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ON COMPACT METRIC SPACES WITH NONCOINCIDING TRANSFINITE DIMENSIONS

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For every no more than countable ordinal number α we shall define an ordinal number $\varphi(\alpha)$ such that for every compact metric space X with $\operatorname{ind} X \leq \alpha$ we have $\operatorname{Ind} X \leq \varphi(\alpha)$ and there exists a compact metric spaces X_α with $\operatorname{ind} X_\alpha = \alpha$, $\operatorname{Ind} X_\alpha = \varphi(\alpha)$, where $\operatorname{ind} X_\alpha$ and $\operatorname{Ind} X_\alpha$ mean small and large transfinite inductive dimensions respectively. In particular we now extend the author's previous result on existence of compact metric spaces with noncoinciding transfinite dimensions.

1. Introduction. In this paper we consider only metric spaces. For instance, by a compact space we mean a compact metric space. All mappings we consider are continuous and I^n denotes the n-dimensional euclidean cube.

1. Definitions and statements of main results.

DEFINITION 1.1. (a) $\operatorname{ind} X = -1 \Leftrightarrow X = \varnothing$.

(b) We assume that for every ordinal number $\alpha < \beta$ the class of spaces X with $\operatorname{ind} X \leqq \alpha$ is defined. Then, we say $\operatorname{ind} X \leqq \beta$ if for every point $x \in X$ and a closed subset F, $x \notin F \subset X$, there exists a neighborhood Ox of x such that:

$$Ox \subset X \backslash F$$

$$\operatorname{ind} FrOx \leqq \alpha < \beta$$ [1]

We put $\operatorname{ind} X = \min\{\beta: \operatorname{ind} X \leqq \beta\}$.

(c) We say that dimension $\operatorname{ind}_x X$ of a space X in a point $x \in X \leqq \beta$ if there exists such a base $\{O_\lambda: \lambda \in A\}$ at this point, so that

$$\operatorname{ind} FrO_\lambda < \beta .$$

We put $\operatorname{ind}_x X = \min\{\beta: \operatorname{ind} X \leqq \beta\}$.

DEFINITION 1.2. (a) $\operatorname{Ind} X = -1 \Leftrightarrow X = \varnothing$

(b) Let, for every ordinal number $\alpha < \beta$, the class of spaces X with $\operatorname{Ind} X \leqq \alpha$ be defined. Then, $\operatorname{Ind} X \leqq \beta$ if for every pair of disjoint closed subsets F and G there exists a partition C[2]

[1] Fr A denotes the boundary of A.

[2] By a partition in X between sets A and B we mean a closed set C in X such that $X \backslash C = U \cup V$, $U \cap V = \varnothing$, $A \subset U$, $B \subset V$ for some open sets U and V in X.

between F and G such that $\operatorname{Ind} C \leqq \alpha < \beta$. We put $\operatorname{Ind} X = \min\{\beta: \operatorname{Ind} X \leqq \beta\}$.

We note that we can also introduce the dimension ind using partitions, because if $x \in \bar{U} \subset X \backslash F$, and U is open, then $Fr U$ is a partition between x and F. Obviously, $\operatorname{ind} X \leqq \operatorname{Ind} X$. For spaces with a countable basis, in particular, for compact spaces dimension $\operatorname{ind} X$ is no more than a countable ordinal (Hurewicz [4], p. 50), and $\operatorname{Ind} X$ is no more than a countable ordinal even for all metric spaces (Smirnov [16], p. 418). Dimensions $\operatorname{ind} X$ and $\operatorname{Ind} X$ are not defined for every metric space. For example, the Hilbert cube I^ω does not have any transfinite dimension (Hurewicz [4], p. 51). Let

$$Z = \bigcup_{n=1}^{\infty} I^n$$

be the discrete union of cubes I^n. Then, obviously, $\operatorname{ind} Z = \omega_0$. However, the dimension $\operatorname{Ind} X$ doesn't exist. But if for a space X the dimension $\operatorname{Ind} X$ exists, then $\operatorname{ind} X$ also exists. In this paper we solve the following problem: to find a function $\psi: \Omega \to \Omega$ defined on the set of all ordinal numbers $\alpha < \omega_1$ and satisfying the following conditions:

(i) *for every compact space X, having dimension* $\operatorname{ind} X$, *we have*

$$\psi(\operatorname{Ind} X) \leqq \operatorname{ind} X \leqq \operatorname{Ind} X .$$

(ii) *for every $\alpha < \omega_1$, there exists a compact space $X = X(\alpha)$ satisfying the following equalities*:

$$\operatorname{Ind} X = \alpha; \qquad \operatorname{ind} X = \psi(\alpha) .$$

We shall also find such a function $\varphi: \Omega \to \Omega$, so that for every compact space X with dimension $\operatorname{ind} X$

(iii) $\operatorname{ind} X \leqq \operatorname{Ind} X \leqq \varphi(\operatorname{ind} X)$.

(iv) *for every $\alpha < \omega_1$, there exists a compact space $X = X(\alpha)$ such that*

$$\operatorname{ind} X = \alpha, \qquad \operatorname{Ind} X = \varphi(\alpha) .$$

The first examples of compacta with noncoinciding transfinite dimensions were constructed by the author in [9]. Let us introduce some notations. In § 1 small greek letters denote ordinal numbers. For every ordinal number β the equality $\beta = \alpha + n$ holds, where α is a limit number or 0, and $n = 0, 1, 2, \cdots$. Then we set $K(\beta) = n$, $J(\beta) = \alpha$. Further, for every $\beta \geqq \omega_0$ by $\tau(\beta)$ we denote an ordinal number, defined by the equality $\omega_0 + \tau(\beta) = \beta$. If $\beta < \omega_0$ we set $\tau(\beta) = 0$.

DEFINITION 1.3. Put $\varphi(\beta) = \omega_0 + \omega_0 \times \tau(\beta)$ for $\beta \geqq \omega_0$ and $\varphi(\beta) = \beta$ for $\beta < \omega_0^3$.

Numbers β such that $\beta = \varphi(\beta)$ shall be called invariant. It is easy to prove that a number $\beta > \omega_0$ is invariant if and only if $\beta = \omega_0^{\omega_0} \times \gamma$[4] for some γ; and, every $\alpha \leqq \omega_0$ is invariant.

DEFINITION 1.4. Put $\psi(\beta) = \min\{\alpha: \varphi(\alpha) \geqq \beta\}$.

LEMMA 1.1. *Functions φ and ψ have the following properties:*

(a) *Let $\alpha = \omega_0 + \beta + n$ where β is a limit number or 0, $n = K(\alpha)$, and ξ is a number such that $\omega_0 \times \xi = \beta$. Then*

$$\psi(\alpha) = \begin{cases} \omega_0 + \xi + 1 & \text{if } n > 0 \\ \omega_0 + \xi & \text{if } n = 0 \end{cases}$$

(b) *$\psi(n) = \varphi(n) = n$ for $n = 0, 1, 2 \cdots$.*

(c) *$\beta + \omega_0 \geqq \varphi \circ \psi(\beta) \geqq \beta$.*

(d) *If $\alpha \geqq \omega_0$ then $\varphi(\alpha)$ is a limit number.*

(e) *If $\gamma > \beta$ then $\varphi(\gamma) > \varphi(\beta)$.*

(f) *Let $\gamma > \omega_0$, then $\psi(\gamma) = \psi(\beta)$ iff $J(\beta) = J(\gamma)$ and the numbers $K(\beta)$, $K(\gamma)$ are either both equal to 0, or both different from 0.*

(g) *If β is a nonlimit number, then $\psi(\beta)$ also is a nonlimit one.*

(h) *If $\gamma \leqq J(\alpha) < \alpha$ then $\psi(\gamma) < \psi(\alpha)$.*

(i) *If α is a limit number, then $\psi(\alpha) < \psi(\alpha + m)$ where $m = 1, 2 \cdots$.*

(j) *If $\alpha \geqq \beta$, then $\psi(\alpha) \geqq \psi(\beta)$.*

(k) *If $\alpha = \sup\{\gamma_i: i = 1, 2, \cdots\}$, then $\varphi(\alpha) = \sup\{\varphi(\gamma_i): i = 1, 2, \cdots\}$.*

(l) *If $\beta < \alpha$ and α is invariant number, then $\varphi(\beta) < \alpha$.*

(m) *If $\alpha = \sup\{\gamma_i: i = 1, 2, \cdots\}$, then $\psi(\alpha) = \sup\{\psi(\gamma_i): i = 1, 2, \cdots\}$.*

(n) *$\psi \circ \varphi(\beta) = \beta$.*

(o) *$\varphi(\alpha) = \omega_0 \times \alpha$ for $\alpha \geqq \omega_0^2$, $\varphi(\omega_0 + p) = \omega_0 \times (p + 1)$, $\varphi(\omega_0 \times q + p) = \omega_0^2 \times (q - 1) + \omega_0 \times p$ for $q = 2, 3, \cdots$, $p = 0, 1, 2, \cdots$.*

Proof. (a) Let $n > 0$. Since $\varphi(\omega_0 + \xi + 1) = \omega_0 + \omega_0 \times \xi + \omega_0$ we have $\omega_0 + \xi + 1 \geqq \psi(\alpha)$. Further, $\varphi(\omega_0 + \xi) = \omega_0 + \omega_0 \times \xi < \alpha$.

[3] We assume $0 \times \alpha = \alpha \times 0 = 0$, $0 + \alpha = \alpha + 0 = \alpha$ for any α.

Let A, B be two well-ordered sets having types α, β respectively. In a product $A \times B$ we introduce the following well ordering: $(a, b) < (a', b)$ if $b' > b$ or if $b' = b$ and $a' > a$. Then the type of $A \times B$ is denoted by $\alpha \times \beta$. Generally speaking $\alpha \times \beta \neq \beta \times \alpha$.

[4] By definition $\omega_0^{\omega_0} = \sup\{\omega_0^n: n = 1, 2, \cdots\}$

Consequently $\omega_0 + \xi + 1 = \psi(\alpha)$. Now let $n = 0$. Then $\varphi(\omega_0 + \xi) = \alpha$ and $\varphi(\omega_0 + \xi') = \omega_0 + \omega_0 \times \xi' < \alpha$ for $\xi' < \xi$. Hence, $\psi(\alpha) = \omega_0 + \xi$.

(b) is evident.

(c) follows from (a) and Definitions 1.3, 1.4.

(d), (e) are evident.

(f), (g), (h), (i) follow from (a).

(j) follows from (b) and Definition 1.4.

(k) If $\alpha < \omega_0$ then the assertion is evident. Let $\alpha = \omega_0 + \beta$, then $\beta = \tau(\alpha) = \sup\{\tau(\gamma_i): i = 1, 2, \cdots\}$. Consequently, $\varphi(\alpha) = \sup\{\omega_0 + \omega_0 \times \tau(\gamma_i) = \varphi(\gamma_i): i = 1, 2\cdots\}$.

(l) If α is invariant, then $\alpha = \varphi(\alpha)$ and $\varphi(\beta) < \varphi(\alpha) = \alpha$ by property (e).

(m) If $\alpha \leqq \omega_0$ or α is nonlimit number then the assertion is evident. Let α be a limit number $> \omega_0$. Then $\alpha = \omega_0 + \omega_0 \times \xi$ for some $\xi > 0$. Then by virtue of (a), $\psi(\alpha) = \omega_0 + \xi$. Obviously, for some $n > 0$ each $\gamma_k (k \geqq n)$ has the representation:

$$\gamma_k = \omega_0 + \omega_0 \times \xi_k + K(\gamma_k) \tag{1}$$

and $\gamma_k < \omega_0$ for $k < n$. Let ξ be a limit number and $\xi > \sup_{i>n}\{\xi_i\}$. Then $\sup\{\xi_i\}+1 < \xi$ and $\sup\{\gamma_i\} \leqq \sup\{\omega_0 + \omega_0 \times \xi_i + \omega_0\} < \omega_0 + \omega_0 \times \xi = \alpha$ which contradicts the condition. Therefore

$$\xi = \sup\{\xi_i: i > n\} . \tag{2}$$

Consequently, by virtue of (a), (1) and (2)

$$\psi(\alpha) = \omega_0 + \xi = \sup\{\omega_0 + \xi_i: i > n\} \leqq \sup\{\psi(\gamma_i): i = 1, 2\cdots\} .$$

The inequality $\psi(\alpha) \geqq \sup\{\psi(\gamma_i): i > n\}$ follows from (j). If $\xi = \xi_0 + 1$, then $\alpha = \omega_0 + \omega_0 \times \xi_0 + \omega_0$ and there is a number γ_i such that $\gamma_i \geqq \omega_0 + \omega_0 \times \xi_0 + 1$. By virtue of (a) $\psi(\gamma_i) = \omega_0 + \xi_0 + 1 = \omega_0 + \xi = \psi(\alpha)$. Hence (m) is proved.

(n) follows from (l), (o) is evident. □

We shall use the addition theorem for inductive dimensions, proved in [8], Levsenko, Theorem 1, p. 255 and Theorem 1, p. 257. Let α, β be limit numbers and p, q integers $\geqq 0$. Then put:

$$k(\alpha, p, \beta, q) = \begin{cases} \alpha + p & \text{for } \beta < \alpha \\ \beta + q & \text{for } \alpha < \beta \\ \alpha + p + q + 1 & \text{for } \alpha = \beta . \end{cases}$$

ADDITION THEOREM (L). *Let the hereditarily normal space R be a union of two closed sets R_1 and R_2, having dimensions* $\operatorname{ind} R_1 \leqq \alpha + p$ *(respectively* $\operatorname{Ind} R_1 \leqq \alpha + p$*) and* $\operatorname{ind} R_2 \leqq \beta + q$ *(respectively*

$\operatorname{Ind} R_2 \leqq \beta + q$). *Then* R *has dimension* $\operatorname{ind} R$ (*respectively* $\operatorname{Ind} R$) *and the following inequality holds*:

$$\operatorname{ind} R(\operatorname{Ind} R) \leqq k(\alpha, p, \beta, q) .$$

THEOREM[5] 1.1. *Let X be a completely normal bicompactum* (*not necessarily metrizable*) *having dimension* $\operatorname{ind} X$. *Then X has dimension* $\operatorname{Ind} X$ *and*

$$\operatorname{Ind} X \leqq \varphi(\operatorname{ind} X) .$$

Proof. If $\operatorname{ind} X < \omega_0$ it is well known (Vedenisov [19] that $\operatorname{ind} X = \operatorname{Ind} X = \varphi(\operatorname{ind} X)$. Suppose that $\operatorname{ind} X = \beta \geqq \omega_0$ and for all $\gamma < \beta$ and for any completely normal bicompactum X having $\operatorname{ind} X = \gamma$ the theorem is proved. Let F and G be closed disjoint subsets of X. Since X is a bicompactum, there exists a finite collection of open sets $O_1, \cdots, O_s$ in X, such that:

$$\bar{O}_i \cap G = \varnothing,\ \operatorname{ind} FrO_i \leqq \gamma_i < \beta,\ \cup\{O_i: i = 1, \cdots, s\} \supset F ,$$
$$(i = 1, \cdots, s) .$$

By the inductive assumption dimensions $\operatorname{Ind} FrO_i$ exist and

$$\operatorname{Ind} FrO_i \leqq \varphi(\operatorname{ind} FrO_i) .$$

By property (e) from Lemma 1.1, $\varphi(\operatorname{ind} FrO_i) < \varphi(\operatorname{ind} X)$. Since by property (d) from Lemma 1.1 $\varphi(\beta)$ is a limit number,

$$\varphi(\operatorname{ind} FrO_i) + \omega_0 \leqq \varphi(\beta) . \tag{3}$$

From Theorem L and (3) it follows that

$$\begin{aligned}\operatorname{Ind}(\cup\{FrO_i: i = 1, \cdots, s\}) &\leqq \max\{J(\operatorname{Ind} FrO_i): i = 1, \cdots, s\} \\ &\quad + \sum_{i=1}^{s} K(\operatorname{Ind} FrO_i) + (s-1) \\ &< \max\{\varphi(\operatorname{ind} FrO_i): i = 1, \cdots, s\} + \omega_0 \leqq \varphi(\beta) .\end{aligned}$$

Since the set $\cup\{FrO_i: i = 1, \cdots, s\}$ obviously contains a partition between F and G, we have

$$\operatorname{Ind} C \leqq \operatorname{Ind} \bigcup_{i=1}^{s} FrO_i < \varphi(\beta) . \qquad \square$$

COROLLARY 1.1. *For any completely normal bicompactum* (*not necessarily metrizable*) X *having dimension* $\operatorname{ind} X$, *we have* $\operatorname{Ind} X \geqq \operatorname{ind} X \geqq \psi(\operatorname{Ind} X)$.

[5] In [8] Theorem 2, p. 260 an upper bound for dimension Ind was also obtained, however it is less exact for completely normal bicompacta.

Proof. Let us suppose that $\operatorname{ind} X < \psi(\operatorname{Ind} X)$; then by definition of the function ψ, $\varphi(\operatorname{ind} X) < \operatorname{Ind} X$ which contradicts Theorem 1.1. □

COROLLARY 1.2. *Let X be a completely normal bicompactum (not necessarily metrizable). Then*

(a) *if* $\operatorname{ind} X$ *is an invariant number we have* $\operatorname{ind} X = \operatorname{Ind} X$.

Let α be an invariant number, then

(b) *If* $\operatorname{Ind} X = \alpha + 1$, *then* $\operatorname{ind} X = \operatorname{Ind} X$.

(c) *If* $\operatorname{Ind} X = \alpha$, *then* $\operatorname{ind} X = \operatorname{Ind} X$.

Proof. (a) follows from Theorem 1.1.

