PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

# GOOD CHAINS WITH BAD CONTRACTIONS

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**Let $R \subset T$ be commutative rings with $T$ integral over $R$. In the study of chains of prime ideals, it is often of interest to know about primes $q \subset q'$ of $T$ such that height $(q'/q) <$ height $(q' \cap R/q \cap R)$. In this paper we will consider a chain of primes $q_1 \subset q_2 \subset \cdots \subset q_m$ in $T$ which is well behaved in that height $(q_m/q_1) = \sum_{i=2}^{m}$ height $(q_i/q_{i-1})$, but which suffers the pathology that height $(q_i \cap R/q_{i-1} \cap R) >$ height $(q_i/q_{i-1})$ for each $i = 2, \cdots, m$. Our goal is to find a bound on how large $m$ can be.**

**Our main result is that if $T$ is generated as an $R$-module by $n$ elements, then there is a bound $b_n$ such that $m \leqq b_n$; moreover $b_2 = 2$ and in general $b_n \leqq b_{n-1}^{n-2} + b_{n-1}^{n-3} + \cdots + b_{n-1} + 2$. Let us quickly add that we do not claim that this formula gives the best bound possible. (We rather suspect not.) If $c = b_{n-1} + 2$, we also have, as part of our main result, that $m \leqq$ height $(q_c/q_1) + b_{n-1}$. (If $m > b_{n-1}$, so that $q_c$ exists.) Finally, if we have the added assumption that height $(q_i/q_{i-1}) \leqq r$ for $i = 2, \cdots, m$, then $m \leqq 2(r+1)^{n-2}$.**

The bulk of our effort is needed to discuss the case that $T = R[u]$ is a simple integral extension of $R$. This is done in § 3. That section also introduces a new "going down" technique of some interest. Section 2 treats a highly special situation in which we obtain a much sharper bound. This case has some interest in its own right and also starts an induction needed in § 3. The fourth section gives the main result mentioned above. Lastly, in § 5, we present some examples. These illustrate the point that there is no bound in general, even in the case of Noetherian domains, on $m$ which is independent of the size of the integral extension $R \subset T$. Specifically, we show that $b_n \to \infty$ as $n \to \infty$. Thus our bounds, while presumably not sharp, have the proper form.

DEFINITION. The chain of primes $P_1 \subset P_2 \subset \cdots \subset P_m$ is *taut* if height $(P_m/P_1) = \sum_{i=2}^{m}$ height $(P_i/P_{i-1})$.

NOTATION. The following notation will be standard throughout except when specifically indicated otherwise. $R \subset T$ will be an integral extension of domains, $q_1 \subset \cdots \subset q_m$ will be a taut chain of primes in $T$ lying over $p_1 \subset \cdots \subset p_m$ in $R$. Height$(p_m/p_1)$ will be finite and height$(p_i/p_{i-1}) >$ height$(q_i/q_{i-1})$, $i = 2, \cdots, m$. Finally, $x$ will be an indeterminate.

**2. Split simple extensions.** In this section, as well as the next, we will assume, in addition to the standard assumptions mentioned in the introduction, that $T$ is a simple integral extension of $R$. In order to be more specific, we make a definition.

DEFINITION. Let the domain $T = R[u]$ be a simple integral extension of $R$ with $u$ a root of a monic polynomial $f(x) \in R[x]$. We will say that *$T$ is a simple integral extension of $R$ via $f(x)$*. Throughout §§ 2 and 3, without further mention, we will assume that $T = R[u]$ is a simple integral extension of $R$ via $f(x)$ with $f(x)$ having degree $n$ and $f(u) = 0$. Furthermore, in the present section we add one more assumption, namely that $f(x)$ is split.

DEFINITION. The polynomial $f(x) \in R[x]$ is said to be *split* if $R[u] = R[u']$ for any two roots $u$ and $u'$ of $f(x)$.

Notice that if $f(x) = x^2 + ax + b = (x - u)(x - u') \in R[x]$, then $-u - u' = a \in R$ so that $R[u] = R[u']$. Thus if $n = 2$, $f(x)$ is split. We will show in this section that when $f(x)$ is split, $m$ is bounded by $\deg f(x)$. Our first lemma is well known. We state it explicitely because it is frequently used in what follows.

LEMMA 2.1. (a) *Let $p$ be prime in a ring $A$. Let $g(x)$ be a monic polynomial in $A[x]$ with $\deg g(x) = d$. Then there are at most $d$ primes of $A[x]$ which lie over $p$ and contain $g(x)$.*

(b) *Let $T = R[u]$ be a simple integral extension of $R$ via $f(x)$ with $\deg f(x) = n$. Let $p$ be prime in $R$. Then at most $n$ primes in $T$ lie over $p$.*

*Proof.* (a) follows from standard facts such as [3, §§ 1-5] and the fact that taken modulo $p$, $g(x)$ has at most $d$ irreducible factors. (b) follows from (a) by considering preimages under the map $R[x] \to R[u] = T$.

THEOREM 2.2. *Let $f(x)$ be split. Let $q$ be prime in $T$ with $p = q \cap R$. In $R[x]$, let $P$ be prime with $P \cap R = p$ and suppose that $f(x) \in P$. Then for some root $u$ of $f(x)$, $q$ is the image of $P$ under the homomorphism $R[x] \to R[u] = T$.*

*Proof.* As is well known, there is a $g(x) \in P$ such that $P = \{h(x) \in R[x]/sh(x) \in (p, g(x))R[x]$ for some $s \in R - p\}$. Since $R[x] \subset T[x]$ is integral and $qT[x] \cap R[x] = pR[x]$, by going up we can find a prime $Q$ of $T[x]$ with $Q \cap T = q$ and $Q \cap R[x] = P$. Thus $f(x) \in P \subset Q$ and as $f(x)$ splits in $T[x]$, for some root $u$ of $f(x)$ we have

$x - u \in Q$. Now $g(x) \in P \subset Q$ and as $x \equiv u \bmod Q$, $g(u) \in Q \cap T = q$. Thus the preimage of $q$ under the map $R[x] \to R[u] = T$ contains $g(x)$, and so is easily seen to be $P$.

