PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 2, 1981

A NECESSARY CONDITION ON THE EXTREME POINTS OF A CLASS OF HOLOMORPHIC FUNCTIONS II

FRANK FORELLI

We correct an oversight in the paper of the same title by pointing out that a theorem holds which is stronger than the theorem of that paper.

1. Let X be a complex manifold. (We agree that X is connected.) It will be convenient to denote by $H(X)$ the class of all holomorphic functions on X. Let $p \in X$, and let:

$$N(X, p) = \{f: f \in H(X), \operatorname{Re} f > 0, f(p) = 1\}$$

$$W(X, p) = \{g: g \in H(X), |g| < 1, g(p) = 0\}$$

$$W(X) = \{g: g \in H(X), |g| \leqq 1\} .$$

Thus

$$N(X, p) = \{(1 + g)/(1 - g): g \in W(X, p)\} .$$

Let $g \in W(X)$. We will say that g is irreducible [1] if whenever $g = \varphi\psi$ where $\varphi, \psi \in W(X)$, then either φ or ψ is a constant of modulus one. The purpose of this brief note is to correct an oversight in [2] by pointing out that the following theorem (which is stronger than the theorem of [2]) holds.

THEOREM. *Let $g \in W(X, p)$ and let $f = (1 + g)/(1 - g)$. If $N(X, p) \neq \{1\}$, and if f is an extreme point of $N(X, p)$, then g is irreducible.*

Proof. Our proof is based on the following three identities.

$$\frac{1 - zw}{(1 - z)(1 - w)} = \frac{1}{2}\frac{1 + z}{1 - z} + \frac{1}{2}\frac{1 + w}{1 - w} . \tag{1.1}$$

$$\frac{1 + zw}{1 - zw} = \frac{1}{2}\left[\frac{(1 - z)(1 - w)}{1 - zw} + s\right] + \frac{1}{2}\left[\frac{(1 + z)(1 + w)}{1 - zw} - s\right] . \tag{1.2}$$

And

$$\frac{1 + [w(z + w)/(1 + zw)]}{1 - [w(z + w)/(1 + zw)]} = \frac{1}{2}(1 - z)\frac{1 - w}{1 + w} + \frac{1}{2}(1 + z)\frac{1 + w}{1 - w} . \tag{1.3}$$

The identity (1.1) proves that

$$\operatorname{Re}\frac{1-zw}{(1-z)(1-w)}>0$$

if $|z|<1$, $|w|<1$. This in turn proves that

$$\operatorname{Re}\frac{(1-z)(1-w)}{1-zw}>0$$

if $|z|<1$, $|w|<1$. Thus if $\operatorname{Re}s=0$, then

$$\operatorname{Re}\left[\frac{(1-z)(1-w)}{1-zw}+s\right]\geqq 0 \tag{1.4}$$

if $|z|<1$, $|w|\leqq 1$.

Let $g=\varphi\psi$ where $\varphi\in W(X,p)$, $\psi\in W(X)$. It is to be proved that ψ is a constant of modulus one. If $t\in \boldsymbol{T}$, then by the identity (1.2),

$$\begin{aligned} f&=\frac{1}{2}\left[\frac{(1-t\varphi)(1-\bar{t}\psi)}{1-\varphi\psi}+s\right]+\frac{1}{2}\left[\frac{(1+t\varphi)(1+\bar{t}\psi)}{1-\varphi\psi}-s\right]\\ &=\frac{1}{2}\alpha+\frac{1}{2}\beta\,. \end{aligned}$$

We have

$$\alpha(p)=1-\bar{t}\psi(p)+s\,. \tag{1.5}$$

Let t in $\boldsymbol{T}$ satisfy $\operatorname{Re}[\bar{t}\psi(p)]=0$ and let $s=\bar{t}\psi(p)$. Then by (1.4) and (1.5) we have $\alpha,\beta\in N(X,p)$. Thus $\alpha=\beta$. This gives

$$\sigma=\bar{t}\psi=\frac{s-t\varphi}{1+st\varphi}=\frac{s-\tau}{1+s\tau}\,. \tag{1.6}$$

Thus

$$f=\frac{1+\tau\sigma}{1-\tau\sigma}=\frac{1+[\tau(s-\tau)/(1+s\tau)]}{1-[\tau(s-\tau)/(1+s\tau)]}\,. \tag{1.7}$$