(b) If $\operatorname{ind} X \leqq \alpha$, then $\varphi(\operatorname{ind} X) \leqq \varphi(\alpha) = \alpha$, by property (e) of Lemma 1.1, and consequently, by Theorem 1.1 $\operatorname{Ind} X \leqq \alpha$, which contradicts our condition. Therefore, $\operatorname{ind} X = \alpha + 1$.

(c) If $\operatorname{ind} X < \alpha$, then by virtue of property (e) of Lemma 1.1, $\varphi(\operatorname{ind} X) < \alpha$. Then by Theorem 1.1 $\operatorname{Ind} X < \alpha$, which contradicts the condition. Hence $\operatorname{ind} X = \alpha$. □

THEOREM 1.2. *For any countable ordinal number $\beta < \omega_1$, there exists a weakly countable dimensional*[6] *compactum X_β, such that* $\operatorname{Ind} X_\beta = \beta$, $\operatorname{ind} X_\beta = \psi(\beta)$.

THEOREM 1.2′. *For any ordinal number $\beta < \omega_1$, there exists a weakly-countable dimensional compactum Y_β such that* $\operatorname{Ind} Y_\beta = \varphi(\beta)$, $\operatorname{ind} Y_\beta = \beta$.

Theorem 1.2′ follows from Theorem 1.2, since $\psi \circ \varphi(\beta) = \beta$ by Lemma 1.1 (n). We can set in Theorem 1.2′ $Y_\beta = X_{\varphi(\beta)}$. Therefore, we shall prove only Theorem 1.2.

Theorem 1.2 and Corollary 1.1 show that the function ψ possesses properties (i) and (ii). Theorem 1.2′ and Theorem 1.1 show that the function φ possesses properties (iii), (iv). We restrict our investigation to the field of compact spaces because every separable space X is contained in a compactum K such that $\operatorname{ind} K = \operatorname{ind} X$; $\operatorname{Ind} K = \operatorname{Ind} X$ (see [10], Luxemburg).

Problem. Let α, β be two ordinal numbers. Under what conditions does there exist a compactum X such that

$$\operatorname{ind} X = \alpha, \quad \operatorname{Ind} X = \beta? \tag{4}$$

[6] A space is called weakly countable dimensional if it is a union of a countable number of its closed finite dimensional subsets. In this work by finite dimensional space we mean a space with finite dimension dim.

From Theorem 1.1 it follows that the condition

$$\alpha \leqq \beta \leqq \varphi(\alpha) < \omega_1 . \tag{5}$$

is necessary. Is this condition sufficient? For this it is necessary and sufficient to prove that for any $\alpha < \omega_1$ there exists a compactum Y_α such that

$$\operatorname{Ind} Y_\alpha = \operatorname{ind} Y_\alpha = \alpha . \tag{6}$$

Indeed, by Theorem 1.2 there exists a compact Z such that

$$\operatorname{Ind} Z = \beta, \quad \operatorname{ind} Z = \psi(\beta) . \tag{7}$$

Let α satisfy the condition (5), then by properties (n), (j) of Lemma 1.1

$$\psi(\beta) \leqq \psi \circ \varphi(\alpha) = \alpha . \tag{8}$$

Put $X = Y_\alpha \cup Z$, $Y_\alpha \cap Z = \varnothing$. Then, by virtue of (6), (7) and (8), the condition (4) holds.

We begin now to prove preliminary results for Theorem 1.2.

2. Systems of general position.

DEFINITION 2.1. A system of finite dimensional sets $A = \{A_\mu: \mu \in \mathscr{M}\}$ is in general position (g. p.) if for any finite number of indexes $\mu_{(1)}, \cdots, \mu_{(k)}$ of $\mathscr{M}$ we have either

$$\dim \cap \{A\mu_{(i)}: i = 1, \cdots, k\} \leqq \max \{\dim A\mu_{(i)}: i = 1, \cdots, k\} - (k - 1)$$

or

$$\cap \{A\mu_{(i)}: i = 1, \cdots, k\} = \varnothing .$$

We shall write A is (g. p.) if A is in general position.

In this section we consider $A = \{A_\mu: \mu \in \mathscr{M}\}$ to be a locally countable system of closed sets in a finite dimensional space X, such that $X \in A$.

LEMMA 2.1. *Let $F = \{F_\nu: \nu \in \mathscr{N}\}$ be a locally countable system of closed sets in X, such that for every $\nu \in \mathscr{N}$ the system $A(\nu) = A \cup \{F_\nu\}$ is* g. p. *Then, for the set $\varphi = \cup \{F_\nu: \nu \in \mathscr{N}\}$ the system $B = A \cup \{\varphi\}$ is* g.p.

Proof. Let $A\mu_{(1)}, \cdots, A\mu_{(k)}$ be a finite subsystem of A. Since $A(\nu)$ is g.p., for every $\nu \in \mathscr{N}$ we have

$$\dim (\cap \{A\mu_{(i)}: i = 1, \cdots, k\} \cap F_\nu) \leqq \max \{\dim A\mu_{(i)}, \dim F_\nu: \\ i = 1, \cdots, k\} - k$$

if $F_\nu \cap \cap \{A\mu_{(i)}: i = 1, \cdots, k\} \neq \varnothing$. Since

$$\cap \{A\mu_{(i)}: i = 1, \cdots, k\} \cap \varphi = \cup\{\cap \{A\mu_{(i)}: i = 1, \cdots, k\} \cap F_\nu: \nu \in \mathscr{N}\}$$

and the closed sets

$$G_\nu = F_\nu \cap \cap \{A\mu_{(i)}: i = 1, \cdots, k\}, \quad \nu \in \mathscr{N}$$

form a locally countable system, then by virtue of the sum-theorem for dim we have:

$$\begin{aligned}\dim \{\cap \{A\mu_{(i)}: i = 1, \cdots, k\} \cap \varphi\} &\leqq \sup \{\dim \{\cap \{A\mu_{(i)}: i = 1, \cdots, k\} \\ &\qquad \cap F_\nu\}: \nu \in \mathscr{N}\} \\ &\leqq \sup \{\max \{\dim A\mu_{(i)}, \dim F_\nu: \nu \in \mathscr{N}, \cap \{A\mu_{(i)}: i = 1, \cdots, k\} \\ &\qquad \cap F_\nu \neq \varnothing\} - k\} \\ &\leqq \max \{\dim A\mu_{(i)}, \dim \varphi: i = 1, \cdots, k\} - k\end{aligned}$$

if $\varphi \cap \cap \{A\mu'_{i)}: i = 1, \cdots, k\} \neq \varnothing$. □

LEMMA 2.2. *Let $F \subset U \subset X$, where F is closed and U is open in X, and let A be a locally countable system of sets such that A is* g.p. *If U intersects no more than a countable number of elements of the system A, then there exists an open set W, such that*

(1) $$F \subset W \subset \bar{W} \subset U$$

and

(2) *the system $A \cup \{Fr W\}$ is* g.p.

Proof. Let C be a subsystem of the system A, consisting of all sets intersecting U, and let D be a system consisting of all intersections of finite collection sets in C. Since, by hypothesis, the system C is no more than countable, the system D is also no more than countable. Then, see [14] Morita, there exists an open set W such that condition (1) holds and

(3) $$\dim (Fr W \cap L) \leqq \dim L - 1 \text{ for } L \in D\,.$$

Let $A\mu_{(1)}, \cdots, A\mu_{(k)}$ be a finite collection of elements of A such that $L = \cap \{A\mu_{(i)}: i = 1, \cdots, k\}$. Then, from property (3), it follows that

$$\begin{aligned}\dim Fr W \cap \cap \{A\mu_{(i)}: i = 1, \cdots, k\} &\leqq \dim \cap \{A\mu_{(i)}: i = 1, \cdots, k\} - 1 \\ &\leqq \max \{\dim Fr W, \dim A\mu_{(i)}: i = 1, \cdots, k\} - k\,.\end{aligned}$$ □

LEMMA 2.3. *The assertion of Lemma 2.2 is true without the assumption that U intersects no more than a countable number of elements of the system A.*

Proof. Let $\mathscr{U} = \{U_\beta: \beta \in B\}$ be a locally finite open covering of X, such that every element $U_\beta \in U$ intersects no more than a countable number of elements of A and

$$\text{if } U_\beta \cap F \neq \varnothing \text{ then } U_\beta \subset U,\ \beta \in B\,. \tag{4}$$

Let $F = \{F_\beta: \beta \in B\}$ be a combinatorial refinement[7] of the covering $\mathscr{U}$, and let F_β be closed for every $\beta \in B$. Then, by Lemma 2.2 for any $\beta \in B$ there exists a set W_β such that

$$F_\beta \subset W_\beta \subset \bar{W}_\beta \subset U_\beta \tag{5}$$

and the system

$$A(\beta) = A \cup \{Fr\, W_\beta\} \text{ is g.p.} \tag{6}$$

We set $W = \cup\{W_\beta: W_\beta \cap F \neq \varnothing; \beta \in B\}$. From properties (5), (6) it follows that $F \subset W \subset \bar{W} \subset U$. Since $\mathscr{U}$ is locally finite and (5) is true it follows that

$$Fr\, W \subset H = \cup\{Fr\, W_\beta: \beta \in B\}\,.$$

From Lemma 2.1 and (6) it follows that the system $A \cup \{H\}$ is g.p. Consequently, $A \cup \{Fr\, W\}$ is g.p.

PROPOSITION 2.1. *Let U be an open covering of a space X. Then there exists a locally finite open covering $W = \{W_\nu: \nu \in N\}$ of X, which is a refinement of U, and such that*

(7) *The system $B = A \cup \{Fr\, W_\nu: \nu \in \mathscr{N}\}$ is* g.p.

(8) $\dim Fr\, W_\nu < \dim X$.

Proof. Let $V = \{V_\nu: \nu \in \mathscr{N}\}$ be a locally finite open refinement of U, and $F = \{F_\nu: \nu \in \mathscr{N}\}$ be a combinatorial closed refinement of V. We can suppose that the set of indexes $\mathscr{N}$ is well ordered. We shall construct by induction open sets W_ν such that properties (7), (8) hold and

$$F_\nu \subset W_\nu \subset \bar{W}_\nu \subset V_\nu \quad \text{for } \nu \in \mathscr{N}\,. \tag{9}$$

Let $\nu = 0$. Then by Lemma 2.3 there exists an open set W_0 satisfying condition (2) for $W = W_0$ and condition (9) for $\nu = 0$. Suppose that for all $\nu < \nu_0$ we have constructed sets W_ν satisfying condition (9) and such that the system $A(\nu) = A \cup \{Fr\, W_{\nu'}: \nu' < \nu\}$ is g.p. Then, obviously, the system

$$C = A \cup \{Fr\, W_\nu: \nu < \nu_0\} \quad \text{is g.p.}$$

[7] i.e. $F_\beta \subset U_\beta$ for every $\beta \in B$.

Since A is locally countable and W is a locally finite system, C is locally countable. By Lemma 2.3 (we now set $A = C$) there exists an open set W_{ν_0} such that the system $A \cup \{Fr W_\nu : \nu \leqq \nu_0\}$ is g.p. Hence, sets W_ν satisfying condition (7) have been constructed. Since $X \in A$, condition (8) follows from (7). □

COROLLARY 2.1. *Let $\mathscr{U}_i$, $i = 1, 2, \cdots$, be a countable collection of open coverings of X. Then there exist locally finite open coverings $\mathscr{V}_i = \{V^i_\mu : \mu \in \mathscr{M}\}$ such that:*

(i) *$\mathscr{V}_i$ is a refinement of $\mathscr{U}_i$*

(ii) *The system $A \cup \{Fr V^i_\mu : i = 1, 2, \cdots, \mu \in \mathscr{M}\}$ is g.p.*

Proof. We can construct the coverings $\mathscr{V}_i$ by induction, using Proposition 2.1. □

COROLLARY 2.2. *Every finite dimensional space X has a σ-locally finite open basis with boundaries in general position and having dimension $<$ dim X.*

Proof. Let $\mathscr{U}_i$ be a covering of mesh$\mathscr{U}_i$[8] $< 1/i$. Then we obtain our assertion using Corollary 2.1. □

We denote by $d_k(X)$ the greatest lower bound of all numbers ε such that there exists an open covering $U(\varepsilon)$ of X with order $U(\varepsilon) \leqq k$ and mesh $U(\varepsilon) < \varepsilon$. The number $d_k(X)$ is called the kth-coefficient of Urysohn of X.

COROLLARY 2.3. *Let $\{\mathscr{U}_i : i = 1, 2, \cdots\}$ be a countable collection of coverings of X and let $\{\varepsilon_i\}$ be a sequence of positive numbers. Then there exist closed sets $C_i \subset X$ such that*

(i) *the system $A \cup \{C_i : i = 1, 2, \cdots\}$ is g.p.*

(ii) *the set $X \backslash C_i$ is a union of disjoint open sets with diameter $< \varepsilon_i$ such that each of them is contained in some set $U_\alpha \in \mathscr{U}_i$.*

(iii) $\dim C_i \leqq \dim X - 1$.

(iv) $d_1(X \backslash C_i) < \varepsilon_i$.

Proof. Let $\mathscr{W}_i$ be an open refinement of $\mathscr{U}_i$ and let mesh $\mathscr{W}_i < \varepsilon_i/2$, then by Corollary 2.1 there exist locally finite open coverings of X $\mathscr{V}_i = \{V^i_\nu : \nu \in \mathscr{N}_i\}$ such that:

$$\begin{aligned} &\mathscr{V}_i \text{ is a refinement of } \mathscr{W}_i \text{ and } \mathscr{U}_i, \text{ mesh } \mathscr{V}_i < \varepsilon_i/2, \\ &\qquad \text{system } B = A \cup \{Fr V^i_\nu : \nu \in \mathscr{N}_i,\ i = 1, 2, \cdots\} \end{aligned} \tag{10}$$

is g.p. Put

[8] i.e., diameter $U < 1/i$ for any $U \in \mathscr{U}_i$

(11) $$C_i = \cup \{Fr V_\nu^i : \nu \in \mathcal{N}\} .$$

Consequently applying Lemma 2.1 we obtain that the system $A \cup \{C_i : i \leqq k\}$ is g.p. for each $k = 1, 2, \cdots$. Therefore, property (i) also holds. Property (ii) follows from (10), (11). Then, property (iii) follows from (i), because $X \in A$, and $X \cap C_i = C_i$. Property (iv) follows from (ii). $\square$

LEMMA 2.4. *Let $F = \{F_\mu : \mu \in \mathcal{M}\}$ be a locally finite collection of closed sets in space X and let $\mathcal{U} = \{U_\mu : \mu \in \mathcal{M}\}$ be a collection of open sets, such that*

(i) *$U_\mu \supseteqq F_\mu$ for any $\mu \in \mathcal{M}$.*

Then there exists a collection $\mathcal{W} = \{W_\mu : \mu \in \mathcal{M}\}$ of open sets such that

(ii) $F_\mu \subset W_\mu \subset \bar{W}_\mu \subset U_\mu (\mu \in \mathcal{M})$.

(iii) *If $\cap \{F\mu_{(i)} : i = 1, \cdots, k\} = \varnothing$ then $\cap \{W_{\mu(i)} : i = 1, \cdots, k\} = \varnothing (k = 1, 2, \cdots)$.*

Proof. Since a collection F is locally finite, we can select for every point $x \in X$ an open set $O_x \ni x$ such that:

(12) $$\text{If } x \notin F\mu, \text{ then } Ox \cap F\mu = \varnothing .$$

Since X is a metric space and, consequently, paracompact, we can find an open covering $\mathcal{V}$ of a space X such that $\mathcal{V}$ is a star refinement of the covering $\{Ox : x \in X\}$. We consider a system of sets: $\{V_\mu = St(F_\mu, \mathcal{V}) : \mu \in \mathcal{M}\}$[9]. We shall prove that

(13) $$\text{If } \cap \{F_{\mu(i)} : i = 1, \cdots, k\} = \varnothing \text{ then } \cap \{V_{\mu(i)} : i = 1, \cdots, k\} , \quad (k = 1, 2, \cdots) .$$

Suppose that $\cap \{V_{\mu(i)} : i = 1, \cdots, k\} \neq \varnothing$ and $x \in \cap \{V_{\mu(i)} : i=1, \cdots, k\}$. Then the set $st(x, V)$ is contained in the open set Oy for some point $y \in Y$. Since $x \in \cap \{V_{\mu(i)} : i = 1, \cdots, k\}$ then

$$st(x, \mathcal{W}) \cap F_{\mu(i)} \neq \varnothing \quad \text{for } i = 1, \cdots, k .$$

Consequently, $Oy \cap F_{\mu(i)} \neq \varnothing$ and by virtue of (12)

$$y \in \cap \{F_{\mu(i)} : i = 1, \cdots, k\} .$$

This proves property (13). Let us take for every $\mu \in \mathcal{M}$ a set W_μ such that $\bar{W}_\mu \subset V_\mu$ and such that property (ii) holds. Then property (iii) follows from (13). $\square$

LEMMA 2.5. *Let $F = \{F_\mu : \mu \in \mathcal{M}\}$ be a locally finite collection*

[9] By $st(A, B)$, where $A \subset X$, and B is a system of sets in X, we denote a star of a set A with respect to a system B.

of closed sets of order r in a space X and $\mathscr{U} = \{U_\mu: \mu \in \mathscr{M}\}$ be a collection of open sets such that $F_\mu \subset U_\mu$ for $\mu \in \mathscr{M}$. Then there exist closed sets $C_i i = 1, \cdots, r$ such that

(i) *Every set $C_i i = 1, \cdots, r$ is a union of closed sets $\{C^i_\nu: \nu \in \mathscr{M}\}$ forming a discrete system in X, and every set C^i_ν is contained in some set $U_\mu \in \mathscr{U}$.*

(ii) $\bigcup_{i=1}^r C_i = \bigcup \{F_\mu: \mu \in \mathscr{M}\}$.