COROLLARY 2.3. *Let $f(x)$ be split. Let $p$ be prime in $R$.*

(a) *If $P_1$ and $P_2$ are prime in $R[x]$ with $P_1 \cap R[x] = p = P_2 \cap R[x]$ and $f(x) \in P_1 \cap P_2$ then $R[x]/P_1 \approx R[x]/P_2$, this isomorphism fixing $R/p$.*

(b) *Let $q_1$ and $q_2$ be primes in $T$ both lying over $p$. Then $T/q_1 \approx T/q_2$, this isomorphism fixing $R/p$.*

*Proof.* (a) Let $q$ be a prime of $T$ lying over $p$. By Theorem 2.2, for roots $u_1$ and $u_2$ of $f(x)$, $q$ is the image of $P_i$ under $R[x] \to R[u_i] = T$, $i = 1, 2$. Thus $R[x]/P_1 \approx R[u_1]/q = R[u_2]/q \approx R[x]/P_2$.

(b) If $P$ is prime in $R[x]$ with $P \cap R = p$ and $f(x) \in P$, and if $q$ is any prime in $T$ lying over $p$, then the proof of (a) shows that $T/q \approx R]x]/P$. Thus $T/q_1 \approx R[x]/P \approx T/q_2$.

THEOREM 2.4. *Let $f(x)$ be split. Then $m \leqq \deg f(x)$.*

*Proof.* We first claim that there are distinct primes $Q_1, \cdots, Q_m$ lying over $p_m$ satisfying $q_1 \subset Q_j$ and $\operatorname{height}(Q_j/q_1) \geqq \operatorname{height}(q_m/q_1)$, $j = 1, \cdots, m$. To do this, we induct on $m$. For $m = 2$, by going up there is a prime $q_2'$ of $T$ with $q_1 \subset q_2'$, $q_2' \cap R = p_2$ and $\operatorname{height}(q_2'/q_1) = \operatorname{height}(p_2/p_1) > \operatorname{height}(q_2/q_1)$. Let $Q_1 = q_2$ and $Q_2 = q_2'$.

For $m > 2$ take $q_2'$ as above. The isomorphism in Corollary 2.3 between $T/q_2$ and $T/q_2'$ carries $q_2 \subset \cdots \subset q_m$ isomorphically to a chain $q_2' \subset \cdots \subset q_m'$ which also lies over $p_2 \subset \cdots \subset p_m$ (since $R/p_2$ is fixed). By induction there are distinct primes $Q_1, \cdots, Q_{m-1}$ of $T$ lying over $p_m$ with $q_2' \subset Q_j$ and $\operatorname{height}(Q_j/q_2') \geqq \operatorname{height}(q_m'/q_2')$, $j = 1, \cdots, m - 1$. Since $q_2 \subset \cdots \subset q_m$ and $q_2' \subset \cdots \subset q_m'$ are "isomorphic", $\operatorname{height}(q_m'/q_2') = \operatorname{height}(q_m/q_2)$. Recall also $\operatorname{height}(q_2'/q_1) > \operatorname{height}(q_2/q_1)$. By the tautness of $q_1 \subset \cdots \subset q_m$ we have for $j = 1, \cdots, m - 1$, $\operatorname{height}(Q_j/q_1) \geqq \operatorname{height}(Q_j/q_2') + \operatorname{height}(q_2'/q_1) \geqq \operatorname{height}(q_m'/q_2') + \operatorname{height}(q_2'/q_1) > \operatorname{height}(q_m/q_2) + \operatorname{height}(q_2/q_1) = \operatorname{height}(q_m/q_1)$. That is, $\operatorname{height}(Q_j/q_1) > \operatorname{height}(q_m/q_1)$, for $j = 1, \cdots, m - 1$. Letting $Q_m = q_m$ proves our claim.

Finally, as the number of primes in $T$ contracting to any given prime in $R$ cannot exceed $\deg f(x)$, the existence of $Q_1, \cdots, Q_m$ shows that $m \leqq \deg f(x)$.

The final result in this section discusses the situation when the bound given by Theorem 2.4 is obtained.

PROPOSITION 2.5. *Let $f(x)$ be split and let $m = \deg f(x)$. Suppose that $p \subseteqq p_1 \subset p_m \subseteqq p'$ with $p, p'$ primes in $R$ and that $q \cap R =$*

*$p$, $q' \cap R = p'$ with $q$, $q'$ primes in $T$. Then $q \subset q'$.*

*Proof.* The proof of Theorem 2.4 shows that there are primes $Q_1, \cdots, Q_m = q_m$ lying over $p_m$, each of which contains $q_1$. By going up, find a prime $q_1'$ of $T$ with $q \subset q_1'$ and $q_1' \cap R = p_1$. Now $q_1$ is contained in $m$ primes lying over $p_m$ (namely $Q_1, \cdots, Q_m$) and so by Corollary 2.3 $q_1'$ is also contained in $m$ primes lying over $p_m$. However, since $\deg f(x) = m$, $Q_1, \cdots, Q_m$ are the only primes lying over $p_m$ and so $q \subset q_1' \subset Q_1 \cap \cdots \cap Q_m$.

Now consider $R[x] \to R[u] = T$ and let $Q^*, Q_1^*, \cdots, Q_m^*$ be the preimages of $q', Q_1, \cdots, Q_m$ respectively. Obviously $Q^* \cap R = p'$, $Q_j^* \cap R = p_m$, $j = 1, \cdots, m$ and $f(x) \in Q^* \cap Q_1^* \cap \cdots \cap Q_m^*$ since $f(u) = 0$. By [4, Lemma 3] (applied to $R/p_m$) we easily see that there is a prime $P$ of $R[x]$ with $P \cap R = p_m$, and $f(x) \in P \subset Q^*$. However since $\deg f(x) = m$, at most $m$ primes in $R[x]$ can contain $f(x)$ and also contract to $P_m$. As each of $Q_1^*, \cdots, Q_m^*$ do just that, obviously $P = Q_j^*$ for some $j = 1, \cdots, m$. Thus $Q_j^* = P \subset Q^*$ from which we see that $Q_j \subset q'$. Thus $q \subset Q_1 \cap \cdots \cap Q_m \subset Q_j \subset q'$ and we are done.

3. **Arbitrary simple extensions.** We now drop the "split" assumption and just assume that $T$ is a simple integral extension of $R$ via $f(x)$ with $\deg f(x) = n$. We will show that there is a number $b_n$ such that $m \leqq b_n$. We do not identify the best such bound although we do give an inequality limiting the size of the best such bound. To be explicit, let us use $b_n$ to denote the smallest number such that $m \leqq b_n$ for all such $m$.

