν_1' and ν_2' are all log-concave probability measures such that ν_1 is mome peaked than ν_1' on $\boldsymbol{R}^{n_1}$ and*

ν_2 is more peaked than ν_2' on $\boldsymbol{R}^{n_2}$. Then $\nu_1 \times \nu_2$ is more peaked than $\nu_1' \times \nu_2'$ on $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$.

For the proof of Lemma 4 we refer to Kanter [7, Corollary 3.2] where the result is obtained for the more general class of unimodal measures.

3. Proof of Theorem 1. We begin by proving the following lemma.

LEMMA 5. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers and $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$. Then $\chi_{Q_N}(\bar{x})$ is more peaked than the normal density function* $\exp\{-\pi|\bar{x}|^2\}$ *on $\boldsymbol{R}^N$.*

Proof. Since the measures $\chi_{Q_N}(\bar{x})d\mu_N(\bar{x})$ and $\exp\{-\pi|\bar{x}|^2\}d\mu_N(\bar{x})$ are both product measures which factor in $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \cdots \times \boldsymbol{R}^{n_J}$ it suffices to prove the peakedness relation in each factor space and then apply Lemma 4. Thus we need only show that for each positive integer n, $\chi_{S_n}(\bar{x})$ is more peaked than $\exp\{-\pi|\bar{x}|^2\}$ on $\boldsymbol{R}^n$. Of course it is trivial to verify that both of the density functions $\chi_{S_n}(\bar{x})$ and $\exp\{-\pi|\bar{x}|^2\}$ are log-concave on $\boldsymbol{R}^n$.

Let $\sum_{n-1} = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| = 1\}$ so that for each $\bar{x} \neq \bar{0}$ in $\boldsymbol{R}^n$ we have the unique polar decomposition $\bar{x} = r\bar{x}'$ where $r = |\bar{x}|$ and $\bar{x}' \in \sum_{n-1}$. *If* U is a closed, convex, symmetric subset of $\boldsymbol{R}^n$ then it follows that

$$\int_U \exp\{-\pi|\bar{x}|^2\}d\mu_n(\bar{x}) = \int_{\Sigma_{n-1}}\int_0^\infty \chi_U(r\bar{x}')\exp\{-\pi r^2\}r^{n-1}drd\bar{x}' \,, \tag{3.1}$$

where $d\bar{x}'$ is the induced Lebesgue measure on $\sum_{n-1}$. Now for each fixed $\bar{x}' \in \sum_{n-1}$ we have either

$$\chi_U(r\bar{x}') \leqq \chi_{S_n}(r\bar{x}') \,, \quad 0 \leqq r < \infty \tag{3.2}$$

or

$$\chi_{S_n}(r\bar{x}') \leqq \chi_U(r\bar{x}') \,, \quad 0 \leqq r < \infty \,, \tag{3.3}$$

since S_n and U are convex. If (3.2) holds at $\bar{x}'$ then

$$\begin{aligned}&\int_0^\infty \chi_U(r\bar{x}')\exp\{-\pi r^2\}r^{n-1}dr \\ &\qquad \leqq \int_0^\infty \chi_U(r\bar{x}')r^{n-1}dr = \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}')r^{n-1}dr \,.\end{aligned} \tag{3.4}$$

If (3.3) holds at $\bar{x}'$ then

$$
\begin{aligned}
&\int_0^\infty \chi_U(r\bar{x}') \exp\{-\pi r^2\} r^{n-1} dr \\
(3.5) \qquad &\leqq \int_0^\infty \exp\{-\pi r^2\} r^{n-1} dr = n^{-1}\pi^{-n/2}\Gamma\left(\frac{n}{2}+1\right) \\
&= \int_0^\infty \chi_{S_n}(r\bar{x}') r^{n-1} dr \\
&= \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}') r^{n-1} dr .
\end{aligned}
$$

Combining (3.1), (3.4) and (3.5) we obtain

$$\int_U \exp\{-\pi|\bar{x}|^2\} d\mu_n(\bar{x}) \leqq \int_{\Sigma_{n-1}} \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}') r^{n-1} dr d\bar{x}' = \int_U \chi_{S_n}(\bar{x}) d\mu_n(\bar{x}) .$$

Thus $\chi_{S_n}(\bar{x})$ is more peaked than $\exp\{-\pi|\bar{x}|^2\}$ on R^n and the lemma is proved.