Proof. We shall prove the lemma by induction on r. Let $\Phi = \bigcup\{F_\mu: \mu \in \mathscr{M}\}$. For $r = 1$ the lemma is true because we can consider $C_1 = \Phi$. Assume that the lemma has been proved for $(r-1)$ and $W = \{W_\mu: \mu \in \mathscr{M}\}$ be a collection of open sets, satisfying the hypothesis of Lemma 2.4. Obviously,

$$\text{(14)} \qquad \text{The system } W(1) = \{\bar{W}_\mu: \mu \in \mathscr{M}\} \text{ has order} \leqq r\,.$$

Moreover, the system $W(2) = \{Fr\, W_\mu: \mu \in \mathscr{M}\}$ has an order $\leqq r-1$ on Φ. Indeed, let $x \in \Phi \cap \bigcap \{Fr\, W_{\mu(i)}: i = 1, \cdots, r\}$ then there exists such index $\mu_{(0)}$ that $x \in W_{\mu(0)} \cap \Phi$. Therefore $x \in \bigcap\{\bar{W}_{\mu(i)}: i=0, \cdots, r\}$ which contradicts the condition (14). By Lemma 2.4 we can construct a locally finite open collection of sets $Q = \{Q_\mu: \mu \in \mathscr{M}\}$ such that

$$\text{(15)} \qquad Q \text{ has an order } \leqq r-1 \text{ on } \Phi \text{ and}$$

$Fr\, W_\mu \subset Q_\mu \subset U_\mu (\mu \in \mathscr{M})$. For any μ we consider an open set P_μ such that

$$\text{(16)} \qquad Fr\, W_\mu \subset P_\mu \subset \bar{P}_\mu \subset Q_\mu \subset U_\mu\,.$$

Set

$$\text{(17)} \qquad C_1 = \Phi \backslash \bigcup \{P_\mu: \mu \in \mathscr{M}\}$$

$$\text{(18)} \qquad D_\nu = (\bar{W}_\nu \cap \Phi) \backslash \bigcup \{P_\mu: \mu \in \mathscr{M}\}\,.$$

Then from conditions (17), (18) it follows that

$$\text{(19)} \qquad C_1 = \bigcup \{D_\mu: \mu \in \mathscr{M}\}\,.$$

Assume that the set $\mathscr{M}$ is well ordered and put

$$\text{(20)} \qquad C'_\mu = D_\mu \backslash \bigcup \{D_\nu: \nu < \mu\}\,.$$

By condition (16) we have $\bar{W}_\nu \cup P_\nu = W_\nu \cup P_\nu$, then, by virtue of (18), (20)

$$\text{(21)} \qquad \begin{aligned} C'_\mu &= D_\mu \backslash (\bigcup\{\bar{W}_\nu: \nu < \mu\} \cup \{P_\mu: \mu \in \mathscr{M}\}) = D_\mu \\ &\backslash \bigcup \{W_\nu \cup P_\nu: \nu < \mu\} \cup (P_\mu: \mu \in M\}\,. \end{aligned}$$

From conditions (20), (21) it follows that sets C'_μ are closed and disjoint. Since $C'_\mu \subset D_\mu \subset W_\mu$ the system $\{C'_\mu: \mu \in \mathscr{M}\}$ is locally finite and, consequently, discrete. Since by virtue of (19), (20)

$$\cup \{C'_\mu: \mu \in \mathscr{M}\} = \cup \{D_\mu: \mu \in \mathscr{M}\} = C_1$$

the set C_1 satisfies the condition (i) for $i = 1$. From (15), (16) it follows that the system $P = \{\bar{P}_\mu: \mu \in \mathscr{M}\}$ has order $\leqq r - 1$ on Φ. Applying inductive assumption to locally finite closed system P and open system $Q = \{Q_\mu: \mu \in \mathscr{M}\}$ we can find closed sets $C_2, \cdots, C_r$ satisfying conditions (i) (since $Q_\mu \subset U_\mu$) and the following condition:

$$\cup \{C_i: i = 2, \cdots, r\} = \cup \{\bar{P}_\mu \cap \Phi: \mu \in \mathscr{M}\} .$$

By property (17) we obtain the equality $\cup \{C_i: i = 1, \cdots, r\} = \Phi$. □

COROLLARY 2.4. *Let U be an open covering of a space X, and* ord $U \leqq r$. *Then there exists a closed refinement of the covering U, consisting of r discrete systems.*

Proof. Using paracompactness of a space X we can get closed locally finite combinatorial refinement F of the covering U with order $F \leqq r$. Then, our corollary follows from Lemma 2.5. □

COROLLARY 2.5. *For a space X the following conditions are equivalent:*

(a) $d_k(X) < \varepsilon$.

(b) *X is a union of k closed sets $C_i (1 \leqq i \leqq k)$ such that every set C_i is a union of closed sets $\{C^i_\mu: \mu \in \mathscr{M}\}$ forming a discrete collection in X, and diameter $C^i_\mu < \varepsilon$ for each pair i, μ.*

(c) *X is a union of k closed sets $C_i (1 \leqq i \leqq k)$ such that $d_1(C_i) < \varepsilon$ for every $i \leqq k$.*

(d) *X is a union of k open sets $U_i (1 \leqq i \leqq k)$ such that $d_1(U_i) < \varepsilon$ for each $i \leqq k$.*

Proof. (a) ⟹ (b). Let $d_k(X) < \varepsilon$, then there exists an open covering U with mesh $U < \varepsilon$ and order $U \leqq k$. Then, using Corollary 2.4 we obtain assertion (b).

(b) ⟹ (c). For proving it is sufficient to note that sets C^i_μ are open in C_i.

(c) ⟹ (d). By virtue of (c) for each $i \leqq k$ there exists a covering $\mathscr{V}_i = \{C^i_\mu: \mu \in \mathscr{M}\}$ where sets C^i_μ are open in C_i and disjoint, and have the diameter $< \varepsilon$. Therefore the collection $\mathscr{V}_i$ is discrete in X. Since X is paracompact, we can find open sets $U^i_\mu\ \mu \in \mathscr{M}$ such that

$$C^i_\mu \subset U^i_\mu, \text{ diam } U^i_\mu < \varepsilon, \ U^i_\mu \cap U^i_{\mu'} = \phi \text{ for } \mu \neq \mu' .$$

Put $U_i = \cup\{U^i_\mu : \mu \in \mathscr{M}\}$. Then, the assertion (d) clearly holds.

Since a union of k system of order $\leqq 1$ has an order $\leqq k$, we obtain the implication (d) $\Rightarrow$ (a). □

3. The Main Lemma.

LEMMA 3.1 (The Main Lemma). *Let $S = \{S_i: i = 1, 2, \cdots\}$ be a countable locally finite system of closed sets in n-dimensional space X and let S be* g.p, *and $\mathscr{U} = \{U_\mu: \mu \in \mathscr{M}\}$ be an arbitrary open covering of X. Let also* $\dim S_i \leqq n-1$, $i = 1, 2, \cdots, k \leqq n-1$, $n = 2, 3, 4, \cdots$, $D(n, k) = [n/(k+1)]$, $\varepsilon > 0$ *and let $\{F_r, G_r\}$ ($r = 1, \cdots, D(n, k)$) be a system of pairs of disjoint closed sets in X such that for any i and for $r \leqq D(n, k)$*

(1) *either $S_i \cap F_r = \varnothing$ or $S_i \cap G_r = \varnothing$.*

Then there exist sets $D_r 1 \leqq r \leqq D(n, k)$ such that

(2) *D_r is a partition between F_r and G_r.*

(3) *For the set $R = \cap\{D_r: 1 \leqq r \leqq D(n, k)\}$ we have $d_k(R \cap S_i) < \varepsilon (i = 1, 2, \cdots)$.*

(4) $\dim R \leqq \dim X - D(n, k)$.

(5) *The set $R \cap S_i$ is a union of k closed sets $L^i_1, \cdots, L^i_k$ such that every set L^i_j is a union of closed sets forming a discrete system, and every element of this system is contained in some set $U_\mu \in \mathscr{U}$.*

Proof. By virtue of Corollary 2.3 we can find closed sets $C^j_i j$, $i = 1, 2, \cdots$, such that for any pair (i, j)

(6) the system $S' = S \cup \{C^j_i: i, j = 1, 2, \cdots\}$ is g.p.

(7) $\dim C^j_i \leqq n - 1$.

(8) $d_1(X \backslash C^j_i) < \varepsilon$.

(9) The set $X \backslash C^j_i$ is a union of collection $B^j_i = \{U^j_{i\alpha}: \alpha \in A_{ij}\}$ of disjoint open sets and each $U^j_{i\alpha}$ is contained in some element $U_\mu \in \mathscr{U}$.

Let us consider a system of closed sets

$$\mathscr{P} = \{P_i: P_i = S_i \cap \cap\{C^j_i: j = 1, 2, \cdots, k\}\} . \tag{10}$$

Since

$$S_i \backslash P_i = \cup\{S_i \backslash C^j_i: j = 1, \cdots, k\} \subset \cup\{X \backslash C^j_i: j = 1, \cdots, k\} \tag{11}$$

by virtue of (8) and Corollary 2.5

$$d_k(S_i \backslash P_i) < \varepsilon . \tag{12}$$

By virtue of (6), (7), (10) we have

$$\begin{aligned}&\dim \cap \{P_{i(m)}: 0 \leqq m \leqq D(n, k)\}\\ &\quad = \dim \cap \{S_{i(m)} \cap \cap \{C^j_{i(m)}: j = 1, \cdots, k\}: 0 \leqq m \leqq D(n, k)\}\\ &\quad \leqq \max \{\dim S_{i(m)}, \dim C^j_{i(m)}: 0 \leqq m \leqq D(n, k)\}\\ &\qquad - (D(n, k) + 1)(k + 1) + 1\\ &\quad = (n - 1) - (D(n, k) + 1)(k + 1) + 1 < n\\ &\qquad - ([n/(k + 1)] + 1)(k + 1) < 0 .\end{aligned}$$

Consequently,

(13) The order of $\mathscr{P} \leqq \mathscr{D}(n, k)$.

Since the system S is locally finite and $P_i \subset S_i$ the system $\mathscr{P}$ is also locally finite. By virtue of (1) there exist such open sets $O_i \supset P_i$ that

(14) either $O_i \cap F_r = \emptyset$ or $O_i \cap G_r = \emptyset$ $1 \leqq r \leqq D(n, k)$.

By Lemma 2.5 there exist $D(n, k)$ closed sets $C_1, \cdots, C_{D(n,k)}$ such that

(15) $\cup \{C_r: r = 1, \cdots, D(n, k)\} = P = \cup \{P_i: i = 1, 2, \cdots\}$.

(16) Every set C_r is a union of a discrete collection of closed sets $\{C^\mu_r: \mu \in \mathscr{M}_r\}$ and each set C^μ_r is contained in some set O_i.

From (14) and (16) it follows that

(17) Either $C^\mu_r \cap F_r = \emptyset$ or $C^\mu_r \cap G_r = \emptyset$ for $r \leqq D(n, k)$.

Put $F'_r = F_r \cup \{C^\mu_r: C^\mu_r \cap F_r \neq \emptyset,\ \mu \in \mathscr{M}_r\}$, $G'_r = G_r \cup \{C^\mu_r: C^\mu_r \cap F_r = \emptyset,\ \mu \in \mathscr{M}_r\}$. Obviously, F'_r and G'_r are disjoint closed sets and

(18) $F'_r \cup G'_r \supset C_r$

(19) $F'_r \supset F_r, \quad G'_r \supset G_r$.

Therefore, we can find sets D_r $r \leqq D(n, k)$ such that D_r is a partition between F'_r and G'_r and

(20) $\dim \cap \{D_r: r = 1, \cdots, D(n, k)\} \leqq \dim X - D(n, k)$.

Put

(21) $R = \cap \{D_r: r = 1, \cdots, D(n, k)\}$.

Then condition (4) follows from (21) and (20). The condition (2) follows from (19). From (18) it follows that $D_r \cap C_r = \emptyset$. Consequently, by virtue of (15), (21)

$$R \cap P = \cap \{D_r : r = 1, \cdots, D(n, k)\} \\ \cap (\cup \{C_r : r = 1, \cdots, D(n, k)\}) = \varnothing .$$

Hence

$$R \cap S_i \subset S_i \backslash P \subset S_i \backslash P_i \tag{22}$$

and by virtue of (12) the condition (3) is satisfied. By virtue of (9), (11), (22)

$$R \cap S_i \subset \cup \{U^j_{i\alpha} \cap R \cap S_i : U^j_{i\alpha} \in B^j_{i'} j = 1, \cdots, k\}$$

and collection $\mathscr{V} = \{U^j_{i\alpha} \cap R \cap S_i : U^j_{i\alpha} \in B^j_{i'} j = 1, \cdots, k\}$ has an order $\leqq k$ and any element of $\mathscr{V}$ is contained in some $U_\mu \in \mathscr{U}$. By virtue of Corollary 2.4 we now obtain property (5). □

LEMMA 3.2. *Let $V \subset Z$ and $V = K \cup \cup \{R_i : i = 1, 2, \cdots\}$, where the sets R_i are open-closed in V, K is a closed set in Z and*

(i) *$R_i \cap R_j = R_i \cap K = \varnothing$ for $i \neq j$.*

If

(ii) $\lim_{i\to\infty} d_1(R_i) = 0$

then for any point $x \in K$ there is an arbitrary small neighborhood Ox in Z such that

(iii) *$FrOx \subset Z \backslash \cup \{R_i : i = 1, 2, \cdots\}$.*

Proof. Let $\varepsilon > 0$. By virtue of (i) and (ii) there exist such disjoint open in V sets $R_{i\alpha} i = 1, 2, \cdots, \alpha \in A_i$ so that $R_i = \cup \{R_{i\alpha} : \alpha \in A\}$, diam $R_{i\alpha} < \varepsilon_i$, $\lim_{i\to\infty} \varepsilon_i = 0$. Let N be an integer such that for $i > N \varepsilon_i < \varepsilon/4$. Then there exists a number $\delta (0 < \delta < \varepsilon/4)$ such that the neighborhood $O_\delta(x)$[10] doesn't intersect R_i for $i < N$. Put

$$Ox = O_\delta(x) \cup \cup \{R_{i\alpha} : R_{i\alpha} \cap O_\delta(x) \neq \varnothing, \ i = 1, 2, \cdots, \alpha \in A_i\} . \tag{24}$$

Obviously the diameter $Ox \leqq \operatorname{diam} O_\delta + 2 \sup \{\operatorname{diam} R_{i\alpha} : i = 1, 2, \cdots, \alpha \in A_i\} < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since sets $R_{i\alpha}$ are disjoint property (iii) follows from equality (24). □

LEMMA 3.3. *Let a space Z have the representation:*

(i) *$Z = K \cup \cup \{L^j_i : j = 1, \cdots, k, i = 1, 2, \cdots\}$ where K is a finite dimensional closed set and*

(ii) *Sets $L^j_i (i = 1, 2, \cdots)$ are open-closed in $K^j = K \cup \cup \{L^j_i : i = 1, 2, \cdots\}$, are open-closed in $K^j = K \cup \cup \{L^j_i : i = 1, 2, \cdots\}$, $K \cap L^j_i = L^j_i \cap L^j_{i'} = \varnothing$ for $i \neq i'$.*

(iii) $\lim_{i\to\infty} d_1(L^j_i) = 0$.

Then, if

[10] For any set $\mathscr{M} \subset X$ by $O_\delta(\mathscr{M})$ we denote a δ-neighborhood of the $\mathscr{M}$ in a space X.

(iv) $\operatorname{ind}(Z\backslash K) \leqq \alpha,\ \alpha \geqq \omega_0$

we have

(v) $\operatorname{ind} Z \leqq \alpha + (k-1)$.

Proof. By virtue of (iv) it is sufficient to prove that

$$\operatorname{ind}_x Z \leqq \alpha + (k-1) \text{ if } x \in K . \tag{25}$$

We shall prove this inequality by induction on k. Suppose that $k = 1$, then by Lemma 3.2 there is an arbitrary small neighborhood of x with boundary in K. Therefore, by virtue of finite dimensionality of K we obtain (25). Suppose now $k > 1$, and $\varepsilon > 0$. Then by Lemma 3.2 for every $x \in K$ there exists a neighborhood $Ox \ni x$ of the diameter $< \varepsilon$ such that $FrOx \subset Z\backslash \cup \{L_i^k: i = 1, 2, \cdots\} = Z_1$. (We consider that in Lemma 3.2 $L_i^k = R_i$.) Since $Z_1 \subset K \cup \cup \{L_i^j: j = 1, \cdots, k-1, i = 1, 2, \cdots\}$, by inductive assumption we obtain $\operatorname{ind} Z_1 \leqq \alpha + (k-2)$. Consequently, $\operatorname{ind} FrOx \leqq \alpha + (k-2)$. Hence, the inequality (25) and Lemma 3.3 are proved. □

LEMMA 3.4. *Let a space Z have the representation*

$$Z = K \cup \cup \{\Theta_i: i = 1, 2, \cdots\}$$

where K is a finite dimensional set, and sets Θ_i are open-closed in Z and disjoint. If $\lim_{i\to\infty} d_k(\Theta_i) = 0$, $K = Z\backslash \cup \{\Theta_i: i = 1, 2, \cdots\}$ then $\operatorname{ind} Z \leqq \alpha + (k-1)$ where $\alpha \geqq \sup\{\operatorname{ind}\Theta_i: i = 1, 2, \cdots\}$.