We have already seen at the start of §2 that if $n = 2$ then $f(x)$ is split, and so by Theorem 2.4 we have $b_2 = 2$. (This is best possible, [5, Example 2, pp. 203-205].) We will now assume inductively that $b_{n-1}$ exists.

In our next lemma we start a chain at $P_2$ rather than $P_1$, since that will be the situation when we apply the lemma.

LEMMA 3.1. *Let $P_2 \subset \cdots \subset P_m$ be a taut chain of primes in $R[x]$ contracting to $p_2 \subset \cdots \subset p_m$ in $R$. Let $P_2' \neq P_2$ with $P_2' \cap R = p_2$. Let $f(x)$ be a monic polynomial of degree $n$ with $f(x) \in P_2 \cap P_2'$. Let $s > 0$ be an integer with $m > b_{n-1}(s-1)+1$. Then for some $i \in \{1, \cdots, m-s\}$ there is a taut chain $P'_{i+1} \subset \cdots \subset P'_{i+s}$ in $R[x]$ lying over $p_{i+1} \subset \cdots \subset p_{i+s}$ with $height(P'_{i+j}/P'_{i+j-1}) = height(P_{i+j}/P_{i+j-1})$, $j = 2, \cdots, s$ and with $P_2' \subseteqq P'_{i+1}$ and $height(P'_{i+1}/P_2') \geqq height(P_{i+1}/P_2)$.*

*Proof.* Obviously we may work modulo $p_2$; so assume that $p_2 = 0$. Since $f(x) \in P_2 \cap P_2'$, $R[x]/P_2$ and $R[x]/P_2'$ are simple integral extensions of $R$ via $f(x)$. Let $R[x]/P_2 \approx R[u]$ and $R[x]/P_2' \approx R[u']$

with $u$ and $u'$ distinct roots of $f(x)$ (distinct since $P_2 \neq P_2'$). Taken modulo $P_2$, $P_2 \subset \cdots \subset P_m$ becomes a taut chain $0 = q_2 \subset \cdots \subset q_m$ in $R[u]$ lying over $0 = p_2 \subset \cdots \subset p_m$. As $R[u] \subset R[u, u']$ is integral, we lift $0 = q_2 \subset \cdots \subset q_m$ to a taut chain $0 = q_2^* \subset \cdots \subset q_m^*$ in $R[u, u']$, with height $q_m^* =$ height $q_m$.

Since $f(u') = 0$, $f(x) = (x - u')g(x)$ with $g(x)$ monic in $R[u'][x]$. As $u \neq u'$, we have $g(u) = 0$ so that $R[u, u']$ is a simple integral extension of $R[u']$ via $g(x)$. Since $\deg g(x) = n - 1$, the induction assumption concerning the existence of $b_{n-1}$ applies to $R[u'] \subset R[u, u']$.

Let $b = b_{n-1}$ and consider a subchain of $q_2^* \subset \cdots \subset q_m^*$, namely $q_2^* \subset q_{2+(s-1)}^* \subset q_{2+2(s-1)}^* \subset \cdots \subset q_{2+b(s-1)}^*$, which, being a subchain of a taut chain, is taut. (Note $q_{2+b(s-1)}^*$ exists since $m > b(s-1) + 1$.) Because this taut (sub)-chain contains $b + 1$ primes, by the induction assumption for some $l = 1, \cdots, b$ we must have $\text{height}(q_{2+l(s-1)}^* \cap R[u']/q_{2+(l-1)(s-1)}^* \cap R[u']) = \text{height}(q_{2+l(s-1)}^*/q_{2+(l+1)(s-1)}^*)$. Thus letting $i = 1 + (l-1)(s-1)$ we see that the tautness of $q_{i+1}^* \subset \cdots \subset q_{i+s}^*$ implies that $q_{i+1}^* \cap R[u'] \subset \cdots \subset q_{i+s}^* \cap R]u']$ is taut, and that $\text{height}(q_{i+j}^* \cap R[u']/q_{i+j-1}^* \cap R[u']) = \text{height}(q_{i+j}^*/q_{i+j-1}^*)$ which in turn equals height $(q_{i+j}/q_{i+j-1})$ $j = 2, \cdots, s$ by the manner in which $q_2^* \subset \cdots \subset q_n^*$ was constructed. Also $\text{height}(q_{i+1}^* \cap R[u']) \geqq$ height $q_{i+1}$ since height $q_{i+1} =$ height $q_{i+1}^*$.

Finally, recalling that $R[u'] \approx R[x]/P_2'$, the chain $q_{i+1}^* \cap R[u'] \subset \cdots \subset q_{i+s}^* \cap R[u']$ gives rise to a chain $P_{1+1}' \subset \cdots \subset P_{i+s}'$ in $R[x]$ with $P_2' \subseteqq P_{i+1}'$. That this chain satisfies the lemma follows easily from what we know about $q_{i+1}^* \cap R[u'] \subset \cdots \subset q_{i+s}^* \cap R[u']$.

COROLLARY 3.2. *Let the domain $T$ be a simple integral extension of $R$ via $f(x)$ with $\deg f(x) = n$. Let $q_2 \subset \cdots \subset q_m$ be a taut chain in $T$ lying over $p_2 \subset \cdots \subset p_m$ in $R$. Let $q_2' \neq q_2$ be prime in $T$ with $q_2' \cap R = p_2$. Let $s > 0$ be an integer with $m > b_{n-1}(s-1) + 1$. Then for some $i \in \{1, \cdots, m-s\}$, there is a taut chain $q_{i+1}' \subset \cdots \subset q_{i+s}'$ in $T$ lying over $p_{i+1} \subset \cdots \subset p_{i+s}$ with $\text{height}(q_{i+j}'/q_{i+j-1}') = \text{height}$ $(q_{i+j}/q_{i+j-1})$, $j = 2, \cdots, s$, and with $q_2' \subseteqq q_{i+1}'$ and $\text{height}(q_{i+1}'/q_2') \geqq \text{height}$ $(q_{i+1}/q_2)$.*

*Proof.* Let $P_2 \subset \cdots \subset P_m$ and $P_2'$ be, respectively, the preimages of $q_2 \subset \cdots \subset q_m$ and $q_2'$ under $R[x] \to R[u] = T$. Then, since $f(x) \in P_2 \cap P_2'$, the hypothesis of Lemma 3.1 is satisfied. We complete the proof by letting $q_{i+1}' \subset \cdots \subset q_{i+s}'$ be the images of $P_{i+1}' \subset \cdots \subset P_{i+s}'$ given by Lemma 3.1.