We now prove Theorem 1. If $N = K$ then (1.1) is trivial so we may suppose that $K' = N - K$ is positive. Let P_K be the K-dimensional subspace of R^N spanned by the columns of A. Next let W be an $N \times N$ matrix whose first K columns are the columns of A and whose next K' columns are the columns of an $N \times K'$ matrix B. We choose the columns of B so that they form an orthonormal basis in $\boldsymbol{R}^N$ of the K'-dimensional subspace which is orthogonal to P_K. Identifying $\boldsymbol{R}^N$ with $\boldsymbol{R}^K \times \boldsymbol{R}^{K'}$ we may write each $\bar{z} \in \boldsymbol{R}^N$ as $\bar{z} = (\bar{x}/\bar{y})$ where $\bar{x} \in \boldsymbol{R}^K$ and $\bar{y} \in \boldsymbol{R}^{K'}$. For each ε, $0 < \varepsilon \leqq 1$ we define

$$H_\varepsilon = \left\{\bar{z} \in \boldsymbol{R}^N : z = \left(\frac{\bar{x}}{\bar{y}}\right), \max_{1 \leqq j \leqq K'} |y_j| \leqq \frac{\varepsilon}{2}\right\}$$

and

$$H'_\varepsilon = \left\{\bar{y} \in \boldsymbol{R}^{K'} : \max_{1 \leqq j \leqq K'} |y_j| \leqq \frac{\varepsilon}{2}\right\} .$$

Clearly H_ε is a closed, convex, symmetric subset of $\boldsymbol{R}^N$ and so is the image of H_ε under the nonsingular linear transformation determined by W. Thus by Lemma 5,

$$\int_{H_\varepsilon} \exp\{-\pi|W\bar{z}|^2\} d\mu_N(\bar{z}) \leqq \int_{H_\varepsilon} \chi_{Q_N}(W\bar{z}) d\mu_N(\bar{z}) . \tag{3.6}$$

Multiplying each side of (3.6) by $\{\mu_{K'}(H'_\varepsilon)\}^{-1} = \varepsilon^{-K'}$ and factoring H_ε into $\boldsymbol{R}^K \times H'_\varepsilon$ we find that

$$
\begin{aligned}
&\varepsilon^{-K'} \int_{\boldsymbol{R}^K} \int_{H'_\varepsilon} \exp\{-\pi|A\bar{x} + B\bar{y}|^2\} d\mu_{K'}(\bar{y}) d\mu_K(\bar{x}) \\
(3.7) \qquad &\leqq \varepsilon^{-K'} \int_{\boldsymbol{R}^K} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) d\mu_K(\bar{x}) .
\end{aligned}
$$

By the orthogonality condition $|A\bar{x} + B\bar{y}|^2 = |A\bar{x}|^2 + |B\bar{y}|^2$ and so as $\varepsilon \to 0+$ the left hand side of (3.7) clearly converges to

$$\int_{R^K} \exp\{-\pi |A\bar{x}|^2\} d\mu_K(\bar{x}) = |\det A^T A|^{-1/2} .$$

To evaluate the corresponding limit on the right hand side of (3.7) we observe that for $0 < \varepsilon \leqq 1$ and each $\bar{x} \in \boldsymbol{R}^K$,

$$\varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \leqq 1 .$$

Since Q_N and H'_ε are both bounded we have

$$\varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = 0$$

for sufficiently large $|\bar{x}|$ independent of ε. Thus by dominated convergence the limit on the right of (3.7) as $\varepsilon \to 0+$ is

$$\int_{R^K} \left\{ \lim_{\varepsilon \to 0+} \varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \right\} d\mu_K(\bar{x}) . \tag{3.8}$$

Clearly

$$\lim_{\varepsilon \to 0+} \varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = \chi_{Q_N}(A\bar{x})$$

except possibly when $A\bar{x}$ is a boundary point of $Q_N \cap P_K$, Since this boundary has K-dimensional measure zero we see that (3.8) is equal to

$$\int_{R^K} \chi_{Q_N}(A\bar{x}) d\mu_K(\bar{x}) .$$

We have now shown that as $\varepsilon \to 0+$ on each side of (3.7) we obtain (1.1) and this proves the theorem.

4. Proof of Theorem 2. By van der Corput's extension of Minkowski's convex body theorem [5] (see also Cassels [4, Chapter III, Theorem II]) the condition (1.6) implies that there exist at least M distinct pairs $\pm\bar{v}_m$, $m = 1, 2, \cdots, M$, of nonzero lattice points such that (1.3) holds. If $\operatorname{rank}(A) < K$ then (1.2) and (1.6) are both trivially satisfied. Thus to eatablish the first part of Theorem 2 it suffices to show that if $\operatorname{rank}(A) = K$ then

$$2^K |\det A^* E^2 A|^{-1/2} \leqq \mu_K(\{\bar{x} \in \boldsymbol{R}^K \colon |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) . \tag{4.1}$$