Proof. Let $d_k(\Theta_i) < \varepsilon_i$ and $\lim_{i\to\infty} \varepsilon_i = 0$ then by Corollary 2.5 there exist closed sets $L_i^j j = 1, \cdots, k$, such that $d_1(L_i^j) < \varepsilon_i$, $\cup \{L_j^i: j = 1, \cdots, k\} = \Theta_i (i = 1, 2, \cdots)$. Therefore $Z = K \cup \cup \{L_i^j: i = 1, 2 \cdots, j = 1, \cdots, k\}$. Obviously, all conditions of Lemma 3.3 are satisfied. Consequently Lemma 3.4 follows from Lemma 3.3. □

4. Compacta R_β.

DEFINITION 4.1. Let $\{Y_\mu: \mu \in \mathscr{M}\}$ be a collection of spaces and $\{p\}$ be some point. Then by $\omega(p, Y_\mu: \mu \in \mathscr{M})$ we denote a space

$$\{p\} \cup \cup \{Y_\mu: \mu \in \mathscr{M}\},\ Y_\mu \cap Y_{\mu'} = Y_\mu \cap \{p\} = \varnothing \text{ for } \mu \neq \mu'$$

where in set $\cup \{Y_\mu: \mu \in \mathscr{M}\}$ the topology is defined as in a discrete union of spaces $Y_{\mu'}$ and in the point $\{p\}$ it is defined by the open basis:

$$\omega(p, Y_\mu: \mu \in \mathscr{M})\backslash \cup \{Y_{\mu(i)}: i = 1, 2, \cdots, k\},\ \mu(i) \in \mathscr{M},\ k = 1, 2, \cdots .$$

We note, that if the set $\mathscr{M}$ is countable, and Y_μ are compact metric spaces, then $\omega(p, Y_\mu\colon \mu \in \mathscr{M})$ is also a compact metric space.

LEMMA 4.0. *If the set $\mathscr{M}$ is countable, then for any metric on the set $\omega(Y_\mu\colon \mu \in \mathscr{M})$ and any $\varepsilon > 0$ the inequality* $\operatorname{diam} Y_\mu < \varepsilon$ *holds for all but only finitely many $\mu \in \mathscr{M}$.*

The lemma is evident.

DEFINITION 4.2. (See [14] Luxemburg). Let $(A_{n+1}, B_{n+1}), \cdots, (A_{n+k}, B_{n+k})$ be a fixed collection of pairs of opposite faces in Euclidean cube I^{n+k}. If C_{n+i} is a partition between A_{n+i} and B_{n+i} $i = 1, \cdots, k$ and

$$\dim \cap \{C_{n+i}\colon i = 1, \cdots, k\} = n$$

then the set $Q = \cap \{C_{n+i}\colon i = 1, \cdots, k\}$ is called n-dimensional pseudocube. (We note that Q is always of dimension $\geqq n$, Hurewicz ([4], p. 40).) The rest n pairs A_i, B_i of opposite faces in cube I^{n+k} are called improper faces of pseudocube. The intersections $F_i = A_i \cap Q$, $G_i = B_i \cap Q (1 \leqq i \leqq n)$ are called opposite faces of the pseudocube Q.

LEMMA 4.1. *The product of n-dimensional and k-dimensional pseudocubes is $(n + k)$ dimensional pseudocube.*

The lemma is evident.

LEMMA 4.2. *Let Q^n be an n-dimensional pseudocube and $(F_i, G_i) i = 1, \cdots, l$, $l \leqq n$ are l-pairs of its opposite faces. Then for any collection of partitions C_i between these pairs*

(i) $\dim \cap \{C_i\colon i = 1, \cdots, l\} \geqq n - l$.

(ii) *If* $\dim \cap \{C_i\colon i = 1, \cdots, l\} \leqq n - l$ *then the set* $R = \cap \{C_i\colon i = 1, \cdots, l\}$ *is an $(n - l)$ dimensional pseudocube.*

Proof. Let the pseudocube Q^n has a representation

$$Q^n = \cap \{D_{n+j}\colon j = 1, \cdots, k\}$$

where sets D_{n+j} are partitions between opposite spaces of some cube I^{n+k}. Let

$$A_i \cap Q^n = F_i, \; B_i \cap Q^n = G_i, \; i = 1, \cdots, n$$

where A_i, B_i are proper faces of the pseudocube Q^n. Let D_i be a partition between A_i and B_i in the cube I^{n+k} such that $D_i \cap Q^n = C_i$. Then

$$\cap \{C_i: i = 1, \cdots, l\} = \cap \{D_{j+n}: j = 1, \cdots, k\} \\ \cap \cap \{D_i: i = 1, \cdots, l\} = R .$$

The set R is an intersection of $(k + l)$ partitions between opposite faces in cube I^{n+k}. Consequently $\dim R \geqq n + k - (k + l) = n - l$ (see Hurewicz [4] p. 40) and if also $\dim R \leqq n - l$ then R is a pseudocube. □

In this and in the next section by α we mean a limit ordinal number or 0, and $n = 0, 1, 2, \cdots$.

DEFINITION 4.3. We shall define for every ordinal number $\beta<\omega$, a class ρ_β consisting of compacta. Suppose that β is a finite number, then ρ_β consists of all β-dimensional pseudocubes. If β is a limit number, then ρ_β consists of all compacta R_β having the following representation

$$R_\beta = \omega(p_\beta; R_\gamma: \gamma \in \Gamma_\beta)$$

where p_β is an extra point, $R_\gamma \in \rho_\gamma$ and Γ_β some cofinal subset in $\phi_\beta = \{\gamma: \gamma < \beta\}$. If $\beta = J(\beta)+K(\beta)$, $K(\beta)>0$, then class ρ_β consists of all compacta R_β such that

$$R_\beta = R_{J(\beta)} \times R_{K(\beta)}, \ R_{J(\beta)} \in \rho_{J(\beta)}, \ R_{K(\beta)} \in \rho_{K(\beta)} .$$

In what follows R_β will denote an element of class ρ_β. Let us introduce some notations. In this and following sections we consider α to be a limit number or 0 and $n = 0, 1, 2, \cdots$. If $\beta = \xi + 1$ then put $\beta - 1 = \xi$. For every compactum $R_{\alpha+n}$ we have by definition: $R_{\alpha+n} = R_\alpha \times R_n$ for some $R_\alpha \in \rho_{\alpha'}$ $R_n \in \rho_n$. Obviously, $R_{\alpha+n}$ has the representation:

$$R_{\alpha+n} = \bar{R}_n \cup \cup \{R_\gamma \times R_n: \gamma \in \Gamma_\alpha\} \tag{1}$$

where Γ_α is some cofinal subset in $\phi_\alpha = \{\gamma: \gamma < \alpha\}$, and $\bar{R}_n = \{p_\alpha\} \times R_n$ and p_α is an extra point in R_α. Representation (1) we shall call a standard representation of a compactum $R_{\alpha+n}$.

DEFINITION 4.4. (See Smirnov [17].) For any $\beta < \omega_1$, we define a class of compacta $\prod_\beta$. If $\beta < \omega_0$, then $\prod_\beta$ consists of all β-dimensional compacta. If β is a limit number, then $\prod_\beta$ consists of all compacta X such that $X = \omega(p; Y_\gamma: Y_\gamma \in \prod_\gamma: \gamma < \beta)$ where p is an extra point. Moreover, if $\beta = J(\beta) + K(\beta)$, $X \in \prod_{J(\beta)}$, $Y \in \prod_{K(\beta)}$ then $X \times Y \in \prod_\beta$. Thus, for any $\beta < \omega_1$, we have defined the class $\prod_\beta$.

In [17], (Smirnov) it was proved, that for any $\beta < \omega_0$

(2) $$\operatorname{Ind} X \leqq \beta \text{ if } X \in \prod_\beta .$$

LEMMA 4.3. *Every compactum R_β is contained in some compactum $X \in \prod_\beta$.*

Proof. Let $Y \in \prod_\beta$, then from Definitions 4.3 and 4.4 it follows that a discrete sum $X = Y \oplus R_\beta$ is an element of $\prod_\beta$. □

COROLLARY 4.1. $\operatorname{ind} R_\beta \leqq \operatorname{Ind} R_\beta \leqq \beta$.

Proof. It is directly derived from (2) and Lemma 4.3. □

LEMMA 4.4. *For any q-dimensional compactum K the following inequality holds*:

$$\operatorname{Ind} R_\beta \times K \leqq \operatorname{Ind} R_\beta + q \leqq \beta + q .$$

Proof. Let $\beta = \alpha + n$, then $R_\beta = R_{\alpha+n} = R_\alpha \times R_n$, $R_\beta \times K = R_\alpha \times (R_n \times K)$. Let $X \supset R_\alpha$ and $X \in \prod_\alpha$ then by Definition 4.4 $X \times (R_n \times K) \in \prod_{\alpha+s}$ $s = \dim R_n \times K \leqq n + q$. From property (2) it follows that $\operatorname{Ind}(X \times R_n \times K) \leqq \alpha + s \leqq \alpha + n + q = \beta + q$. Since, obviously $R_\beta \times K \subset X \times R_n \times K$, $\operatorname{Ind} R_\beta \times K \leqq \beta + q$. □

Let (1) be a standard representation of a compactum $R_{\alpha+n}$, then by i we denote the natural homeomorphism $i: \bar{R}_n \to R_n$.

LEMMA 4.5. *Let* (1) *be a standard representation of a compactum $R_{\alpha+n}$ and F be a closed subset in $R_{\alpha+n}$ such that $F \subseteqq \bar{R}_n \cup \bigcup \{R_\gamma \times C_\gamma : \gamma \in \Gamma_\alpha\}$*

(3) $$\textit{where } C_\gamma \subset R_n \dim C_\gamma \leqq k,\ k = -1, 0, 1, \cdots .$$

Then for any pair of closed disjoint sets (A, A'), $A \cup A' \subset \bar{R}_n$ there exists a partition D in compactum $R_{\alpha+n}$ between A and A' such that

(4) $$D \cap R_\gamma \times C_\gamma \subset R_\gamma \times C'_\gamma$$

where $C'_\gamma \subset R_{n'} \dim C'_\gamma \leqq k - 1$ for $k \geqq 1$ and $D \cap R_\gamma \times C_\gamma = \varnothing$ for $k \leqq 0$.

Proof. Let D_0 be an arbitrary partition in R_n between $i(A)$ and $i(A')$, then there exist open in R_n sets U and V such that

(5) $$R_n \backslash D_0 = U \cup V,\ A \subset U,\ A' \subset V .$$

Consequently, there exists a sequence of pairs of open sets U_n, V_n such that

$$A \subset \bar{U}_n \subset U_{n+1},\ A' \subset \bar{V}_n \subset V_{n+1},\ \cup \{U_n: n = 1, 2, \cdots\} = U, \quad \cup \{V_n: n = 1, 2, \cdots\} = V. \tag{6}$$

Let $\tau: N \to \Gamma_\alpha$ be a bijective mapping of the set of all integers N onto Γ_α. Then there exists a partition D_n in R_n between $i(\bar{U}_n)$ and $i(\bar{V}_n)$ such that:

$$\dim (D_n \cap C_{\tau(n)}) \leqq k - 1 \text{ for } k \geqq 1 \text{ and } \dim D_n \cap C_{\tau(n)} = -1 \quad \text{for } k \leqq 0. \tag{7}$$

We put

$$D = D_0 \cup \cup \{R_{\tau(n)} \times D_n: n = 1, 2, \cdots\}. \tag{8}$$

Then by virtue of (7)

$$D \cap R_\gamma \times C_\gamma = R_\gamma \times (D_{\tau^{-1}(\gamma)} \cap C_\gamma) = R_\gamma \times C'_\gamma$$

where $C'_\gamma = D_{\tau^{-1}(\gamma)} \cap C_\gamma$ and $\dim C'_\gamma \leqq k - 1$ for $k \geqq 1$ and $C'_\gamma = \varnothing$ for $k \leqq 0$. By virtue of (5), (6), (8) D is a partition between A and A' in $R_{\alpha+n}$. □

LEMMA 4.6. *If conditions of the Lemma 4.5 are satisfied, then for the set F* $\operatorname{Ind} F \leqq \alpha + k$ *for* $k \geqq 0$ *and* $\operatorname{Ind} F \leqq n$ *for* $k < 0$.

Proof. We shall prove the lemma by induction on k. Suppose $k = -1$ or $\alpha = 0$ then the assertion is evident. Let $k \geqq 0$, $\alpha \geqq \omega_0$ and (A, A') be an arbitrary pair of disjoint closed sets in F. By virtue of Lemma 4.5 there exists a partition D between $A \cap \bar{R}_n$ and $A' \cap \bar{R}_n$ such that property (4) holds. Consequently, $F \cap D \subset \bar{R}_n \cup \cup \{R_\gamma \times C'_\gamma: \gamma \in \Gamma_\alpha\}$, $\dim C'_\gamma \leqq k - 1$ for $k \geqq 1$ and $D \cap F \subset \bar{R}_n$ for $k = 0$. Therefore, by inductive assumption

$$\operatorname{Ind} D \cap F \leqq \alpha + k - 1, \text{ or } \operatorname{Ind} D \cap F \leqq n. \tag{9}$$

Since F is a compactum, there exists a finite collection of ordinal numbers $\gamma(1), \cdots, \gamma(s)$, $s = 1, 2, \cdots$ such that for the set $X = F \backslash \cup \{R_{\gamma(i)} \times C_{\gamma(i)}: i = 1, \cdots, s\}$ we have

(10) The set $D \cap X$ is a partition in X between $A \cap X$ and $A' \cap X$. Let $Y = F \backslash X = \cup R_{\gamma(i)} \times C_{\gamma(i)}$, then from condition (3) and Lemma 4.4 it follows that

$$\begin{aligned}\operatorname{Ind} Y &\leqq \max \{\operatorname{Ind} R_{\gamma(i)} \times C_{\gamma(i)}: i = 1, \cdots, s\} \\ &\leqq \max \{\gamma(i) + k: i = 1, \cdots, s\} < \alpha.\end{aligned}$$

Therefore, an arbitrary partition D' in Y between $Y \cap A$ and $Y \cap A'$ has the dimension $\operatorname{Ind} D' < \alpha$. Since $X \cap Y = \varnothing$, $X \cup Y = F$, then $(F \cap D) \cup D'$ is a partition in F between A and A', and

$$\begin{aligned} &\operatorname{Ind}(D \cap F) \cup D' \leqq \max(\operatorname{Ind}(D \cap F), \operatorname{Ind} D') \leqq \alpha + k - 1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } k \geqq 1 , \\ &\operatorname{Ind}(D \cap F) \cup D' < \alpha \qquad \text{for } k = 0 . \end{aligned}$$

In any case $\operatorname{Ind}(D \cap F) \cup D' < \alpha + k$. □

LEMMA 4.7. *Let $(F_i, G_i) i = 1, \cdots, n$ be a fixed system of pairs of closed sets in a space X, $F_i \cap G_i = \varnothing$. If for any partition C_i between these pairs we have*

$$\operatorname{Ind} \cap \{C_i : i = 1, \cdots, n\} \geqq \beta \tag{11}$$

then for any $k \leqq n$

$$\operatorname{Ind} \bigcap_{i=1}^{k} C_i \geqq \beta + n - k \tag{12}$$

and

$$\operatorname{Ind} X \geqq \beta + n . \tag{13}$$

Proof. Let inequality (12) be false. Then there exist partitions $C_{k+1}, \cdots, C_n$ such that C_l separates F_l and G_l $k+1 \leqq l \leqq n$ and $\operatorname{Ind} \cap \{C_i : i = 1, \cdots, k\} < \beta$. This contradicts inequality (11). From property (12) it follows that

$\operatorname{Ind} C_1 \geqq \beta + (n-1)$ for every partition between F_1 and G_1. Hence, inequality (13) holds. □

LEMMA 4.8. *Let R_n be a pseudocube, and $(A_i, B_i)(i = 1, \cdots, k)$ $k \leqq n$ be any system of its opposite faces. Consider a system of pairs of closed subsets $(A_i \times R_\beta, B_i \times R_\beta)$ in the compactum $R_{\beta+n} \in \rho_{\beta+n}$, $R_{\beta+n} = R_\beta \times R_n$ $\beta < \omega_1$. Then for any collection of partitions between these pairs we have:*

$$\operatorname{Ind} \cap \{C_i : i = 1, \cdots, k\} \geqq \beta + n - k .$$

We shall prove the lemma by induction on β. For $\beta < \omega_0$ our assertion follows from Lemmas 4.1, 4.2. Suppose $\beta = \alpha + q \geqq \omega_0$, $q = 0, 1, \cdots$ and for all $\gamma < \beta$ our lemma is proved. We can suppose $R_\beta = R_\alpha \times R_q$. Let A be a system of all opposite faces of pseudocube $R_{n+q} = R_n \times R_q$. Then the following system $\mathscr{B} = \{A_i \times R_q, B_i \times R_q : i = 1, \cdots, k\}$ is a subsystem of A. By virtue of Lemma 4.7 it is sufficient to prove that for any collection $\{D_i : i = 1, \cdots, n+q\}$ of partitions in $R_{\beta+n}$ between distinct pairs of the system $(X \times R_\alpha, Y \times R_\alpha)$, $(X, Y) \in A$ we have

$$\text{Ind} \cap \{D_i : i = 1, \cdots, n + q\} \geqq \alpha . \tag{14}$$

Let $\bar{R}_{n+q} \cup \cup \{R_\gamma \times R_{n+q} : \gamma \in \Gamma_\alpha\}$ be a standard representation of a compactum $R_{\beta+n} = R_{\alpha+n+q}$. Let $\gamma \in \Gamma_\alpha$ then

$R_\gamma \times R_{n+q} = R_{\gamma+n+q} \subset R_{\beta+n}$, $\cap \{D_i : i = 1, \cdots, n + q\} \cap R_{\alpha+n+q} \subset \cap \{D_i : i = 1, \cdots, n + q\}$. Since for every $i \leqq n + q$ the set $D_i \cap R_{\gamma+n+q}$ is a partition between a pair $(X \times R_\gamma, Y \times R_\gamma)$, $(X, Y) \in A$ by inductive assumption we obtain $\text{Ind} \cap \{D_i \cap R_{\gamma+n+q} : i=1, \cdots, n + q\} \geqq \gamma$. Since $\sup \{\gamma : \gamma \in \Gamma_\alpha\} = \alpha$ we obtain inequality (14). □

The following lemma is evident.