PROPOSITION 3.3. *Let $b = b_{n-1}$. Let $l \geqq 0$ be an integer and let $m \geqq b^l + b^{l-1} + \cdots + b + 2$. Then for some $r = 1, \cdots, m$, $p_r$ has lying over it distinct primes $Q_1, \cdots, Q_{l+1}$ in $T$ such that $q_1 \subset Q_1 \cap \cdots \cap Q_{l+1}$*

*and* $height(Q_j/q_1) > height(q_r/q_1)$ *for* $j = 1, \cdots, l+1$.

*Proof.* We induct on $l$. First, since $\text{height}(p_2/p_1) > \text{height}(q_2/q_1)$, by going up there is a prime $q'_2$ of $T$ with $q_1 \subset q'_2$ and $\text{height}(q'_2/q_1) = \text{height}(p_2/p_1)$. If $l = 0$ then $r = 2$ and $Q_1 = q'_2$ satisfy the proposition.

For $l > 0$, we apply Corollary 3.2 with $s = b^{l-1} + b^{l-2} + \cdots + b + 2$. Since $m > b(s-1)+1$ we have for some $i \in \{1, \cdots, m-s\}$ a taut chain $q'_{i+1} \subset \cdots \subset q'_{i+s}$ in $T$ lying over $p_{i+1} \subset \cdots \subset p_{i+s}$ with height $(q'_{i+j}/q'_{i+j-1}) = \text{height}(q_{i+j}/q_{i+j-1})$ which is less than $\text{height}(p_{i+j}/p_{i+j-1})$ for $j = 2, \cdots, s$.

We apply the case $l-1$ of the induction assumption to the chain $q'_{i+1} \subset \cdots \subset q'_{i+s}$ (recalling that $s = b^{l-1} + b^{l-2} + \cdots + b + 2$), to produce an $r \in \{i+1, \cdots, i+s\}$ and distinct primes $Q_1, \cdots, Q_l$ of $T$ lying over $p_r$, with $q'_{i+1} \subset Q_1 \cap \cdots \cap Q_l$ and $\text{height}(Q_j/q'_{i+1}) > \text{height}(q'_r/q'_{i+1})$ for $j = 1, \cdots, l$. If we now let $Q_{l+1} = q'_r$, obviously $Q_{l+1}$ is distinct from $Q_1, \cdots, Q_l$ and we now have $q'_{i+1} \subset Q_1 \cap \cdots \cap Q_{l+1}$ and height $(Q_j/q'_{i+1}) \geqq \text{height}(q'_r/q'_{i+1})$ for $j = 1, \cdots, l+1$.

We have $q_1 \subseteqq q'_2 \subseteqq q'_{i+1}$ by Corollary 3.2. To complete the proof, we must only show that $\text{height}(Q_j/q_1) > \text{height}(q_r/q_1)$ for $j = 1, \cdots, l+1$. To do this, we collect various facts.

(i) $\text{height}(q'_r/q'_{i+1}) = \text{height}(q_r/q_{i+1})$. This follows from the fact that $\text{height}(q'_{i+j}/q'_{i+j-1}) = \text{height}(q_{i+j}/q_{i+j-1})$ $j = 2, \cdots, s$ by Corollary 3.2 and the tautness of $q_{i+1} \subset \cdots \subset q_{i+s}$ and $q'_{i+1} \subset \cdots \subset q'_{i+s}$.

(ii) $\text{height}(Q_j/q'_{i+1}) \geqq \text{height}(q_r/q_{i+1})$. This follows from (i) and the previously noted fact that $\text{height}(Q_j/q'_{i+1}) \geqq \text{height}(q'_r/q'_{i+1})$.

(iii) $\text{height}(q'_{i+1}/q'_2) \geqq \text{height}(q_{i+1}/q_2)$ by Corollary 3.2.

(iv) $\text{height}(q'_2/q_1) > \text{height}(q_2/q_1)$ by choice of $q'_2$.

Finally, from the tautness of $q_1 \subset \cdots \subset q_r$ and (ii), (iii), and (iv), we have $\text{height}(q_r/q_1) = \text{height}(q_r/q_{i+1}) + \text{height}(q_{i+1}/q_2) + \text{height}(q_2/q_1) < \text{height}(Q_j/q'_{i+1}) + \text{height}(q'_{i+1}/q'_2) + \text{height}(q'_2/q_1) \leqq \text{height}(Q_j/q_1)$ for $j = 1, \cdots, l+1$ to complete the proof.

At this point we can prove that $b_n$ exists and show that $b_n \leqq b^{n-1} + b^{n-2} + \cdots + b + 1$ with $b = b_{n-1}$. To see this, with the notation of Proposition 3.3, if $m > b^{n-1} + b^{n-2} + \cdots + b + 1$ we would have primes $q_r, Q_1, \cdots, Q_n$ lying over $p_r$ which are distinct (by the inequality in that proposition). However, as $\deg f(x) = n$, at most $n$ primes can lie over $p_r$, a contradiction. Thus $m \leqq b^{n-1} + \cdots + b + 1$.

We wish to introduce a "going down" technique which will let us improve this inequality somewhat, giving $b_n \leqq b^{n-2} + b^{n-3} + \cdots + b + 2$, $b = b_{n-1}$, and which, in certain circumstances, allows us to give a more substantial improvement on the bound on $b_n$.

DEFINITION. Let $p$ be a prime in the ring $R$. Let $I$ be an ideal in $R[x]$. Define $k(p, I) = n$ if $IR_p[x]$ contains a monic polynomial of degree $n$ but no monic polynomial of lesser degree. (If $IR_p[x]$ contains no monic polynomial let $k(p, I) = \infty$.)