Let $G_j(\bar{x})$, $j = 1, 2, \cdots, N$ be linear forms defined by $G_j(\bar{x}) = L_j(\bar{x})$ for $j = 1, 2, \cdots, r$ and

$$G_{r+2j-1}(\bar{x}) = \sqrt{2}\,\mathrm{Re}\{L_{r+2j-1}(\bar{x})\}\ ,$$
$$G_{r+2j}(\bar{x}) = \sqrt{2}\,\mathrm{Im}\{L_{r+2j-1}(\bar{x})\}$$

for $j = 1, 2, \cdots, s$. We write $B = (b_{jk})$ for the corresponding real $N \times K$ matrix so that

$$G_j(\bar{x}) = \sum_{k=1}^{K} b_{jk}x_k\ .$$

Next we let $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_{r+s}}$ where $n_j = 1$ for $j = 1, 2, \cdots, r$ and $n_j = 2$ for $j = r+1, r+2, \cdots, r+s$. It follows that $|L_j(\bar{x})| \leqq \varepsilon_j$ if and only if $1/2\varepsilon_j^{-1}G_j(\bar{x}) \in S_{n_j}$, $j = 1, 2, \cdots, r$, and

$$|L_{r+2j-1}(\bar{x})| = |L_{r+2j}(\bar{x})| \leqq \varepsilon_{r+2j}$$

if and only if

$$(2\pi)^{-1/2}\varepsilon_{r+2j}^{-1}\begin{pmatrix} G_{r+2j-1}(\bar{x}) \\ G_{r+2j}(\bar{x}) \end{pmatrix} \in S_{n_{r+j}}\ ,$$

$j = 1, 2, \cdots, s$. Therefore

$$\begin{aligned} &\mu_K(\{\bar{x} \in \boldsymbol{R}^K: |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) \\ &\quad = \mu_K\Big(\Big\{\bar{x} \in \boldsymbol{R}^K: \frac{1}{2}EB\bar{x} \in Q_N\Big\}\Big) = \int_{\boldsymbol{R}^K} \chi_{Q_N}\Big(\frac{1}{2}EB\bar{x}\Big)d\mu_K(\bar{x}) \\ &\quad \geqq \Big|\det\Big(\frac{1}{2}EB\Big)^T\Big(\frac{1}{2}EB\Big)\Big|^{-1/2} = 2^K|\det B^TE^2B|^{-1/2}\ . \end{aligned}$$

An easy computation shows that $B^TE^2B = A^*E^2A$ and so completes the proof of (4.1).

To prove the second part of Theorem 2 we choose $\varepsilon_j = |\det A^*A|^{1/2K}$ for $j = 1, 2, \cdots, r$ and $\varepsilon_j = (2/\pi)^{1/2}|\det A^*A|^{1/2K}$ for $j = r+1, r+2, \cdots, N$. Then

$$|\det A^*E^2A| = 1$$

and so (1.4) and (1.5) follow from the first part of the theorem.

5. Lower bounds for arbitrary convex bodies. In this section we suppose that Q_N is a closed, convex, symmetric subset of $\boldsymbol{R}^N$ with $\mu_N(Q_N) = 1$. If A is an $N \times K$ matrix, $\mathrm{rank}(A) = K$, we will be interested in the problem of finding a lower bound for

$$\int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x})d\mu_K(\bar{x})\ . \tag{5.1}$$

The method used to deduce Theorem 1 from Lemma 5 will also lead to a lower bound in this more general situation, provided that we can find a suitable normal density function on $\boldsymbol{R}^N$ which is less peaked than $\chi_{Q_N}(\bar{x})$. We succeeded in proving Lemma 5 because the special structure imposed on Q_N allowed us to appeal to Lemma 4. We now describe an alternative method which leads to a conjectured lower bound for (5.1).

We write Q for Q_N and we assume that Q is a fixed, closed, convex, symmetric subset of $\boldsymbol{R}^N$, $\mu_N(Q) = 1$. For each positive integer m let

$$\chi_Q^{(m)}(\bar{x}) = \chi_Q^* \chi_Q^* \cdots \chi_Q(\bar{x})$$

be the m-fold convolution of χ_Q. We define the dilation operator D_λ for $\lambda > 0$ and for integrable real valued functions f on $\boldsymbol{R}^N$ by

$$D_\lambda(f)(\bar{x}) = \lambda^N f(\lambda\bar{x}) .$$

Next we define a sequence of positive numbers λ_m, $m = 1, 2, \cdots$ by

$$(\lambda_m)^N \chi_Q^{(m)}(\bar{0}) = 1 .$$

With this notation we have the following

CONJECTURE 6. *For each positive integer* m, $\chi_Q(\bar{x})$ *is more peaked than* $D_{\lambda_m}(\chi_Q^{(m)})(\bar{x})$.