LEMMA 4.9. *Let* $Y = \omega(p; Y_i : i = 1, 2, \cdots)$ *then*

$$\text{Ind}\, Y = \sup \{\text{Ind}\, Y_i : i = 1, 2, \cdots\}$$
$$\text{ind}\, Y = \sup \{\text{ind}\, Y_i : i = 1, 2 \cdots\} .$$

COROLLARY 4.2. *For any* $\beta < \omega_1$ *we have* $\text{Ind}\, R_\beta = \beta$.

We shall prove the corollary by induction of β. Suppose β is nonlimit ordinal number, then $R_\beta = R_\alpha \times R_n (n = 1, 2, \cdots)$. Let (A, B) be a pair of opposite faces of pseudocube R_n. Then by Lemma 4.8 for any partition C between $A \times R_\alpha$ and $B \times R_\alpha$ in R_β $\text{Ind}\, C \geqq \alpha + n - 1$. Consequently, $\text{Ind}\, R_\beta \geqq \beta$. The inequality $\text{Ind}\, R_\beta \leqq \beta$ follows from Corollary 4.1. Let β be a limit number, then by definition $R_\beta = \omega(p_\beta : \beta \in \Gamma_\beta)$ and our assertion follows from the inductive assumption and Lemma 4.9. □

LEMMA 4.10. *Every compactum* $R_\beta \in \rho_\beta$ *is weakly countable dimensional.*

We shall prove this lemma by induction on β. If $\beta < \omega_0$, then our lemma is evident. Suppose all compacta R_γ are countable dimensional for $\gamma < \beta$. If β is the limit number, then $R_\beta = \omega(p_\beta : R_\gamma : \gamma \in \Gamma_\beta)$. Since $\beta < \omega_1$, Γ_β is a countable set. Consequently, R_β is a union of countable number of its weakly-countable dimensional closed subsets. Therefore, R_β is also countable dimensional. If $\beta = \alpha + n$, then $R_\beta = R_\alpha \times R_n$ and R_α is weakly countable dimensional by inductive assumption. Consequently, R_β is also weakly countable dimensional. □

5. The proof of Theorem 1.2. First we introduce some notations. For $k = 0, 1, 2, \cdots$, put $\bar{m}(k) = k - [k/2]$. Let R_n be a pseudocube. Let us number all pairs of its opposite faces. Then by $A(s, t, R_n)$ we denote a subsystem of a system of all pairs of opposite faces such that $A(s, t, R_n)$ contains all pairs with numbers $s, \cdots, t (s \leqq t)$. By definition the system $A(s, t, R_n) \times R_\alpha$ consists of all pairs $(F \times R_\alpha, G \times R_\alpha)$ where $(F, G) \in A(s, t, R_n)$[11]. Let A be a system of pairs of sets, then for the set X, $A \wedge X$ denotes the system of pairs $(F \cap X, G \cap X)$ where $(F, G) \in A$. In this section we consider $\Gamma_\alpha = \{\gamma: \gamma < \alpha\}$.

PROPOSITION 5.1. *Let $R_{\alpha+n} = R_\alpha \times R_n$, $n \geqq 2$. Then for any system $A(1, m, R_n)$, $m = \bar{m}(n)$, there exists a compactum $L_{\alpha+m} \subset R_{\alpha+n}$ such that*

(i) $\operatorname{Ind} L_{\alpha+m} \leqq \alpha + m$.

(ii) *For any collection of partitions $D_i (i = 1, \cdots, m)$ between the pairs of the system $A(1, m, R_n) \times R_\alpha$ we have:*

(iii) $\operatorname{ind} L_{\alpha+m} = \psi(\alpha + m)$.

$$\operatorname{Ind} \cap \{D_i: i = 1, \cdots, m\} \cap L_{\alpha+m} \geqq \alpha$$

We note, that from Lemma 4.7 and conditions (i), (ii) it follows that $\operatorname{Ind} L_{\alpha+m} = \alpha + m$. We shall prove the proposition by induction on α. Suppose $\alpha = 0$ and $D_{m+1}, \cdots, D_n$ is a collection of partitions between the pairs of the system $A(m + 1, n, R_n)$ such that for the set $L_m = \cap \{D_{m+i}: i = 1, \cdots, n - m\}$ we have $\dim L_m = m$. Then by Lemma 4.2 L_m is m-dimensional pseudocube and $A(1, m, R_n) \cap L_m$ is a system of pairs of its opposite faces. Thus property (ii) follows from Lemma 4.2, and properties (i), (iii) are evident.

Suppose $\alpha = \alpha_0 \geqq \omega_0$ and for any $\alpha < \alpha_0$ our proposition holds. Let $\tilde{\Gamma}_\alpha \subset \Gamma_\alpha$ be a subset of Γ_α consisting of all ordinal numbers γ with

(1) $K(\gamma) > m$.

Let $f: N \to \tilde{\Gamma}_\alpha$ be a bijection of the set of all integers $n > 0$ onto $\tilde{\Gamma}_\alpha$. Further, since σ-locally finite open base in a compact space is obviously countable, by virtue of Corollary 2.2 we obtain an open basis $\mathscr{V} = \{V_k: k = 1, 2, \cdots\}$ in compactum R_n such that

(2) $\dim Fr V_k \leqq n - 1$.

(3) the system $\{Fr V_k: k = 1, 2, \cdots\}$ is g.p.

Obviously we can also require the following condition: either $V_k \cap F_i = \varnothing$ or $V_k \cap G_i = \varnothing$ for $(F_i, G_i) \in A(1, n, R_n)$, $(k = 1, 2, \cdots)$. If we apply Lemma 3.1 to the system $S_p = \{V_k: k \leqq p\}$, $k = 1$, $\varepsilon = 1/p$, we obtain for any $p \in N$ a closed set $C_p \subset R_n$ such that

(4) C_p is an intersection of $[n/2]$ partitions between pairs of

[11] For $\alpha = 0$ we consider that $A(1, m, R_n) \times R_\alpha = A(1, m, R_n)$.

the system $A(m+1, n, R_n)$ in R_n.

(5) $d_1(C_p \cap Fr V_k) < 1/p$ $(1 \leqq k \leqq p)$.

(6) $\dim C_p \leqq m = n - [n/2]$.

From conditions (4), (6) and Lemma 4.2 it follows that C_p is an m-dimensional pseudocube. Let $R_{\alpha+n} = \bar{R}_n \cup \cup \{R_\gamma \times R_n : \gamma \in \Gamma_\alpha\}$ be a standard representation of compactum $R_{\alpha+n}$. Put

(7) $S_{\alpha+n} = \bar{R}_n \cup \cup \{R_{f(p)} \times C_p : p = 1, 2, \cdots\} \subset R_{\alpha+n}$.

We set $J(f(p)) = \xi_p$, $K(f(p)) = s(p)$, $m + s(p) = n(p)$. Then by virtue of (1)

(8) $m \leqq [n(p)/2]$, $m \leqq n(p) - [n(p)/2] = \bar{m}(n(p))$

and $R_{f(p)} \times C_p \in \rho_{f(p)+m} = \rho_{\xi p+n(p)}$. Let $R_f(p) = R_{\xi_p} \times R_{s(p)}$. By virtue of Lemma 4.1 the set $R_{s(p)} \times C_p$ is $n(p)$-dimensional pseudocube. If $A(1, s(p), R_{s(p)}) = \{A_j, B_j : j \leqq s(p)$, $A(1, m, C_p) = \{F_i, G_i : i \leqq m\}$, then obviously

$$A(1, n(p), R_{s(p)} \times C_p) = \{A_j \times C_p,\ B_j \times C_p,\ R_{s(p)} \times F_i,\ R_{s(p)} \times G_i : i \leqq m,\ j \leqq s(p)\}\ .$$

Let us number elements of the system $A(1, n(p), R_{s(p)} \times C_p)$ by such a way that a pair $R_{s(p)} \times F_i$, $R_{s(p)} \times G_i$ gets a number $i (i \leqq m)$. Let us apply inductive assumption to the system of pairs $A(1, m(p), R_{s(p)} \times C_p)$ and to the compactum $R_{f(p)} \times C_p \in \rho_{\xi_p+n(p)}$ where $m(p) = n(p) - [n(p)/2] = \bar{m}(n(p))$. Then there exists a compactum $L_{\xi_p+m(p)} \subset R_{f(p)} \times C_p \subset R_{\alpha+n}$ such that:

(i_p) $\operatorname{Ind} L_{\xi_p+m(p)} \leqq \xi_p + m(p)$.

(ii_p) For any collection of partitions $D_i^p (i = 1, \cdots, m(p))$ in a compactum $R_{\xi_p+n(p)} = R_{f(p)} \times C_p = R_{\xi_p} \times (R_{s(p)} \times C_p)$ between pairs of the system $A(1, m(p), R_{s(p)} \times C_p) \times R_{\xi_p}$ we have:

$$\operatorname{Ind}(L_{\xi_p+m(p)} \cap \cap \{D_i^p : i = 1, \cdots, m(p)\}) \geqq \xi_p\ .$$

(iii_p) $\operatorname{ind} L_{\xi_p+m(p)} = \psi(\xi_p + m(p))$

and $L_{\xi_p+m(p)} \subset R_{f(p)} \times C_p = R_{\xi_p+n(p)} \subset R_{\alpha+n}$. We put

$$(9) \qquad L_{\alpha+m} = \bar{R}^n \cup \cup \{L_{\xi_p+m(p)} : p = 1, 2, \cdots\} \subset S_{\alpha+n} \subset R_{\alpha+n}\ .$$

By virtue of (6), (7) and Lemma 4.6 $\operatorname{Ind} L_{\alpha+m} \leqq \operatorname{Ind} S_{\alpha+m} \leqq \alpha + m$. Hence, the condition (i) holds. Let us prove property (ii). Let $D_i (i \leqq m)$ be a collection of partitions between the pairs of a system $A(1, m, R_n) \times R_\alpha$ and p be an arbitrary integer > 0. By virtue of chosen numeration of elements of the system $A(1, m, R_{s(p)} \times C_p)$ we have

$$(10) \quad A(1, m, R_{s(p)} \times C_p) \times R_{\xi_p} = (A(1, m, R_n) \times R_\alpha) \wedge (R_{f(p)} \times C_p)\ .$$

Put

(11) $$D_i^p = D_i \cap R_{f(p)} \times C_p \subset L_{\alpha+m} .$$

Then D_i^p is a partition in compactum $R_{f(p)} \times C_p$ between a pair $A(i, i, R_{s(p)} \times C_p) \times R_{\xi_p}$. By virtue of (8) $m \leqq m(p) = n(p) - [n(p)/2]$. Therefore, by Lemma 4.7 and condition (ii_p)

(12) $$\operatorname{Ind}(\cap \{D_i^p: i = 1, \cdots, m\} \cap L_{\xi_p+m(p)}) \geqq \xi_p + (m(p) - m) .$$

Since $\xi_p + (m(p) - m) = J(\gamma) + (K(\gamma) + m) - [(K(\gamma) + m)/2] - m = J(\gamma) + K(\gamma) + [(K(\gamma) + m)/2]$ for $\gamma = f(p)$ we have $\sup\{\xi_p + (m(p) - m): p \in N\} = \sup\{J(\gamma) + K(\gamma) + [(K(\gamma) + m)/2]: \gamma < \alpha, K(\gamma) > m\} = \alpha$. Therefore by virtue of conditions (11), (12), property (ii) holds. We have noted above that from properties (i), (ii) follows the equality $\operatorname{Ind} L_{\alpha+m} = \alpha + m$. Then, by virtue of Corollary 1.1

(13) $$\operatorname{ind} L_{\alpha+m} \geqq \psi(\alpha + m) .$$

Therefore, we have only to prove inequality:

(14) $$\operatorname{ind}_x L_{\alpha+m} \leqq \psi(\alpha + m)$$

for any $x \in L_{\alpha+m}$. If $x \in L_{\xi_p+m(p)} \subset L_{\alpha+m}$ then inequality (14) follows from the inductive assumption. Indeed, by virtue of Lemma 1.1 (j) $\psi(\xi_p + m(p)) \leqq \psi(\alpha + m)$ and since $L_{\xi_p+m(p)}$ is obviously open in $L_{\alpha+m}$, inequality (14) follows from (iii_p). Therefore, by virtue of (9) it is sufficient to prove inequality (14) for $x \in \bar{R}_n$. Let us consider an open in $L_{\alpha+m}$ set $U = U(k, q)$, where

(15) $$U = V_k \times (R_\alpha \backslash \cup \{R_{f(p)}: p = 1, \cdots, q\}) \cap L_{\alpha+m}, \quad V_k \in \mathscr{V} .$$

Since the system of open sets $\{U(k, q): k, 1 = 1, 2, \cdots\}$ forms a basis in any point $x \in R_n$ it is sufficient to prove that

(16) $$\operatorname{ind}(Fr\, U \cap L_{\alpha+m}) < \psi(\alpha + m) .$$

By virtue of (9), (7) and (15) we have

(17) $$\begin{aligned} Fr\, U \cap L_{\alpha+m} &\subset (Fr\, V_k \times R_\alpha) \cap L_{\alpha+m} \subset \bar{R}_n \cup \\ &\cup \{(Fr\, V_k \times R_\alpha \cap C_p \times R_{f(p)}) \cap L_{\xi_p+m(p)}: p = 1, 2, \cdots\} . \end{aligned}$$

Further, we obviously can consider that metric ρ in a space $R_{\alpha+n} = R_\alpha \times R_n$ is defined by the equality

$$\rho((x, y), (x', y')) = \rho_1(x, x') + \rho_2(y, y'), \ x, x' \in R_\alpha, \ y, y' \in R_n ,$$

where ρ_1 and ρ_2 are metrics in R_α and R_n respectively. Therefore, by virtue of the equality $Fr\, V_k \times R_\alpha \cap C_p \times R_{f(p)} = (Fr\, V_k \cap C_p) \times R_{f(p)}$ we have

$$d_1((Fr\,V_k \times R_\alpha \cap C_p \times R_{f(p)}) \cap L_{\xi_p+m(p)}) \leqq d_1(Fr\,V_k \times R_\alpha \cap C_p \times R_{f(p)}) \leqq d_1(Fr\,V_k \cap C_p) + \operatorname{diam} R_{f(p)} .$$

Since $R_\alpha = \omega(p_\alpha, R_\gamma\colon \gamma \in \Gamma_\alpha)$, we have by Lemma 4.0 $\lim_{p\to\infty} \operatorname{diam} R_{f(p)} = 0$, therefore by virtue of (5)

$$\lim_{p\to\infty} d_1((Fr\,V_k \times R_\alpha \cap C_p \times R_{f(p)}) \cap L_{\xi_p+m(p)}) = 0 . \tag{18}$$

Put

$$\Theta_p = (Fr\,V_k \times R_\alpha \cap C_p \times R_{f(p)}) \cap L_{\xi_p+m(p)} . \tag{19}$$

By inductive assumption we have

$$\operatorname{ind} \Theta_p \leqq \operatorname{ind} L_{\xi_p+m(p)} = \psi(\xi_p + m(p)) . \tag{20}$$

Moreover, by Lemma 1.1 (j)

$$\sup\{\psi(\xi_p + m(p))\colon p = 1, 2, \cdots\} \leqq \psi(\alpha) . \tag{21}$$

Since the sets Θ_p are open-closed and disjoint in

$$M = \bar{R}_n \cup \bigcup \{\Theta_p\colon p = 1, 2, \cdots\} , \tag{22}$$

by virtue of (18), (19), (20), (21) and Lemma 3.4 we obtain

$$\operatorname{ind} M \leqq \psi(\alpha) . \tag{23}$$

By virtue of Lemma 1.1 (j) $\psi(\alpha) < \psi(\alpha + m)$. Therefore, by virtue of (17), (23) $Fr\,U \cap L_{\alpha+m} \subset M$ and inequality (16) holds. Thus, inequality (14) is proved. The proposition is completely proved. □

Proof of Theorem 1.2. For any nonlimit number β there exists such a number $\alpha + n (n \geqq 2)$ so that $\beta = \alpha + \bar{m}(n)$. We can merely put $\alpha + n = \beta + K(\beta)$. Therefore, by virtue of Proposition 5.1 there exists a compactum $X_\beta = L_{\alpha+\bar{m}(n)} \subset R_{\beta+K(\beta)}$ such that

$$\operatorname{Ind} X_\beta = \beta,\ \operatorname{ind} X_\beta = \psi(\beta) . \tag{24}$$

Since by Lemma 4.10 $R_{\beta+K(\beta)}$ is weakly countable dimensional, X_β is also weakly countable dimensional. Let β_0 be a limit number. Then we put

$$X_{\beta_0} = \omega(p, X_\beta\colon \beta < \beta_0, K(\beta) > 1) . \tag{25}$$

Since X_β are weakly countable dimensional, X_{β_0} is also weakly countable dimensional. By virtue of Lemma 4.9 and (24) $\operatorname{Ind} X_{\beta_0} = \sup\{\operatorname{Ind} X_\beta = \beta\colon \beta < \beta_0, K(\beta) > 1\} = \beta_0$

$$\operatorname{ind} X_{\beta_0} = \sup\{\operatorname{ind} X_\beta = \psi(\beta)\colon \beta < \beta_0, K(\beta) > 1\} . \tag{26}$$

By virtue of Lemma 1.1 (m) and (26) $\operatorname{ind} X_{\beta_0} = \psi(\beta_0)$. Theorem 1.2 is completely proved. As it was mentioned in §1 Theorem 1.2′ follows from Theorem 1.2. Therefore, Theorem 1.2′ is also proved. □

From Proposition 5.1 and an equality (25) it follows that compactum X_β imbeds in some compact $R_{\beta+K(\beta)} \in \rho_{\beta+K(\beta)}$ for nonlimit β, and X_β imbeds in R_β for limit β. Therefore the following assertion holds:

COROLLARY 5.1. *For any $\beta < \omega_1$ there exists a compactum X_β satisfying condition* (24) *and having an imbedding in some compact $R_{\beta+K(\beta)}$.*

6. On small inductive dimension of product of spaces. This section is auxiliary. We shall prove here some results available for estimation of the small inductive dimension.