LEMMA 3.4. *Let $p$ be prime in a ring $R$ and let $I$ be an ideal in $R[x]$. Suppose that $k(p, I) = n < \infty$.*

(a) *If $g(x) \in I$ and $\deg g(x) < n$ then $g(x) \in pR[x]$.*

(b) *Let $h(x) \in I$ with $\deg h(x) = n$ and the leading coefficient of $h(x)$ outside of $p$. Let $P$ be prime in $R[x]$ with $P \cap R = p$. Then $I \subseteqq P$ if and only if $h(x) \in P$.*

(c) *The number of primes $P$ in $R[x]$ satisfying $P \cap R = p$ and $I \subseteqq P$ does not exceed $n$.*

*Proof.* Without loss we may localize at $p$.

(a) Since $k(p, I) = n < \infty$ and $(R, p)$ is quasi-local, there is in $I$ a monic polynomial $h(x)$ of degree $n$, and no monic polynomial of lesser degree. If the result is false, then for some $g(x) = a_k x^k + \cdots + a_i x^i + \cdots + a_0 \in I$ with $k < n$ we have $a_i \notin p$ for some $i$. Assume that $g(x)$ and $i$ have been chosen so as to make $i$ as large as possible. Now $a_k \in p$ since $g(x)$ is not monic. We have $a_k h(x) - x^{n-k} g(x) \in I$. Its degree is clearly less than $n$ and its $(i + n - k)$th coefficient is not in $p$. This is a contradiction since $i + n - k > i$.

(b) Since $h(x)$ (in part (a)) is monic, clearly $I$ is generated by $h(x)$ together with those polynomials in $I$ having degree less than $n$. By part (a), each of these latter polynomials is in $pR[x] \subset P$. Thus $I \subseteqq P$ if and only if $h(x) \in P$.

(c) This is immediate from Lemma 2.1 and (b).

PROPOSITION 3.5. *Let $p \subset p'$ be primes in a ring $R$. Let $I$ be an ideal of $R[x]$, and suppose that $k(p, I) = k(p', I) < \infty$. If $P'$ is prime in $R]x]$ with $P' \cap R = p'$ and $I \subset P'$, then there is a prime $P$ in $R[x]$ with $P \cap R = p$ and $I \subseteqq P \subset P'$.*

*Proof.* We may localize at $p'$. If $k(p', I) = n$ then $I$ contains a monic polynomial $h(x)$ of degree $n$. Thus $h(x) \in I \subset P'$. By [4, Lemma 3] (applied to $R/p$) there is a prime $P$ of $R[x]$ with $P \cap R = p$ and $h(x) \in P \subset P'$. By Lemma 3.4, $I \subseteqq P$.

We apply Proposition 3.5 to our special situation of $R \subset R[u] = T$ a simple integral extension of domains, $u$ a root of the monic polynomial $f(x)$.

COROLLARY 3.6. *Let $p \subset p'$ be primes in $R$. Let $I = \ker(R[x] \to R[u] = T)$ and suppose that $k(p, I) = k(p', I)$. If $q'$ is prime in $T$*

*with* $q' \cap R = p'$ *then there is a prime* $q$ *of* $T$ *with* $q \cap R = p$ *and* $q \subset q'$.

*Proof.* Since $f(x) \in I$, $k(p', I) < \infty$. Let $P'$ be the preimage of $q'$ under $R[x] \to R[u]$. Then $P' \cap R = p'$ and $I \subset P'$. With $P$ as in Proposition 3.5 take $q$ to be the image of $P$ in $T$.

THEOREM 3.7. $b_n \leqq b^{n-2} + b^{n-3} + \cdots + b + 2$ *where* $b = b_{n-1}$.

*Proof.* Let $B = b^{n-2} + b^{n-3} + \cdots + b + 2$ and assume that $m > B$. We will derive a contradiction. Applying Proposition 3.3 to the chain $q_1 \subset \cdots \subset q_B$ we see that for some $r \in \{1, \cdots, B\}$ there are distinct primes $Q_1, \cdots, Q_{n-1}$ of $T$ lying over $p_r$ with $q_1 \subset Q_1 \cap \cdots \cap Q_{n-1}$ and $\text{height}(Q_j/q_1) > \text{height}(q_r/q_1)$ $j = 1, \cdots, n-1$. Obviously $q_r$ is distinct from $Q_1, \cdots, Q_{n-1}$ and if we let $Q_n = q_r$ then, since deg $f(x) = n$, $Q_1, \cdots, Q_n$ are all of the primes of $T$ lying over $p_r$ and we have $\text{height}(Q_j/q_1) \geqq \text{height}(q_r/q_1)$ $j = 1, \cdots, n$.

We claim that if $p$ is prime in $R$ with $p_r \subseteqq p$, then $k(p, I) = n$ where $I = \ker(R[x] \to R[u] = T)$. Since $f(x) \in I$, $k(p, I) \leqq n$. Also $p_r \subseteqq p$ implies $k(p_r, I) \leqq k(p, I)$ and so we must only show that $k(p_r, I) \geqq n$. That this is true follows from Lemma 3.4 (c) and the existence of $Q_1, \cdots, Q_n$.

We now consider a chain of maximal length between $p_r$ and $p_m$. Since $k(p, I) = n$ for each prime $p$ in that chain, we can use Corollary 3.6 iteratively to find a prime $q$ of $T$ with $q \cap R = p_r$, $q \subset q_m$ and $\text{height}(q_m/q) = \text{height}(p_m/p_r)$. Since $q_r \subset \cdots \subset q_m$ is taut and height $(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$ $i = r+1, \cdots, m$, obviously $\text{height}(q_m/q) = \text{height}(p_m/p_r) > \text{height}(q_m/q_r)$, (here we use $m > B \geqq r$). As $Q_1, \cdots, Q_n$ are all of the primes which lie over $p_r$, we must have $q = Q_j$, some $j = 1, \cdots, n$. Thus $\text{height}(q/q_1) = \text{height}(Q_j/q_1) \geqq$ height $(q_r/q_1)$. Thus $\text{height}(q_m/q_1) \geqq \text{height}(q_m/q) + \text{height}(q/q_1) > \text{height}(q_m/q_r) + \text{height}(q_r/q_1)$ contradicting the tautness of $q_1 \subset \cdots \subset q_m$. This completes the proof.

We repeat that we doubt that equality holds in Theorem 3.7. Let us note that $b_2 \leqq b_3 \leqq b_4 \leqq \cdots$. To see this, observe that if $T$ is a simple integral extension of $R$ via $f(x)$, then it is also a simple integral extension of $R$ via $xf(x)$. The examples at the end of this paper show that $b_n \to \infty$ as $n \to \infty$.