Now let Ω be the $N \times N$ covariance matrix determined by a random vector which is uniformly distributed on the convex body Q. That is $\Omega = (\omega_{rs})$ is the $N \times N$ matrix defined by

$$\omega_{rs} = \int_{R^N} y_r y_s \chi_Q(\bar{y}) d\mu_N(\bar{y}) ,$$

where y_r and y_s are the rth and sth co-ordinate functions of $\bar{y}$, $r = 1, 2, \cdots, N$, and $s = 1, 2, \cdots, N$. It is clear that Ω is symmetric and nonsingular since Q has a nonempty interior. By the Central Limit Theorem (Breiman [3, Theorem 11.10]) we have

$$\lim_{m\to\infty} D_{\sqrt{m}}(\chi_Q^{(m)})(\bar{x}) = (2\pi)^{-N/2}(\det \Omega)^{-1/2} \exp\left\{-\frac{1}{2}\bar{x}^T \Omega^{-1}\bar{x}\right\}$$

uniformly for $x \in \boldsymbol{R}^N$. It follows that

$$\lim_{m\to\infty} \frac{\lambda_m}{\sqrt{m}} = (2\pi)^{1/2}(\det \Omega)^{1/2N}$$

and hence

$$\lim_{m\to\infty} D_{\lambda_m}(\chi_Q^{(m)})(\bar{x}) = \exp\{-\pi(\det \Omega)^{1/N}\bar{x}^T\Omega^{-1}\bar{x}\}$$

uniformly for $x \in \boldsymbol{R}^N$. If the Conjecture 6 is true then for each positive integer m and each closed, convex, symmetric subset U of $\boldsymbol{R}^N$

$$\int_U D_{\lambda_m}(\chi_Q^{(m)})(\bar{x})d\mu_N(\bar{x}) \leqq \int_U \chi_Q(\bar{x})d\mu_N(\bar{x}) \; . \tag{5.2}$$

Letting $m \rightarrow \infty$ on the left hand side of (5.2) and we have proved that $\chi_Q(\bar{x})$ is more peaked than $\exp\{-\pi(\det \Omega)^{1/N}\bar{x}^T\Omega^{-1}\bar{x}\}$ on $\boldsymbol{R}^N$. By the same method used to prove Theorem 1 we obtain

THEOREM 7. *Assume that the Conjecture* 6 *holds and let A be a real $N \times K$ matrix,* $\operatorname{rank}(A) = K$. *Then*

$$(\det \Omega)^{-K/2N}|\det A^T\Omega^{-1}A|^{-1/2} \leqq \int_{\boldsymbol{R}^K} \chi_Q(A\bar{x})d\mu_K(\bar{x}) \; . \tag{5.3}$$

If the set Q in Theorem 7 is such that Ω is a constant multiple of the identity matrix then the left hand side of (5.3) is simply $|\det A^TA|^{-1/2}$. Just as in our proof of the corollary to Theorem 1, we deduce that in this case $\mu_K(Q \cap P_K) \geqq 1$, where P_K is a K-dimensional subspace of $\boldsymbol{R}^N$. There is also an application of Theorem 7 to linear forms. If $L_j(\bar{x})$, $j = 1, 2, \cdots, N$, are N linear forms in K-variables we could determine precise conditions under which

$$\left(\sum_{j=1}^N |L_j(\bar{v})|^p\right)^{1/p} \leqq \varepsilon$$

at a nonzero lattice point $\bar{v}$ for any $p \geqq 1$ and $\varepsilon > 0$. At present, however, these results remain hypothetical since they depend on the open problem stated in Conjecture 6.

REFERENCES

1. Z. W. Birnbaum, *On random variables with comparable peakedness*, Ann. Math. Stat., **19** (1948), 76–81.
2. C. Borell, *Convex set functions in d-space*, Period. Math. Hungar., **6** (1975), 111–136.
3. L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., 1968.
4. J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Grundl. Math. Wiss. 99. Springer-Verlag, 1971, Berlin.
5. J. G. van der Corput, *Verallgemeinerung einer Mordellschen Beweiismethode in der Geometrie der Zahlen*, Acta Arith., **2** (1936), 145–146.
6. D. Hensley, *Slicing the cube in R^n and probability*, Proc. Amer. Math. Soc., (to appear).
7. M. Kanter, *Unimodality and Dominance for symmetric random vectors*, Trans. Amer. Math. Soc., **229** (1977), 65–85.
8. H. Minkowski, *Geometrie der Zahlen*, 1910, Leipzig.
9. A. Prekopa, *On logarithmically concave measures and functions*, Acta. Math. Acad. Hungar, (1973), 335–343.
10. S. Sherman, *A theorem on convex sets with applications*, Ann. Math. Stat., **26** (1955), 763–766.

Received November 21, 1978 and in revised form February 19, 1979. This research was supported by the National Science Foundation, grant number MCS77-01830.

THE UNIVERSITY OF TEXAS
AUSTIN, TX 78712