DEFINITION 6.1. (See [6] Katetov). A mapping $f: X \to Y$ is called uniformly zero-dimensional if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if the diameter of a set $M \subset Y$ is less than δ, then $f^{-1}(M)$ is a union of a discrete collection of sets of the diameter $< \varepsilon$. We need the following assertions:

(K1) If $f: X \to Y$ is uniformly zero-dimensional mapping, then $\operatorname{ind} Y \geqq \operatorname{ind} X$. (See [20], Zarelua.)

(K2) (See [6], Katetov.) $\dim X \leqq n$ if and only if there exists a uniformly zero-dimensional mapping $f: X \to I^n$ of the space X into n-dimensional cube.

We also need the following theorem (see [18], Toulmin).

(T1) If a space X has a dimension $\operatorname{ind} X$ then $\operatorname{ind} X \times I \leqq \operatorname{ind} X + I$ where $I = [0, 1]$ is a segment.

LEMMA 6.1.[12] *Let $f: X \to Y$ be a uniformly zero-dimensional mapping and $g: X \times Z \to Y \times Z$ be a mapping defined by the equality $g(x, z) = f(x), z (x \in X, z \in Z)$. Then g is also uniformly zero-dimensional.*

The lemma is evident.

PROPOSITION 6.1. *For any finite dimensional space X and for a space Y having the dimension* $\operatorname{ind} Y$ *the following inequality holds*:

[12] In what follows, we consider that on product $X \times Y$ of spaces X, Y the metric is given by the following equality: $\rho((x, y), (x', y')) = \rho_x(x, x') + \rho_y(y, y')$, $(x, x' \in X, y, y' \in Y)$, ρ_x, ρ_y are metrics in X and Y respectively.

$$\operatorname{ind}(X \times Y) \leqq \operatorname{ind} Y + \dim X . \tag{1}$$

Proof. By Theorem (K2) there exists a uniformly zero-dimensional mapping $f: X \to I^n$, where $\dim X = n$. By Theorem (T1)

$$\operatorname{ind}(Y \times I^n) \leqq \operatorname{ind} Y + n = \operatorname{ind} Y + \dim X . \tag{2}$$

Let $g: X \times Y \to I^n \times Y$ be a mapping defined by the equality

$$g(x, y) = f(x), y \qquad x \in X,\ y \in Y .$$

Then, by virtue of Lemma 6.1 g is uniformly zero-dimensional mapping. From Theorem (K1) and inequality (2) follows (1). □

PROPOSITION 6.2. *Let $\{U_s: s = 1, 2, \cdots\}$ be a collection of open sets in a space X such that*

$$U_s \supset \bar{U}_{s+1},\ U_1 = X \tag{3}$$

$$\lim_{s\to\infty} \operatorname{diam} U_s = 0 \textit{ and } \cap \{U_s: s = 1, 2, \cdots\} = \{p\}, \textit{ where } p \textit{ is a point in } X . \tag{4}$$

$$\operatorname{ind}(U_s \backslash \bar{U}_{s+1}) \times I^n \leqq \alpha \tag{5}$$

for some $n = 1, 2, \cdots, \alpha \geqq \omega_0$ and for any $s = 1, 2, \cdots$. Then for any space K with $\dim K \leqq n$ the following inequality holds: $\operatorname{ind}(X \times K) \leqq \alpha + [(n + 3)/2]$.

At first we need some preliminary lemmas. Let $(n + 1)$ dimensional cube I^{n+1} be a product of segments $[0, 1] = I$. Then $I^{n+1} = I \times I^n$. We denote by $\bar{I}^n$ the set $\{0\} \times I^n \in I^{n+1}$. We suppose that there is a collection of open sets $\mathscr{V} = \{V_s: s = 1, 2, \cdots\}$ in cube I^{n+1} such that

$$\text{The collection } \mathscr{V}_1 = \{V_s \cap (I^{n+1} \backslash \bar{I}^n): s = 1, 2, \cdots\} \text{ is locally finite in } (I^{n+1} \backslash \bar{I}^n) . \tag{6}$$

$$\text{The collection } \mathscr{V}_2 = \{Fr\, V_s: s = 1, 2, \cdots\} \text{ is g.p. and } \dim Fr\, V_s \leqq n . \tag{7}$$

$$\text{The system } \mathscr{V} \text{ forms a basis in all points } x \in \bar{I}^n . \tag{8'}$$

Then the following lemma holds:

LEMMA 6.2. *Let (F, G) be a pair of disjoint closed sets in cube I^{n+1} and*

$$\rho(F, G) > \sup \{\operatorname{diam} V_s: s = 1, 2, \cdots\} \tag{8}$$

where $\rho(F, G) = \inf\{\rho(x, y): x \in F,\ y \in G\}$, ρ *is a metric in* I^{n+1}. *If* $\mathscr{U} = \{U_\mu: \mu \in \mathscr{M}\}$ *is a locally finite in* $U = (I^{n+1} \backslash \bar{I}^n)$ *open covering of the set* U *and* $k = [(n+1)/2]$, *then there exists a partition* C *in cube* I^{n+1} *between* F *and* G *such that for any* $s = 1, 2, \cdots$.

(9) *The set* $C \cap Fr\, U_s \cap U$ *is a union of* k *closed sets* $L^1, \cdots, L^k$ *such that every set* $L^j (j \leqq k)$ *is a union of a discrete in* U *countable collection* L^j *of closed sets* $L_i^j (i = 1, 2, \cdots)$ *and every set* L_i^j *is contained in some set* $U_\mu \in \mathscr{U}$.

Proof. By virtue of (8) there exist open (in cube I^{n+1}) neighborhoods OF and OG of sets F and G respectively such that

(10) $$OF \cap OG = \phi\ .$$

(11) Either $V_s \cap OF = \phi$ or $V_s \cap OG = \phi$ for any $V_s \in \mathscr{V}$.

Then, obviously

(12) $$\overline{(I^{n+1} \backslash OF)} \cap F = \phi\ , \qquad \overline{(I^{n+1} \backslash OG)} \cap G = \phi\ .$$

We note that by virtue of (6), (7) the collection $\mathscr{V}_3 = \{Fr\, V_s \cap U: s = 1, 2, \cdots\}$ is locally finite, and is g.p., and $\dim Fr\, V_s \cap U \leqq n$. Therefore, by virtue of (10), (11) the conditions of Lemma 3.1 are satisfied for

(13) $$F_1 = \overline{OF} \cap U\ , \qquad G_1 = \overline{OG} \cap U$$

$$D(n+1, k) = [(n+1)/([(n+1)/2]+1)] = 1,\ k = [(n+1)/2],\ X = U.$$

By Lemma 3.1 we obtain a partition C' in U between F_1 and G_1 such that the condition (9) holds for $C = C'$. (We note, that collection $\mathscr{L}^j$ is countable, because U is separable.) For proving our lemma it is sufficient to show the existence of partition C between F and G such that

(14) $$C' = C \cap U\ .$$

Since C' is a partition between F and G in U there exist disjoint open sets H_1 and H_2 such that: $H_1 \supset F_1$, $H_2 \supset G_1$, $H_1 \cup H_2 = U \backslash C'$, $H_1 \cap H_2 = \phi$. Then by virtue of (13) $H_1 \subset U \backslash G_1 \subset U \backslash \overline{OG} \subset I^{n+1} \backslash \overline{OG}$. Similarly, $H_2 \subset I^{n+1} \backslash \overline{OF}$. Therefore by virtue of (12) $\bar{H}_1 \cap G = \phi$, $\bar{H}_2 \cap F = \phi$. Since I^{n+1} is hereditarily normal space, there exist open in I^{n+1} sets $\tilde{H}_1$, and $\tilde{H}_2$ such that

$$\tilde{H}_1 \supset H_1 \cup F\ , \quad \tilde{H}_2 \supset H_2 \cup F\ , \quad \tilde{H}_1 \cap \tilde{H}_2 = \phi$$

therefore the set $C = I^{n+1} \backslash (\tilde{H}_1 \cup \tilde{H}_2)$ is a partition between F and G and condition (14) holds.

LEMMA 6.3. *Let* $f: X \to Y$ *be a mapping and* $Y = K \cup \bigcup \{L_i^j: i = 1, 2, \cdots, j = 1, \cdots, k\}$. *If the set* $f^{-1}(K)$ *is closed and finite dimensional, sets* $L_i^j (i = 1, 2, \cdots)$ *are open-closed in* $K^j = K \cup \bigcup \{L_i^j: i = 1, 2, \cdots$ *and* $L_i^j \cap L_{i'}^j = \phi$ *for* $i \neq i'$, $L_i^j \cap K = \phi$ *for any* i. $\lim_{i\to\infty} \operatorname{diam} f^{-1}(L_i^j) = 0$, $\operatorname{ind}(X \backslash f^{-1}(K)) \leqq \alpha$, $\alpha \geqq \omega_0$ *then* $\operatorname{ind} f^{-1}(y) \leqq \alpha + (k-1)$.

Proof. Put $Z = f^{-1}(X)$, then obviously all conditions of Lemma 3.3 are satisfied. We have only to change notations. Consequently, our lemma follows from Lemma 3.3. □

In this section we shall consider that a space X, satisfies the conditions of Proposition 6.2.

LEMMA 6.4. *There is a mapping* $f: X \times I^n \to I \times I^n = I^{n+1}$ *such that*:

(A) *If for a sequence of sets* M_i, $M_i \in I^{n+1} \lim_{i\to\infty} \operatorname{diam} M_i = 0$, $\lim_{i\to\infty} \rho(\bar{I}^n, M_i) = 0$, *where* $\rho(\bar{I}^n, M_i) = \inf \{\rho(x, y): x \in \bar{I}^n, y \in M_i, \rho$ *is a metric in* I^{n+1}, $\bar{I}^n = 0 \times I^n\}$, *then* $\lim_{i\to\infty} \operatorname{diam} f^{-1}(M_i) = 0$.

(B) *For any point* $x \in \{p\} \times I^n \subset X \times I^n$ *and a closed set* F, $x \notin F \subset X \times I^n$ *we have*:

$$\overline{f(F)} \not\ni f(x) .$$

(C) *The restriction of* f *to* $f^{-1}(\bar{I}^n)$ *is a homeomorphism and* $f^{-1}(\bar{I}^n) = \{p\} \times I^n$.

Proof. (A) Let us put $E = \{p\} \cup \bigcup \{Fr\, U_s: s = 1, 2, \cdots\}$. Then E is a closed subset of space X. We define a continuous function $g: E \to I$ by the equalities: $g(p) = 0$, $g(Fr\, U_s) = 1/s$. By Urysohn's theorem there is a continuous function $h: X \to I = [0, 1]$ such that the restriction of h to E coincides with g and if $x \in \bar{U}_s \backslash U_{s+1}$ then $1/(s+1) \leqq h(x) \leqq 1/s$. We shall consider a continuous mapping f: $X \times I^n \to I^n = I^{n+1}$ defined by the equality: $f(x, y) = h(x), y; x \in X$, $y \in I^n$. Let $\pi_1: X \times I^n \to X$, $\pi_2: X \times I^n \to I^n$, $\pi_3: I \times I^n \to I^n$, $\pi_4: I \times I^n \to I$ be projections. As it was mentioned above, we can consider $d((x, y), (x', y')) = \rho_X(x, x') + \rho_{I^n}(y, y')$, $x, x' \in X$, $y, y' \in I^n$, where d, ρ_X, ρ_{I^n} are metrics in $X \times I^n$, X, I^n respectively. Therefore, for proving (A) it is sufficient to show that

$$\lim_{i\to\infty} \operatorname{diam} \pi_1(f^{-1}(M_i)) = 0 \tag{15}$$

$$\lim_{i\to\infty} \operatorname{diam} \pi_2(f^{-1}(M_i)) = 0 . \tag{16}$$

It is evident that if $\rho(\bar{I}^n, M_i) < 1/s$ then $\pi_1 \circ f^{-1}(M_i) = h^{-1} \circ \pi_4(M_i) \subset U_s$.

Consequently, by virtue of (4), equality (15) holds. Moreover, we obviously have $\pi_2(f^{-1}(M_i)) = \pi_3(M_i)$. Since $\lim_{i\to\infty} \operatorname{diam} M_i = 0$ we obtain $\lim_{i\to\infty} \operatorname{diam} \pi_3(M_i) = 0$. Therefore, equality (16) holds, and property (A) is proved.

(B) If $x \in p \times I^n$, $x \notin F$ then there is such a set U_s, and an open in I^n set V such that $x \in U_s \times V \subset X \times I^n \backslash F$, then obviously $f(x) \in [0, 1/s + 1] \times V \subset I \times I^n \backslash f(F)$ and since $[0, 1/(s+1)] \times V$ contains an open neighborhood of a point $f(x)$ in I^{n+1}, property (B) holds. Property (C) is evident. □

Definition 6.2. (See [1] Borsuk). A covering $\mathscr{U}$ of an open set $U \subset Y$ is called canonical if for any point $x \in Y \backslash U$ and its any neighborhood V there is a neighborhood $W \ni x$ in Y such that if $G \in \mathscr{U}$ and $G \cap W \neq \phi$ then $G \subset V$.

In [1] (Borsuk) it was proved that for any open set $U \subset Y$ there is an open canonical covering of U.

Lemma 6.5. *Let $f: X \times I^n \to I^{n+1}$ be a mapping satisfying the condition of Lemma 6.4 and let in Lemma 6.2 a covering U be canonical. Then, for the set C satisfying the condition* (9) *we have* $\operatorname{ind} f^{-1}(C) \leqq \alpha + [(n+1)/2]$.

Proof. By virtue of property (C) in Lemma 6.4 and by virtue of (4)

$$f^{-1}(C) = (f^{-1}(\bar{I}^n) \cap f^{-1}(C)) \cup \bigcup \{f^{-1}(C) \cap ((U_s \backslash \bar{U}_{s+2})2 \times I^n) : s = 1, 2, \cdots\} . \tag{17}$$

Since the sets $f^{-1}(C) \cap ((U \backslash \bar{U}_{s+2}) \times I^n)$ are open in $f^{-1}(C)$, then by virtue of (5) we have only to prove

$$\operatorname{ind}_x f^{-1}(C) \leqq \alpha + [(n+1)/2] \quad \text{for } x \in f^{-1}(C) \cap f^{-1}(\bar{I}^n) . \tag{18}$$

From Lemma 6.4 and condition (8′) it follows that the collection $\{f^{-1}(V_s): s = 1, 2, \cdots\}$ forms a basis in each point $x \in f^{-1}(C) \cap f^{-1}(\bar{I}^n)$. Since $Fr(f^{-1}(V_s)) \subset f^{-1}(Fr\, V_s)$ it is sufficient to prove that for any $s = 1, 2, \cdots$

$$\operatorname{ind} f^{-1}(Fr\, V_s \cap C) < \alpha + [(n+1)/2] . \tag{19}$$

If the condition (9) is satisfied and U is canonical covering, then obviously $\lim_{i\to\infty} \operatorname{diam} L_i^j = 0$, $\lim_{i\to\infty} \rho(\bar{I}^n, L_i^j) = 0$ for $j \leqq k = [(n+1)/2]$. From Lemma 6.4 it follows that

$$\lim_{i\to\infty} \operatorname{diam} f^{-1}(L_i^j) = 0 . \tag{20}$$

Put

$$(21) \qquad K = (C \cap Fr V_s \cap \bar{I}^n) .$$

Then, obviously, $C \cap Fr V_s = K \cup (C \cap Fr V_s \cap U)$. From condition (9) it follows that

$$(22) \qquad C \cap Fr V_s = K \cup \cup \{L_i^j : i = 1, 2, \cdots, j = 1, \cdots, k\} .$$

(23) Sets L_i^j are open-closed in $K^j = K \cup \bigcup_{i=1}^{\infty} L_i^j$ and $L_i^j \cap L_{i'}^j = \phi$ for $i \neq i'$ and $K \cap L_i^j = \phi$ for any i.
From Lemma 6.4 (C) it follows that $\dim f^{-1}(K) \leqq n$. Moreover, since

$$f^{-1}(C \cap Fr V_s \cap U) = f^{-1}(C \cap Fr V_s) \backslash f^{-1}(K) \subset X \times I^n \backslash \{p\} \times I^n$$

we obtain by conditions (4), (5)

$$(24) \qquad \operatorname{ind} (f^{-1}(C \cap Fr V_s) \backslash f^{-1}(K)) \leqq \alpha .$$

We put $X = f^{-1}(C \cap Fr V_s)$, $Y = C \cap Fr V_s$, then $f^{-1}(Y) = X$ and by virtue of conditions (20)-(24) and by Lemma 6.3 we have $\operatorname{ind} f^{-1}(C \cap Fr V_s) \leqq \alpha + (k - 1) < \alpha + k$. Therefore, the inequality holds. The lemma is proved. □

LEMMA 6.6. *There exists a collection of open sets $\mathscr{V} = \{V_s : s = 1, 2, \cdots\}$ in cube I^{n+1} such that conditions* (6), (7), (8′) *hold, and* $\operatorname{diam} V_s < \varepsilon$ *for given* $\varepsilon > 0$.