We now consider situations in which we can give other bounds on the size of $m$.

LEMMA 3.8. *Suppose that* $m > b_{n-1}$. *Let* $c = b_{n-1} + 1$. *If* $p$ *is any prime of* $R$ *containing* $p_c$, *then* $k(p, I) = n$ *where* $I = \ker(R[x] \to$

$R[u] = T$).

*Proof.* Since $f(x) \in I$, obviously $k(p_c, I) \leqq k(p, I) \leqq n$. We must show $k(p_c, I) \geqq n$. For this we may localize at $p_c$. If $k(p_c, I) < n$ then $I$ contains a monic polynomial $g(x)$ with $\deg g(x) = d < n$. Clearly $T$ is a simple integral extension of $R$ via $g(x)$. However the existence of the chain $q_1 \subset \cdots \subset q_c$ with $c > b_{n-1} \geqq b_d$ contradicts the definition of $b_d$.

LEMMA 3.9. *Suppose that $m > b_{n-1}$ and let $c = b_{n-1} + 1$. Let $p$ be any prime of $R$ containing $p_c$ and let $q$ be any prime in $T$ lying over $p$. Then $q_1 \subset q$.*

*Proof.* Let $P_1$ and $P$ be the preimages of $q_1$ and $q$, respectively, under the map $R[x] \to R[u] = T$. We claim that $k(p, P_1) = n$. The result follows, since obviously $f(x) \in P_1 \cap P$ and so by Lemma 3.4 (b) (with $h(x) = f(x)$ and $I = P_1$) $P_1 \subset P$. Thus $q_1 \subset q$.

To show that $k(p, P_1) = n$, we may work modulo $p_1$. That is we go to $R/p_1 \subset T/q_1$ and so assume that $p_1 = 0 = q_1$. Now $P_1 = \ker(R[x] \to T)$ and Lemma 3.8 gives $k(p, P_1) = n$.

THEOREM 3.10. *Suppose that $m > b_{n-1}$ and let $c = b_{n-1} + 1$. Then $m \leqq height(q_c/q_1) + b_{n-1}$.*

*Proof.* Consider a chain of maximal length between $p_c$ and $p_m$. By Lemma 3.8, for each prime $p$ in that chain, $k(p, I) = n$ with $I = \ker(R[x] \to T)$. By iteration of Corollary 3.6, we can find a prime $q$ of $T$ with $q \cap R = p_c$, $q \subseteqq q_m$ and $\text{height}(q_m/q) = \text{height}(p_m/p_c)$. By Lemma 3.9, $q_1 \subset q$. Since $q_1 \subset \cdots \subset q_m$ is taut we have $\sum_{c+1}^{m} \text{height}(q_i/q_{i-1}) + \text{height}(q_c/q_1) = \text{height}(q_m/q_1) \geqq \text{height}(q_m/q) + \text{height}(q/q_1) = \text{height}(p_m/p_c) + \text{height}(q/q_1) \geqq \sum_{c+1}^{m} \text{height}(p_i/p_{i-1}) + \text{height}(q/q_1)$. Thus $\text{height}(q_c/q_1) \geqq \sum_{c+1}^{m}[\text{height}(p_i/p_{i-1}) - \text{height}(q_i/q_{i-1})] + \text{height}(q/q_1)$. By our underlying assumption concerning how $q_1 \subset \cdots \subset q_m$ contracts to $p_1 \subset \cdots \subset p_m$, each term in this last summation is at least one. Thus $\text{height}(q_c/q_1) \geqq (m - c) + \text{height}(q/q_1) \geqq m - c + 1 = m - b_{n-1}$. Thus $m \leqq \text{height}(q_c/q_1) + b_{n-1}$.

COROLLARY 3.11. *Suppose that $m > b_{n-1}$ and that $height(q_i/q_{i-1}) \leqq r$ for $j = 2, \cdots, b_{n-1} + 1$. Then $m \leqq (r + 1)b_{n-1}$.*

*Proof.* Immediate from Theorem 3.10 and the tautness of $q_1 \subset \cdots \subset q_c$.

Suppose that we fix $r > 0$ and restrict our attention to chains

$q_1 \subset \cdots \subset q_m$ with height$(q_i/q_{i-1}) \leqq r$, $i = 2, \cdots, m$. Let $b'_n$ denote the best possible bound on $m$ for such chains when $\deg f(x) = n$. Then Lemma 3.8 through Corollary 3.11 can be repeated, replacing $b_{n-1}$ with $b'_{n-1}$, thus showing that $b'_n \leqq (r+1)b'_{n-1}$. Since $b'_2 = 2$, by induction we get $b'_n \leqq 2(r+1)^{n-2}$.

THEOREM 3.12. *If* $height(q_i/q_{i-1}) \leqq r$ *for* $i = 2, \cdots, m$, *then* $m \leqq 2(r+1)^{n-2}$.

## 4. Finitely generated modules.

We give our main result, assuming only that $T$ is a finitely generated $R$-module.

THEOREM 4.1. *Let* $R \subset T$ *be domains with* $T$ *a finitely generated* $R$*-module, generated by* $n$ *elements. Let* $q_1 \subset \cdots \subset q_m$ *be a taut chain of primes in* $T$ *lying over* $p_1 \subset \cdots \subset p_m$ *with* $height(p_m/p_1)$ *finite. Suppose that* $height(p_i/p_{i-1}) > height(q_i/q_{i-1})$ $i = 2, \cdots, m$. *Then* $m$ *is subject to the following*:

(i) $m \leqq b_n$,

(ii) *if* $m > b_{n-1}$, *then* $m \leqq height(q_c/q_1) + b_{n-1}$ *with* $c = b_{n-1} + 1$,

(iii) $m \leqq 2(r+1)^{n-2}$ *with* $r = \max\{height(q_i/q_{i-1}) \mid i = 2, \cdots, m\}$.