We have $s=i\gamma$, $-1\leqq\gamma\leqq 1$. By (1.7) and the identity (1.3),

$$f=\frac{1}{2}(1-\gamma)\frac{1-i\tau}{1+i\tau}+\frac{1}{2}(1+\gamma)\frac{1+i\tau}{1-i\tau}\,. \tag{1.8}$$

If $-1<\gamma<1$, then by (1.8), $i\tau=-i\tau$, hence $\tau=0$. Thus $f=1$ which contradicts the fact that 1 is not an extreme point of $N(X,p)$ if $N(X,p)\neq\{1\}$. Thus $\gamma=\pm 1$, hence by (1.6), $\bar{t}\psi=s$ which proves that g is irreducible.

2. Let $X=\boldsymbol{D}$. If $g\in W(\boldsymbol{D},0)$, then by the lemma of Schwarz, $g(z)=z\psi(z)$ where $\psi\in W(\boldsymbol{D})$. Thus by the foregoing we have a

quite elementary proof of the fact that if f is an extreme point of $N(\boldsymbol{D}, 0)$, then

$$f(z) = (1 + tz)/(1 - tz)$$

where $t \in \boldsymbol{T}$. There is a different elementary proof of this in [3].

3. The identity (1.1) states that if

$$f(z, w) = (1 - z)(1 - w)/(1 - zw) , \tag{3.1}$$

then $1/f$ is not an extreme point of $N(\boldsymbol{D} \times \boldsymbol{D}, 0)$. We will prove that f on the other hand is extreme. Thus if

$$g = (f - 1)/(f + 1) , \tag{3.2}$$

then the Cayley transform of g is extreme, whereas the Cayley transform of $-g$ is not.

3.1. If A is a convex set, then we will denote by ∂A the class of all extreme points of A. If B is a compact Hausdorff space, then we will denote by $M_+(B)$ the class of all Radon measures on B. Thus if $\mu \in M_+(B)$ and $E \subset B$, then $\mu(E) \geqq 0$.

Let $f \in N(\boldsymbol{D} \times \boldsymbol{D}, 0)$. Then $\operatorname{Re} f$ is the Poisson integral $\mu^\#$ of a measure μ in $M_+(\boldsymbol{T} \times \boldsymbol{T})$. It will be convenient to denote this measure by f^*. Thus

$$\operatorname{Re} f = (f^*)^\# .$$

Let F be a closed subset of the torus $\boldsymbol{T} \times \boldsymbol{T}$. We will denote by N_F the class of those f in $N(\boldsymbol{D} \times \boldsymbol{D}, 0)$ for which $\operatorname{spt}(f^*) \subset F$.

PROPOSITION. $\partial N_F \subset \partial N(\boldsymbol{D} \times \boldsymbol{D}, 0)$.

Proof. Let $f \in \partial N_F$. It is to be proved that $f \in \partial N(\boldsymbol{D} \times \boldsymbol{D}, 0)$. Thus let $f = 1/2g + 1/2h$ where $g, h \in N(\boldsymbol{D} \times \boldsymbol{D}, 0)$. Then $g^* + h^* = 2f^*$, hence $g^* \leqq 2f^*$. This proves that $g \in N_F$. Likewise $h \in N_F$. Thus $f = g = h$.

3.2. Henceforth we let

$$F = \{(t, \bar{t}) : t \in \boldsymbol{T}\} ,$$

and we define $\pi: \boldsymbol{T} \to F$ by $\pi(t) = (t, \bar{t})$. Let $f \in N_F$ and let $\mu = f^*$. Then $\mu = \pi_* \lambda$ where $\lambda \in M_+(\boldsymbol{T})$. We have

$$\hat{\mu}(j, k) = \int \bar{z}^j \bar{w}^k d\mu(z, w) = \int \bar{t}^j t^k d\lambda(t) = \hat{\lambda}(j - k) .$$