Proof. By virtue of Corollary 2.2 there exists an open basis $A = \{U_\alpha : \alpha \in \mathscr{A}\}$ with boundaries of dimension $\dim Fr U_\alpha \leqq n$ and a collection $\{Fr U_\alpha : \alpha \in A\}$ is g.p. We can select for any $i = 1, 2, \cdots$ a finite covering A_i of the set $\bar{I}^n$, consisting of elements of collection A and satisfying the condition: $\operatorname{diam} T < \varepsilon / i$ for $T \in A_i$. Put $\mathscr{V} = \cup \{A_i : i = 1, 2, \cdots\}$. Then $\mathscr{V}$ obviously satisfies conditions (6), (7), (8′).

LEMMA 6.7. $\operatorname{ind} (X \times I^n) \leqq \alpha + [(n + 3)/2]$.

Proof. From conditions (4), (5) it follows that $\operatorname{ind}_x (X \times I^n) \leqq \alpha$ for any $x \in X \times I^n \backslash \{p\} \times I^n$. Therefore we have only to prove that

$$(25) \qquad \operatorname{ind}_x (X \times I^n) \leqq \alpha + [(n + 3)/2] \text{ for } x \in \{p\} \times I^n .$$

Let $f : X \times I^n \to I^{n+1}$ be a mapping, satisfying conditions A, B, C of Lemma 6.4 and let $x \in \{p\} \times I^n$ be any point. If F is a closed set in $X \times I^n x \notin F$, then by virtue of (B) $y = f(x) \notin \overline{f(F)} = G$. By virtue of one theorem in [1] Borsuk, there exists an open canonicla covering $\mathscr{U}$ of a set $U = I^{n+1} \backslash \bar{I}^n$. By virtue of Lemma 6.6 there exists an open collection $\{V_s\colon s = 1, 2, \cdots\}$ satisfying the conditions (6), (7), (8′) and diam $V_s < \varepsilon = \rho(f(x), \overline{f(F)}) = \rho(y, G)$. Hence, the conditions of Lemma 6.2 hold and there is a partition C in I^{n+1} between y and G, satisfying the condition (9). Then $f^{-1}(C)$ is obviously a partition between x and F in $X \times I^n$ and by Lemma 6.5

$$\operatorname{ind} f^{-1}(C) \leqq \alpha + [(n + 1)/2] < \alpha + [(n + 3)/2] .$$

Consequently, inequality (25) holds.

Proof of proposition 6.2. By virtue of Theorem K2 there exists a uniformly zero-dimensional mapping $g\colon K \to I^n$. Let $r\colon X \times K \to X \times I^n$ be a mapping defined by the equality $r(x, y) = x, g(y)$ $x \in X$, $y \in K$. Then by Lemma 6.1 r is uniformly zero dimensional. Consequently, by Theorem K1 and Lemma 6.7 $\operatorname{ind}(K \times X) \leqq \operatorname{ind}(X \times I^n) \leqq \alpha + [(n + 3)/2]$. □

7. On dimensions of Smirnov's compacta. It is easy to show for a space $X = R_\alpha$, where α is a limit ordinal number, that all conditions of Proposition 6.2 are satisfied and, consequently, $\operatorname{ind} R_{\alpha+n} \leqq \alpha + [(n + 3)/2]$. However, we can obtain more accurate estimation for $\operatorname{ind} R_{\alpha+n}$.

PROPOSITION 7.1. *For a compact space $R_{\alpha+n}$ we have*

$$\operatorname{ind} R_{\alpha+n} \leqq \alpha + [(n + 2)/2] \tag{1}$$

where α is a limit number, $n = 0, 1, 2, \cdots$.

Proof. Since for $n \leqq 2$ inequality (1) follows from Corollary 4.1, we can assume $n \geqq 3$. We consider a standard representation of the compactum $R_{\alpha+n}$ $R_{\alpha+n} = \bar{R}_n \cup \bigcup\{R_\gamma \times R_n\colon \gamma \in \Gamma_\alpha\}$. It follows from Corollary 4.1 that

$$\operatorname{ind} R_{\gamma+n} = \operatorname{ind} R_\gamma \times R_n \leqq \gamma + n < \alpha \quad (\gamma \in \Gamma_\alpha) . \tag{2}$$

Since the set $R_\gamma \times R_n$ is open in $R_{\alpha+n}$ for any $\gamma \in \Gamma_\alpha$ we have only to prove that

$$\operatorname{ind}_x R_{\alpha+n} \leqq \alpha + [(n + 2)/2] \quad \text{for } x \in \bar{R}_n . \tag{3}$$

Let $f\colon N \to \Gamma_\alpha$ be a bijection of the set of all integers N onto Γ_α. We put

$$X_p = R_\alpha \backslash \bigcup\{R_{f(i)}\colon i = 1, 2, \cdots\} \quad (p \in N) . \tag{4}$$

Let $\delta > 0$ then we put

$$(5)\qquad F_1 = \bar{O}_\delta(x)^{13}, \quad F_2 = \bar{R}_n \backslash O_{2\delta}(x), \quad G = \bar{O}_{2\delta}(x) \backslash O_\delta(x)$$

By Corollary 2.2 there is an open basis $\mathscr{V} = \{V_s : s = 1, 2, \cdots\}$ in $\bar{R}_n$ such that $\mathscr{V}$ is a σ-locally finite collection (and since $\bar{R}_n$ is a compactum, the collection $\mathscr{V}$ is countable) and

$$(6)\qquad \begin{array}{l} \text{collection } \mathscr{V}_1 = \{Fr\,V_s : V_s \in \mathscr{V},\ s = 1, 2, \cdots\} \\ \qquad\qquad \text{is g.p., } \dim Fr\,V_s \leqq n - 1 \, . \end{array}$$

We can obviously consider that

$$(7)\qquad \text{either } \bar{V}_s \cap F_1 = \phi \text{ or } \bar{V}_s \cap F_2 = \phi \quad (s \in N) \, .$$

Let $k = [n/2]$, $\varepsilon = 1/p$, $D(n, k) = [n/([n/2] + 1)] = 1$ and $S_p = \{V_s : s \leqq p\}$ be a subcollection of a collection $\mathscr{V}$. Then by virtue of (6), (7), the conditions of Lemma 3.1 are satisfied and by this lemma there exists a partition C_p in $\bar{R}_n$ between F_1 and F_2 such that

$$(8)\qquad d_k(C_p \cap Fr\,V_l) < 1/p \quad l = 1, \cdots, p \, .$$

Let $q \in N$, then we set

$$(9)\qquad D = D(\delta, q, x) = G \cup \bigcup \{R_{f(p)} \times C'_p : p = q, q + 1, \cdots\} \subset R_{\alpha+n}$$

where C'_p is an image of C_p under the homeomorphism: $i: \bar{R}_n \to R_n$. Then D is obviously a compactum. Let us show that

$$(10)\qquad \operatorname{ind} D \leqq \alpha + [n/2] \, .$$

By virtue of (2) it is sufficient to prove that

$$(11)\qquad \operatorname{ind}_x D \leqq \alpha + [n/2] \quad \text{for } x \in G \, .$$

We consider a collection of open sets $A = \{X_p \times V'_p : p = 1, 2, \cdots\}$ ($V'_p = i(V_p)$) where X_p is defined by (4). Then, obviously, A forms a base in points of the compactum $\bar{R}_n$ and, consequently, in points $G \subset \bar{R}_n$. Therefore we have only to prove that

$$(12)\qquad \operatorname{ind}(Fr(X_l \times V'_l) \cap D) \leqq \alpha + ([n/2] - 1) \, .$$

It follows from (8), (9) that

$$(13)\qquad (Fr X_l \times V'_l) \cap D \subset \bar{R}_n \cup \bigcup \{R_{f(p)} \times (Fr\,V'_l \cap C'_p) : p = q, q+1, \cdots\}$$

$$(14)\qquad d_k(Fr\,V'_l \times C'_p) \leqq 1/p \qquad l \leqq p \, .$$

Moreover, from Definition 4.3 and Lemma 4.0 it follows that

$$(15)\qquad \lim_{p \to \infty} \operatorname{diam} R_{f(p)} = 0 \, .$$

[13] Here by $O_\delta(x)$ we mean a δ-neighborhood of x in $\bar{R}_n$ ($\varepsilon = \delta, 2\delta$).

Consequently, from (14), (15) it follows that

$$\lim_{p\to\infty} d_k(R_{f(p)} \times (Fr\,V_l' \cap C_p')) = 0\,. \tag{16}$$

Since $R_{f(p)} \times (Fr\,V_l' \cap C_p') \subset R_{f(p)} \times R_n$, from condition (2) it follows that

$$\sup\{\operatorname{ind} R_{f(p)} \times (Fr\,V_l' \cap C_p') \colon p = q, q+1, \cdots\} \leqq \alpha\,. \tag{17}$$

From condition (13), (16), (17) and Lemma 3.4 it follows that

$$\begin{aligned}\operatorname{ind}\,(Fr(X_l \times V_l')) \cap D < \operatorname{ind} \bar{R}_n \cup \bigcup \{R_{f(p)} \times (Fr\,V_l' \cap C_p') \colon p \\ = q, q+1, \cdots\} \leqq \alpha + (k-1) = \alpha + [n/2] - 1\,.\end{aligned}$$

Thus, inequality (12) holds; consequently inequalities (11), (10) also hold. From construction of the set $D = D(\delta, q, x)$ it follows that

(18) for any closed set F and a point $x \in \bar{R}_n$ there exists a $\delta > 0$ and $q = 1, 2, \cdots$, such that the set $D(\delta, q, x)$ is a partition between F and x.

From conditions (10), (18) follows (3) and, consequently (1). □

DEFINITION 7.1. (See [17] Smirnov). For any ordinal number $\beta < \omega_1$, we shall define a compactum K_β. For $\beta < \omega_0 K_\beta$ is a β-dimensional cube. If β is a limit number we put $K_\beta = \omega(p_\beta; K_\gamma\colon \gamma < \beta)$. If $\beta = \alpha + n$, $\alpha = J(\beta)$, $n = K(\beta) > 0$ we put $K_\beta = K_\alpha \times I^n$.

It is evident that $K_\beta \in \rho_\beta$ (see Definition 4.3). In what follows K_β will denote a compactum defined above. In [17] (Smirnov) it was proved that

$$\operatorname{Ind} K_\beta = \beta \tag{19}$$

however, the equality

$$\operatorname{ind} K_\beta = \operatorname{Ind} K_\beta\,. \tag{20}$$

From Proposition 7.1 it follows that for $K(\beta) \geqq 3$, $\beta \geqq \omega_0$ equality is not true (20) is false. However, for some β it is true.

THEOREM 7.1. *If $n = 3, 4, \cdots \alpha$ is a limit number $< \omega_1$, then for compactum $K_{\alpha+n}$, we have*

$$\operatorname{ind} K_{\alpha+n} \leqq \alpha + [(n+2)/2] < \alpha + n = \operatorname{Ind} K_{\alpha+n}\,. \tag{21}$$

If α is an invariant ordinal number then for $i = 0, 1, 2,$

$$\operatorname{ind} K_{\alpha+i} = \operatorname{Ind} K_{\alpha+i} = \alpha + i \tag{22}$$

and besides that

$$\operatorname{ind} K_{\alpha+3} = \alpha + 2 \, . \tag{23}$$

Inequality (21) follows from inclusion $K_\beta \in \rho_\beta$ and Proposition 7.1. If $i = 0, 1$ then equality (22) follows from (19) and Corollary 1.2 (B), (C). Let us suppose that

$$\operatorname{ind} K_{\alpha+2} \geqq \alpha + 2 \, . \tag{24}$$

Since obviously $K_{\alpha+2}$ is topologically contained in $K_{\alpha+3}$ we have $\operatorname{ind} K_{\alpha+3} \geqq \alpha + 2$. The opposite inequality follows from condition (21). Since by virtue of (19) $\operatorname{ind} K_{\alpha+2} \leqq \operatorname{Ind} K_{\alpha+2} = \alpha + 2$, we have only to prove (24). To prove this inequality we need some preliminary results.

DEFINITION 7.2. Let X be a set in a product $E^2 \times Y$ of a plane E^2 and an arbitrary space Y, and $p\colon E^2 \to E^2$ be a reflection with respect to a straight line $\pi \subset E^2$. Let us consider a mapping $p(\pi)$: $E^2 \times Y \to E^2 \times Y$, defined by the equality

$$p(\pi)(x, y) = p(x), y \qquad x \in E^2, y \in Y \, .$$

The mapping $p(\pi)$ is called a reflection in $E^2 \times Y$ with respect to π. The mapping $p(\pi)$ is obviously a homeomorphism, and $p(\pi)(c) = c$ for $c \in \pi \times Y$.

DEFINITION 7.3. Let $X \subset E^2 \times Y$ be an arbitrary set. We define $O(X, Y)$ as a minimal collection of sets in $E^2 \times Y$ such that

(a) $X \in O(X, Y)$.

(b) If $Z \in O(X, Y)$ then for any reflection $q\colon E^2 \times Y \to E^2 \times Y$ we have $q(Z) \cup Z \in O(X, Y)$.

LEMMA 7.1. *Let A, B be a pair of parallel straight lines in E^2, α be a limit number and p_α be an extra point in a compactum $K_\alpha = \omega(p_\alpha\colon K_\beta\colon \beta < \alpha)$. Let M be a set in $E^2 \times K_\alpha$ satisfying the condition*

$$\bar{M} \cap B \times K_\alpha = \phi \, . \tag{25}$$

If the set M contains a set $Q \times V$, where Q is a square in E^2 such that one of its faces is contained in A, and $V \subset K_\alpha$, then for any compactum $K \subset A$ there exists a set $X \in O(M, K_\alpha)$ such that

$$\bar{X} \cap B \times K_\alpha = \phi, \quad X \supset K \times V \, . \tag{26}$$

Proof. Let R be a rectangle in E^2 such that one of its faces is contained in A, and μ, ν be a pair of two parallel straight lines containing two faces of R which are perpendicular to A. Let $r_\nu(\tau_\mu)\colon E^2 \to E^2$, $p_\nu(p_\mu)\colon E^2 \times K_\alpha \to E^2 \times K_\alpha$ be reflections in E^2 and in $E^2 \times K_\alpha$ with respect to ν (respectively μ). We put

$$T^1(R) = r_\nu((r_\mu(R) \cup R)) \cup (r_\mu(R) \cup R) ,$$
$$L^1 = (R \times V) = p_\nu(p_\mu(R \times V) \cup (R \times V)) \cup (p_\mu(R\times V) \cup (R\times V)) .$$

Then obviously $L^1(R \times V) = T^1(R) \times V$ and the set $T^1(R)$ is also a rectangle such that one of its faces is contained in A. Therefore, sets $L^1(L^1(R \times V))$, $T^1(T^1(R))$ are defined. We put

$$L^{(n+1)}(R \times V) = L^1(L^n(R \times V)) ,$$
$$T^{(n+1)}(R) = T^1(T^n(R)) .$$

Therefore, the following condition holds

$$A \subset \cup \{T^{n+1}(R),\ n = 0, 1, \cdots\}, \quad T^{n+1}(R) \times V = L^{n+1}(R \times V) .$$

Consequently $A \times V \subset \cup L^{n+1}(R)$. Therefore, for any compactum $K \subset A$, there exists a number n such that

$$K \times V \subset L^n(R \times V) . \tag{27}$$

Now let $R = O$. Let $O_\perp(M, K_\alpha)$ be a minimal collection of sets, satisfying the following conditions:

$$M \in O_\perp(M, K_\alpha) .$$

If $X \in O_\perp(M, K_\alpha)$, $\pi \perp A$, then $(p(\pi)(X) \cup X) \in O_\perp(M, K_\alpha)$, where $p(\pi)$ is a reflection with respect to π. Then, obviously $O_\perp(M, K_\alpha) \subset O(M, K_\alpha)$ and for any $n = 1, 2, \cdots$ there exists a set $X \in O_\perp(M, K_\alpha)$ such that $L^n(Q \times V) \subset X$. By virtue of (27) we have only to prove that

$$\text{If } X \in O_\perp(M, K_\alpha) \text{ then } \bar{X} \cap B \times K_\alpha = \phi . \tag{28}$$

By virtue of (25) it is sufficient to prove that

$$\text{If } X \cap B \times K_\alpha = \phi,\ \mu \perp A, \text{ then } \overline{P(\mu)(X) \cup X} \cap B \times K_\alpha = \phi . \tag{29}$$

Since $\mu \perp A$, $A \| B$ and $P(\mu)$ is a homeomorphism, we have $P(\mu)(B \times K_\alpha) = B \times K_\alpha$,

$$\phi = P(\mu)(\bar{X}) \cap P(\mu)(B \times K_\alpha) = P(\mu)(X) \cap B \times K_\alpha .$$

Therefore $B \times K_\alpha \cap \overline{(P(\mu)(\bar{X}) \cup \bar{X})} = \overline{P(\mu)(X) \cup X} \cap B \times K_\alpha = \phi$. Thus property (29) and Lemma 7.1 are proved. □

LEMMA 7.2. *Let U be a subset in $E^2 \times Y$. If $W \in O(U, Y)$ then there exists a set $L \in O(Fr\,U, Y)$ such that $Fr\,W \subset L$.*

Proof. Let $R: E^2 \times Y \to E^2 \times Y$ be a reflection, and $Z \subset E^2 \times Y$. Then since R is homeomorphism, we have

$$Fr(Z \cup R(Z)) \subseteqq FrZ \cup FrR(Z) = FrZ \cup R(FrZ) .$$

Therefore, if $FrZ \subset L \in O(FrU, Y)$ then $Fr(Z \cup R(Z)) \subset (L \cup R(L)) \in O(FrU, Y)$. Our lemma now follows from Definition 7.3. $\square$

COROLLARY 7.1. *Let the conditions of Lemma 7.1 be satisfied, and M be an open set. Then there exists an open set X such that*

(30) *FrX is a partition between $B \times K_\alpha$ and $K \times V$.*

(31) *There exists a set $C \in O(FrM, K_\alpha)$ containing FrX.*

Proof. Since the set M is open, the collection $O(M, K_\alpha)$ consists of open sets. By Lemma 7.1 there is a set $X \in O(M, K_\alpha)$ satisfying the condition (26). Since X is open, the condition (30) holds. The property (31) follows from Lemma 7.2. $\square$

We shall use the following proposition proved in [18], (Toulmin).