*Proof.* Since $T$ is a finitely generated $R$-module only finitely many primes of $T$ lie over $p_m$, and we may choose $u \in q_m$ but in no other prime lying over $p_m$. Obviously $q_m$ is the only prime of $T$ lying over $q_m \cap R[u]$ and so height$(q_m \cap R[u]/q_1 \cap R[u]) =$ height$(q_m/q_1)$ (by going up since height$(q_m \cap R[u]/q_1 \cap R[u]) \leqq$ height$(p_m/p_1) < \infty$). Clearly we have $(q_1 \cap R[u]) \subset \cdots \subset (q_m \cap R[u])$, a taut chain in $R[u]$ with height $(q_i \cap R[u]/q_{i-1} \cap R[u]) =$ height $(q_i/q_{i-1}) <$ height $(p_i/p_{i-1})$ $i = 2, \cdots, m$. A standard determinant argument shows that $u$ satisfies a monic polynomial of degree $n$ over $R$, and our result follows from the existence of $b_n$ and Theorems 3.10 and 3.12.

COROLLARY 4.2. *Let* $R$ *be a domain with integral closure* $R'$. *Suppose that* $R'$ *is a finitely generated* $R$*-module with* $n$ *generators. Let the domain* $T$ *be an integral extension of* $R$. *Let* $0 = q_1 \subset \cdots \subset q_m$ *be a taut chain of primes in* $T$ *lying over* $0 = p_1 \subset \cdots \subset p_m$ *in* $R$ *with height* $p_m$ *finite. Suppose that* $height(p_i/p_{i-1}) > height(q_i/q_{i-1})$ $i = 2, \cdots, m$. *Then* (i) $m \leqq b_n$; (ii) *if* $m > b_{n-1}$, *then* $m \leqq height$ $(q_c/q_1) + b_{n-1}$; *and* (iii) $m \leqq 2(r+1)^{n-1}$ *with* $r = \max\{height(q_i/q_{i-1}) \mid i = 2, \cdots, m\}$.

*Proof.* If $T'$ is the integral closure of $T$, we may lift $0 = q_1 \subset \cdots \subset q_m$ to a taut chain $0 = q'_1 \subset \cdots \subset q'_m$ in $T'$ with height $q'_m =$ height $q_m$. By going down in $R' \subset T'$, height $q' \cap R' =$ height $q'$ and we see that $0=(q'_1\cap R')\subset\cdots\subset(q'_m\cap R')$ is taut in $R'$ and height $(q'_i\cap R'/q'_{i-1}\cap R')=\text{height}(q'_i/q'_{i-1})=\text{height}(q_i/q_{i-1})<\text{height}(p_i/p_{i-1})$ $i=2,\cdots,m$. Applying Theorem 4.1 to $0=(q'_1\cap R')\subset\cdots\subset(q'_m\cap R')$, we are done.

5. **Examples.** In this section, we construct a family of examples which demonstrate that $b_n\to\infty$ as $n\to\infty$. We also show that if $R\subset T$ is an infinite integral extension, no bound need exist at all. This construction is a generalization of Nagata's Example 2 [5, pp. 203-205] and is very similar to [2]. However, except for the quotation of one key theorem, the presentation of the basic example will be self-contained.

EXAMPLE 5.1. Retaining the previous notation, we show any $m$ can be realized in some finite integral extension $R\subset T$ (which depends on $m$). Moreover, our example is Noetherian.

Fix an integer $m\geqq 2$. Let $K$ be a countable field of characteristic zero and let $y_1,\cdots,y_{m-1},z_0^{(1)},\cdots,z_0^{(m-1)}$ be indeterminates. We iteratively define a sequence of Noetherian domains $K=T_1\subset\widetilde{T}_2\subset T_2\subset\widetilde{T}_3\subset T_3\subset\cdots\subset T_m=T$ as follows: Set $\widetilde{T}_{i+1}=T_i[y_i]$ for each $i=1,\cdots,m-1$. Suppose $Z_i\in K[[y_i]]$ is a formal power series, say $Z_i=a_1^{(i)}y_i+a_2^{(i)}y_i^2+\cdots$. If we set $z_n^{(i)}=(z_0^{(i)}-\sum_{j=1}^n a_j^{(i)}y_i^j)/y_i^n$ for each $n\geqq 0$, then

$$(*)\qquad z_n^{(i)}=(z_{n+1}^{(i)}+a_{n+1}^{(i)})y_i\in\widetilde{T}_{i+1}]z_{n+1}^{(1)}]\ .$$

Thus we may define a direct union of simple transcendental extensions of $\widetilde{T}_{i+1}$, $T_{i+1}=\lim_{n\to\infty}\widetilde{T}_{i+1}[z_n^{(i)}]$, for each $i=1,\cdots,m-1$. Moreover, by [2, Corollary 1.6], we may choose the formal power series $Z_i$ in such a way that $T_{i+1}$ will be Noetherian.

The nature of the construction makes it very easy to determine the primes; primes in the intermediate rings extend to primes in $T$. Hence we easily see, for each $i=1,\cdots,m$, $q_i=(y_1,\cdots,y_{i-1})T$ is prime. Also, by (*), $z_n^{(i)}\in y_iT$ for each $i,n$. By the Krull Altitude Theorem, height $q_i\leqq i-1$. $(0)=q_1\subset q_2\subset\cdots\subset q_m$ is a taut chain and $\text{height}(q_{i+1}/q_i)=1$ $i=1,\cdots,m-1$. Before leaving this chain, we make one additional observation, also apparent from the construction. The quotient $T/q_i$ is canonically isomorphic to the subring $S_i=K[y_i,\cdots,z_n^{(i)},\cdots,y_{m-1},\cdots,z_n^{(m-1)},\cdots]$ for each $i=1,\cdots,m$.

Next we iteratively define a second chain $(0)=Q_1\subset\widetilde{Q}_2\subset Q_2\subset\cdots\subset Q_m$. First note that, using (*) again, $z_n^{(i)}=(z_{n+1}^{(i)}+a_{n+1}^{(i)})(y_i-1)+z_{n+1}^{(i)}+a_{n+1}^{(i)}$. Thus $z_{n+1}^{(i)}\equiv z_n^{(i)}-a_{n+1}^{(i)}(\text{mod}(y_i-1))$. So if we set, for each $i=1,\cdots,n-1$, $\widetilde{Q}_{i+1}=Q_i+(y_i-1)T$ and $Q_{i+1}=\widetilde{Q}_{i+1}+z_0^{(i)}T$, we have (using equality to denote canonical isomorphism) $T/\widetilde{Q}_{i+1}=S_{i+1}[z_0^{(i)}]$ and $T/Q_{i+1}=S_{i+1}$. So these ideals are prime as required and another application of the Krull Altitude Theorem guarantees that this chain is taut.