Thus $\hat{\lambda}(j-k)=0$ if $jk<0$, hence $\hat{\lambda}(n)=0$ if $n \neq 0, \pm 1$. This proves that

$$d\lambda = \left(\frac{\bar{a}}{2}e^{-i\theta} + 1 + \frac{a}{2}e^{i\theta}\right)\frac{d\theta}{2\pi} \tag{3.3}$$

where $a \in \boldsymbol{C}$. We have

$$\frac{\bar{a}}{2}e^{-i\theta} + 1 + \frac{a}{2}e^{i\theta} \geqq 0 ,$$

hence $1 - |a| \geqq 0$. Thus we see that N_F may be identified with

$$\bar{\boldsymbol{D}} = \{a\colon a \in \boldsymbol{C},\ |a| \leqq 1\}$$

and that ∂N_F may be identified with

$$\boldsymbol{T} = \partial\boldsymbol{D} = \{a\colon a \in \boldsymbol{C},\ |a| = 1\} .$$

3.3. Let (3.1) hold. We have

$$f(z, w) = 1 + 2\left(\sum_1^\infty z^k w^k - \frac{1}{2}\sum_0^\infty z^{k+1}w^k - \frac{1}{2}\sum_0^\infty z^k w^{k+1}\right) ,$$

hence $f^* = \pi_*\lambda$ if in (3.3) we let $a = -1$. This proves that $f \in \partial N_F$, hence by Proposition 3.1, $f \in \partial N(\boldsymbol{D} \times \boldsymbol{D}, 0)$. Furthermore, we see that

$$\partial N_F = \{(1 - az)(1 - \bar{a}w)/(1 - zw)\colon a \in \boldsymbol{T}\} .$$

3.4. A comment on the foregoing. Let (3.1) and (3.2) hold, let $t \in \boldsymbol{T}$, and let

$$h = (1 + tg)/(1 - tg) .$$

Let $t \neq -1$, let $s = (\bar{t} - 1)/2$, and let

$$\varphi(z) = t(z + s)/(1 + \bar{s}z) .$$

Then

$$f(\varphi(z), w) = ah(z, w) + ib$$

where $a + ib = f(\varphi(0), 0)$. By Proposition 3.3 of [2], this proves that $h \in \partial N(\boldsymbol{D} \times \boldsymbol{D}, 0)$.

4. A concluding comment. Let G be a region in $\boldsymbol{D}$. It will be convenient to say that $\boldsymbol{D} - G$ is a Painleve null set if every bounded holomorphic function on G has a holomorphic extension to $\boldsymbol{D}$. By way of a corollary to Theorem 1, we have the following converse of the lemma of Schwarz.

THEOREM. *Let $W(X)$ separate the points of X, and let φ in $W(X, p)$ satisfy*

$$W(X, p) = \varphi W(X) .$$

Then the complex manifold X may be identified with the open unit disc $\boldsymbol{D}$ modulo a Painleve null set.

Proof. If $g \in W(X, p)$, then $g = \varphi\psi$ where $\psi \in W(X)$. Thus by Theorem 1,

$$\partial N(X, p) \subset \{(1 + t\varphi)/(1 - t\varphi) : t \in \boldsymbol{T}\} .$$

This proves, by the Krein-Milman theorem, that if $f \in N(X, p)$, then

$$f = \int \frac{1 + t\varphi}{1 - t\varphi} d\mu(t) \tag{4.1}$$

where $\mu \in M_+(\boldsymbol{T})$. This in turn proves, since $W(X)$ separates the points of X, that φ is univalent. Thus we may identify X with $\varphi(X)$, in which case (4.1) becomes

$$f(z) = \int \frac{1 + tz}{1 - tz} d\mu(t) \tag{4.2}$$

if $z \in X$. The right side of (4.2), however, belongs to $N(\boldsymbol{D}, 0)$. Thus $\boldsymbol{D} - X$ is a Painleve null set, which completes the proof of Theorem 4.

REFERENCES

1. P. Ahern and W. Rudin, *Factorizations of bounded holomorphic functions*, Duke Math. J., **39** (1972), 767-777.

2. F. Forelli, *A necessary condition on the extreme points of a class of holomorphic functions*, Pacific J. Math., **73** (1977), 81-86.

3. F. Holland, *The extreme points of a class of functions with positive real part*, Math. Ann., **202** (1973), 85-87.

Received February 16, 1979 and in revised form July 3, 1979. Partially supported by the National Science Foundation.

THE UNIVERSITY OF WISCONSIN-MADISON
MADISON, WI 53706