Our next step is to construct $R$. Again we construct a chain of rings $T = R_1 \supset R_2 \supset \cdots \supset R_m = R$. For each $i = 1, \cdots, m-1$, set $R_{i+1} = S_{i+1} + (q_{i+1} \cap Q_{i+1} \cap R_i)$. Since $S_{i+1} \subset S_i \subset R_i$, $R_{i+1} \subset R_i$ as desired. We claim that $R_i$ is an integral extension of $R_{i+1}$, generated by two elements as an $R_{i+1}$-module. To verify the claim, consider the canonical $R_{i+1}$-module homomorphism $\pi_i \colon R_i \to (R_i/q_{i+1} \cap R_i) \oplus (R_i/Q_{i+1} \cap R_i) = S_{i+1} \oplus S_{i+1}$. Note $\pi_i(1) = (1, 1)$ and $\pi_i(y_i) = (0, 1)$ together generate $S_{i+1} \oplus S_{i+1} = \operatorname{image}(\pi_i)$ and so $R_i = (1)R_{i+1} + (y_i)R_{i+1} + \operatorname{kernel}(\pi_i)$. However, $\operatorname{kernel}(\pi_i) = q_{i+1} \cap Q_{i+1} \cap R_i \subset R_{i+1}$ and so $R_i = R_{i+1} + y_i R_{i+1}$, proving our claim. Therefore $T$ is generated as an $R$-module by $2^{m-1}$ elements. Consequently, by Eakin's Theorem [1, p. 281], $R$ is a Noetherian domain.

It now only remains to show $R \subset T$ exhibits the desired chain behavior. As $\dim \widetilde{T}_{i+1} = (\dim T_i) + 1$ and $\dim T_{i+1} = (\dim \widetilde{T}_{i+1}) + 1$ for each $i = 1, \cdots, m-1$, $\dim T = 2(m-1)$. So, by going up, $\dim R = 2(m-1)$. Thus $(0) = Q_1 \cap R \subset \widetilde{Q}_2 \cap R \subset Q_2 \cap R \subset \cdots \subset Q_m \cap R$ is taut; then $Q_1 \cap R \subset Q_2 \cap R \subset \cdots \subset Q_m \cap R$ is likewise taut and $\operatorname{height}(Q_{i+1} \cap R)/(Q_i \cap R) = 2$ for each $i = 1, \cdots, m-1$. However, by construction, $Q_i \cap R = q_i \cap R$ and so $\operatorname{height}(q_{i+1} \cap R)/(q_i \cap R) = 2$. As $(0) = q_1 \subset \cdots \subset q_m$ is a taut chain in $T$ and $\operatorname{height}(q_{i+1}/q_i) = 1$, we have the desired chain.

In particular, this example shows $b_{2m-1} \geqq m$ and so $b_n \to \infty$ as $n \to \infty$.

EXAMPLE 5.2. There is an infinite integral extension $R \subset T$ and an infinite taut chain in $T$, $(0) = q_1 \subset q_2 \subset \cdots$, such that $(0) = q_1 \cap R \subset q_2 \cap R \subset \cdots$ is taut and $\operatorname{height}(q_{i+1}/q_i) = 1 < 2 = \operatorname{height}(q_{i+1} \cap R/q_i \cap R)$ for each $i$. Necessarily, $R$ is not Noetherian.

Example 5.2 will be a direct union of domains constructed in the manner of (5.1). We begin as in (5.1) with a sequence of domains $K = T_1 \subset \widetilde{T}_2 \subset T_2 \subset \cdots \subset T_m \subset \cdots$, this time choosing an infinite sequence. For each fixed $m$, we perform the construction in (5.1), superscripting our symbols with $(m)$ when confusion is possible. Thus $T_m = T^{(m)}$ and we have $R^{(m)} \subset T^{(m)}$.

Noting $T^{(m)} \subset T^{(m+1)}$, $q_i^{(m)} \subset q_i^{(m+1)}$ and $Q_i^{(m)} \subset Q_i^{(m+1)}$ for each $i = 1, \cdots, m$, and $S_i^{(m)} \subset S_i^{(m+1)}$ for each $i = 1, \cdots, m$, we have direct unions $T = \bigcup T^{(m)}$, $q_i = \bigcup q_i^{(m)}$, $Q_i = \bigcup Q_i^{(m)}$, and $S_i = \bigcup S_i^{(m)}$ with $T/q_i = S_i = T/Q_i$. Next, using the fact that $S_m^{(m)} = K = S_{m+1}^{(m+1)}$ and some obvious containments, we have $R^{(m)} = R_m^{(m)} = S_m^{(m)} + (q_m^{(m)} \cap Q_m^{(m)} \cap R_{m-1}^{(m)}) = S_m^{(m)} + (q_m^{(m)} \cap Q_m^{(m)} \cap R_m^{(m)}) \subset S_{m+1}^{(m+1)} + (q_{m+1}^{(m+1)} \cap Q_{m+1}^{(m+1)} \cap R_m^{(m+1)}) =$

$R^{(m+1)}$. Thus, we also have a direct union $R = \cup R^{(m)}$. We claim $R \subset T$ is the desired extension.

If $y \in T$, $y \in T^{(m)}$ for some $m$ and so is integral over $R^{(m)}$ and consequently $R$. Thus we have an integral extension. Since $R$ and $T$ are direct unions, the statement about the $q_i$'s is valid because it holds in $R^{(m)} \subset T^{(m)}$ for each $m$.

EXAMPLE 5.3. There is a Noetherian domain $R$ such that, for each $m$, we may find an integral extension $T$ of $R$ and a taut chain of primes $(0) = q_1 \subset q_2 \subset \cdots \subset q_m$ in $T$ such that $q_1 \cap R \subset \cdots \subset q_m \cap R$ is taut and $\mathrm{height}(q_{i+1}/q_i) < \mathrm{height}(q_{i+1} \cap R/q_i \cap R)$ for each $i = 1, \cdots, m-1$.

This example will not be formally constructed. It is obtained by combining two construction ideas. One constructs a family of local Example (5.1)'s and combines them in the manner of Nagata's Example 1 [5, p. 203] (the Noetherian ring with infinite Krull dimension). This is a useful and straightforward way of obtaining this sort of infinite bad behavior.

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Received July 12, 1978 and in revised form March 15, 1978.

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