(T2) *Let A, B be a pair of closed sets in a space S, $A \cup B = S$. If there is a homeomorphism $f: B \to A$ such that $f(c) = c$ for any $c \in A \cap B$ and dimension* $\operatorname{ind} S$ *is defined, then* $\operatorname{ind} S = \operatorname{ind} A$.

LEMMA 7.3. *Let Z be an arbitrary set in a space $E^2 \times Y$, then for any $X \in O(Z, Y)$ we have*: $\operatorname{ind} X = \operatorname{ind} Z$.

Proof. Since any reflection $R: E^2 \times Y \to E^2 \times Y$ is a homeomorphism and $R(c) = c$ for any point $c \in R(A) \cap A$ and any set $A \subset E^2 \times Y$, Lemma 7.3 follows from Theorem T2 and Definition 7.3. $\square$

LEMMA 7.4. *Let (F, G) be a pair of opposite faces of a square I^2. Then there is a partition C in compactum $I^2 \times K_\alpha = K_{\alpha+2}$ between $F \times K_\alpha$ and $G \times K_\alpha$ such that*

(32) $$\operatorname{ind} C < \operatorname{ind} K_{\alpha+2} .$$

Proof. Since spaces $E^2 \times K_\alpha$ and $K_\alpha \times I^2$ have imbeddings into each other

(33) $$\operatorname{ind} K_{\alpha+2} = \operatorname{ind} K_\alpha \times I^2 = \operatorname{ind} E^2 \times K_\alpha .$$

Let $\varphi: I^2 \to E^2$ be a linear imbedding and A, B be a pair of straight lines in E^2 such that $A \supset F$, $B \supset G$. We consider imbedding ψ: $I^2 \times K_\alpha \to E^2 \times K_\alpha$ defined by the equality

$$\psi(x, y) = \varphi(x), y \quad (x \in I^2, y \in K_\alpha) .$$

Now we can consider that $I^2 \times K_\alpha$ is imbedded in $E^2 \times K_\alpha$ by means of ψ. For any point $p \in F = F \times p_\alpha$ (where $p_\alpha \in K_\alpha$ is an extra point) there exists a neighborhood $M \ni p$ in $E^2 \times K_\alpha$ such that

$$\text{(34)} \qquad \operatorname{ind} FrM < \operatorname{ind} E^2 \times K_\alpha \quad M \cap B \times K_\alpha = \phi .$$

Then there exist sets V and Q such that: V is open in K_α, Q is a square in E^2, such that one of its faces is contained in A, $Q \times V \subset M$ and

$$\text{(35)} \qquad V = K_\alpha \backslash \bigcup_{i=1}^{s} K_{\gamma i} \quad (s = 1, 2, \cdots, \gamma_i < \alpha) .$$

By virtue of Corollary 7.1 there exists a set C^1 such that

(36) C^1 is a partition in $E^2 \times K_\alpha$ between $B \times K_\alpha$ and $F \times V$.

(37) C^1 is contained in some set $R \in O(FrM, K_\alpha)$.

By virtue of Lemma 7.3 and conditions (34), (37) we have:

$$\text{(38)} \qquad \operatorname{ind} C^1 \leqq \operatorname{ind} R = \operatorname{ind} FrM < \operatorname{ind} E^2 \times K_\alpha .$$

Let $T = \bigcup_{i=1}^{s} K_{\gamma i} = K_\alpha \backslash V$. Since obviously sets $E^2 \times T$ and $I^2 \times T$ have imbeddings into each other and $K_{\gamma i} \times I^2 = K_{\gamma i+2}$, by virtue of (21), (35) we have $\operatorname{ind} E^2 \times T < \alpha$. Therefore for any partition C^2 in $E^2 \times T$ between $B \times K \cap E^2 \times T$ and $F \times T$

$$\text{(39)} \qquad \operatorname{ind} C^2 < \alpha \leqq \operatorname{ind} E^2 \times K_\alpha .$$

We put $C^3 = C^1 \cup C^2$. Then by virtue of (36) the set C^3 contains a partition C^4 in $E^2 \times K_\alpha$ between $B \times K_\alpha$ and $F \times K_\alpha$. Moreover,

$$\text{(40)} \qquad C^1 \cap C^2 = \phi .$$

Consequently, by virtue of (38), (39), (40):

$$\text{(41)} \qquad \operatorname{ind} C^4 \leqq \operatorname{ind} C^3 \leqq \max(\operatorname{ind} C^1, \operatorname{ind} C^2) < \operatorname{ind} E^2 \times K_\alpha .$$

We put $C = C^4 \cap I^2 \times K_\alpha$, then C is a partition in $I^2 \times K_\alpha$ between $F \times K_\alpha$ and $G \times K_\alpha = B \times K_\alpha \cap I^2 \times K_\alpha$, and by virtue of (41), (33) $\operatorname{ind} C \leqq \operatorname{ind} C^4 < \operatorname{ind} E^2 \times K_\alpha = \operatorname{ind} K_{\alpha+2}$. □

LEMMA 7.5. *For any partition C in $K_{\alpha+2}$ between $F \times K_\alpha$ and $G \times K_\alpha$ (where (F, G) are opposite faces of I^2) we have* $\operatorname{ind} C \geqq \alpha + 1$.

Proof. Since $K_{\alpha+2} \in \rho_{\alpha+2}$, by Lemma 4.8 $\operatorname{Ind} C \geqq \alpha + 1$. Let $\operatorname{ind} C \leqq \alpha$. Since α is invariant number and by virtue of Theorem

1.1, $\operatorname{Ind} C \leqq \varphi(\operatorname{ind} C) \leqq \varphi(\alpha) = \alpha$. Consequently, $\operatorname{ind} C \geqq \alpha + 1$. □

As we mentioned above, for proving Theorem 7.1 it is sufficient to prove inequality (24). However, this inequality directly follows from Lemmas 7.4, 7.5. Thus Theorem 7.1 is proved. □

COROLLARY 7.2. *The equality* $\operatorname{ind} X \times I = \operatorname{ind} X + 1$, *where* $I = [0, 1]$ *is false even for a compact space* X.

Proof. We put $X = K_{\omega_0+2}$. Since ω_0 is an invariant number and $K_{\omega_0+3} = K_{\omega_0+2} \times I$ then by Theorem 7.1 $\operatorname{ind} K_{\omega_0+2} \times I < \operatorname{ind} K_{\omega_0+2} + 1$. □

THEOREM 7.2. *There exists a compactum* X *such that for any finite dimensional separable space* Y *with dimension* $\operatorname{ind} Y > 0$ *we have* $\operatorname{ind} X \times Y < \operatorname{ind} X + \operatorname{ind} Y$.

Proof. We put again $X = K_{\omega_0+2}$. Since Y is separable space, $\operatorname{ind} Y = \dim Y$ and by Theorem K2 (§ 6) there exists a uniformly zero-dimensional mapping $f\colon Y \to I^n$ where $n = \operatorname{ind} Y$. Let $g\colon K_{\omega_0+2} \times Y \to K_{\omega_0+2} \times I^n = K_{\omega_0+n+2}$ be a mapping, defined by the equality $g(x, y) = x, f(y)$. Then by virtue of Lemma 6.1 g is a uniformly zero-dimensional mapping. Consequently, by Theorem K1 § 6 and by Theorem 7.1

$$\operatorname{ind} X \times Y = \operatorname{ind} K_{\omega_0+2} \times Y \leqq \operatorname{ind} K_{\omega_0+n+2} \leqq \omega_0 + [(n+2)/2] < \omega_0 + n\ .$$

The last inequality holds because $n > 0$. □

8. On D-dimension. In [2] Henderson defined a transfinite D-dimension in the class of all metric spaces[14] For any space X, $D(X)$ is either ordinal number or abstract symbol Δ[15].

DEFINITION 8.1. We put $D(\phi) = -1$. If $X \neq \phi$ then $D(X)$ is the smallest ordinal number β such that there exists a collection of sets $\{A_\xi\colon 0 \leqq \xi \leqq \gamma\}$ satisfying the following conditions:

(a) $X = \cup\{A_\xi\colon 0 \leqq \xi \leqq \gamma\}$.
(b) Every set A_ξ is closed and finite dimensional.
(c) For any $\delta \leqq \gamma$ the set $\cup\{A_\alpha\colon \delta \leqq \alpha \leqq \gamma\}$ is closed in X.
(d) $J(\beta) = \gamma$, $\dim A_\gamma \leqq K(\beta)$.
(e) For any point $x \in X$ there exists the greatest number $\delta \leqq \gamma$

[14] Some results on D-dimension see also in [**11**], (Luxemburg).
[15] It is considered for every ordinal number β that $\beta < \Delta$, $\Delta + \beta = \Delta$.

such that $x \in A_\delta$. If there is no such number β we put $D(X) = \Delta$. If the conditions (a)-(e) hold then equality (a) is called a β-D-representation of a space X.

In [2] Henderson proved that for any compact space X having the dimension Ind X we have

(1) $\operatorname{ind} X \leqq \operatorname{Ind} X \leqq D(X)$.

(2) $|D(X)| \leqq$ weight X, where $|D(X)|$ is a cardinality of $D(X)$.

(3) If X is a finite dimensional space then $\dim X = D(X) = \operatorname{Ind} X$.

PROPOSITION 8.1. (see also [12] (Luxemburg)). *For any compactum Y such that* $\operatorname{Ind} Y \geqq \omega_0$, $D(Y) < \Delta$ *and for any ordinal number γ, $D(Y) \leqq \gamma < \omega_1$ there exists a compactum Y_γ containing Y and satisfying the following condition*

$$D(Y_\gamma) = \gamma, \quad \operatorname{Ind} Y_\gamma = \operatorname{Ind} Y, \quad \operatorname{ind} Y_\gamma = \operatorname{ind} Y .$$

Proof. In [2] D. W. Henderson constructed for any $\gamma < \omega_1$ a compactum X_γ such that $\operatorname{ind} X_\gamma = \operatorname{Ind} X_\gamma = \omega_0$, $D(X_\gamma) = \gamma$. Let us put $Y_\gamma = X_\gamma \cup Y$, $Y \cap X_\gamma = \phi$. Then, obviously Y_γ satisfies the conditions of Proposition 8.1. □

COROLLARY 8.1. *There exist compacta X such that* $\operatorname{ind} X < \operatorname{Ind} X < D(X)$.

Proof. By virtue of Theorem 1.2 there exists a weakly-countable dimensional compactum X such that $\operatorname{Ind} X > \operatorname{ind} X$. In [3] (Henderson) proved that every countable dimensional separable space X has dimension $D(X) < \Delta$. From (2) it follows that $D(X) < \omega_1$. Consequently, Corollary 8.1 follows from Proposition 8.1. □

THEOREM 8.1. *If for a space X, $D(X) < \Delta$ then X has dimension* $\operatorname{ind} X$ *and*

(4) $$\operatorname{ind} X = \leqq D(X) + 1^{16} .$$

LEMMA 8.1. *If for any point $y \in Y$ there exists a neighborhood $O_y \ni y$ such that* $\operatorname{ind} O_y < \alpha$, *then* $\operatorname{ind} Y \leqq \alpha$.

The lemma is evident.

[16] There is a mistake in [7] Kozlovsky, Proposition 11, where the author asserts the inequality $\operatorname{ind} X \leqq D(X)$ in equivalent statement.

LEMMA 8.2. *Let* (a) *be a β-D-representation of a space X, $K(\beta)= n > 0$ and $C \subset X$ be a closed set such that*

$$\dim C \cap A_\gamma \leqq n-1 \,. \tag{5}$$

Then $D(C) \leqq J(\beta) + (n-1)$.

Proof. We put

$$B_\xi = C \cap A_\xi \quad \xi \leqq \gamma \,. \tag{6}$$

Then by virtue of (5)

$$\dim B_\gamma \leqq (n-1) \tag{7}$$

and

$$C = \cup \{B_\xi : \xi \leqq \gamma\} \,. \tag{8}$$

Since a) is a β-D-representation of X, from (6), (8), (7) it follows that (8) is a $(\beta - 1)$-D-representation of C. □

LEMMA 8.3. *Let* (a) *be a β-D-representation of a space X and $\gamma \geqq \omega_0$. We put*

$$U_\delta = X \backslash \cup \{A_\xi : \xi \geqq \delta : \delta < \gamma\} \,. \tag{9}$$

Then the set U_δ is open and $D(U_\delta) \leqq \delta + s < \gamma$ for some $s = 1, 2 \cdots$.

Proof. Since (c) holds, U_δ is open. Let $\delta = T(\delta) + K(\delta)$. We put

$$B_{J(\delta)} = \cup \{A_{(J(\delta)+i)} \cap U_\delta : i = 0, \cdots, K(\delta)\} \tag{10}$$

$$B_\mu = A_\mu \cap U_\delta \quad (\mu < J(\delta)) \tag{11}$$

$$s = \dim B_{J(\delta)} \,. \tag{12}$$

Then

$$U_\delta = \cup \{B_\mu : \mu \leqq J(\delta) = J(\delta + s)\} \,. \tag{13}$$

Since (a) is a β-D-representation of X and by virtue of (10), (11), (12), the equality (13) is a $(\delta + s)$-D-representation of U_δ. □

Proof of Theorem 8.1. We shall prove the theorem by induction on $\beta = D(X)$. If $D(X) < \omega_0$ then by virtue of (3)

$$\operatorname{Ind} X = D(X) = \dim X \geqq \operatorname{ind} X \,.$$

Let $D(X) = \beta \geqq \omega_0$ and (a) be a β-D-representation of X. Let us show that

$$\text{ind}\,(X\backslash A_\gamma) \leqq \gamma = J(\beta)\ . \tag{14}$$

Let $x \in X\backslash A_{T(\beta)}$, then by virtue of (e) there exists such $\delta_0 < \gamma$ that $x \notin A_\delta$, for $\delta' > \delta_0$. Consequently,

$$x \in U_{\delta_0+1} = X\backslash \cup \{A_{\delta'}\colon \delta' \geqq \delta_0 + 1;\, \delta' < \gamma\}\ .$$

By virtue of Lemma 8.3 $D(U_{\delta_0+1}) < \gamma$. Consequently, by inductive assumption $\text{ind}\, U_{\delta_0+1} \leqq D(U_{\delta_0+1}) + 1 < \gamma$. Inequality (14) now follows from Lemma 8.1. Let $x \in X$ and $F \ni x$ be a closed set in X. If $K(\beta) = n > 0$ then there is a partition C between x and F such that inequality (5) holds. By virtue of Lemma 8.2 and by inductive assumption $\text{ind}\, C \leqq D(C) + 1$. Consequently, inequality (4) holds. If $K(\beta) = 0$ then by virtue of (d) $\dim A_\gamma \leqq 0$ and we can find a partition C between x and F such that $C \cap A_\gamma = \varnothing$. Consequently, $C \subset X\backslash A_\gamma$. By virtue of (14) $\text{ind}\, C \leqq \text{ind}\,(X\backslash A_\gamma) \leqq J(\beta) \leqq D(X)$. The theorem is proved. □

Inequality in Theorem 8.1 cannot be improved. We give an example of a space X such that

$$\text{ind}\, X = D(X) + 1\ . \tag{15}$$

(A) *Construction*. Let I^n $n = 1, 2, \cdots$ be a collection of cubes. We can consider that on each cube I^n is defined a metric ρ_n such that there exist two points $x_n, y_n \in I^n$ with $\rho_n(x_n, y_n) = 1$. We identify all points $\{x_n\}$ in disjoint sum $\cup\{I^n\colon n = 1, 2, \cdots\}$ with the point $\{p\} = x_1$. We obtain the set $X = \bigcup_{n=1}^\infty I^n$, $I^n \cap I^{n'} = \{p\}$ for $n \neq n'$. In X we define the following metric:

$$\rho(x, y) = \begin{cases} \rho_n(x, y) & \text{for } x, y \in I^n \\ \rho_n(x, p) + \rho_m(y, p) & \text{if } x \in I^n,\ y \in I^m,\ m \neq n\ . \end{cases}$$

LEMMA 8.4. $\text{ind}\, X = \omega_0 + 1$.

Proof. Let us show that

$$\text{ind}\, X \geqq \omega_0 + 1\ . \tag{17}$$

We put $F = \bigcup_{n=1}^\infty \{y_n\}$. Then F is a closed set in X and $p \notin F$. Let C be a partition between $\{p\}$ and F. Then $C \cap I^n \subset C$ is a partition between $\{p\}$ and $\{y_n\}$ in cube I^n. Consequently, for every n $\text{ind}\,(C \cap I^n) \geqq (n-1)$ and inequality (17) holds. The inequality $\text{ind}\, X \leqq \omega_0 + 1$ follows from Theorem 8.1 and the following lemma.

LEMMA 8.5. *For any n-dimensional cube I^n $n = 1, 2, \cdots$ we* *have*

$$D(X \times I^n) = D(X) + n \tag{18}$$

and $D(X) = \omega_0$.

Proof. The equality (18) holds if there is such a point $x \in X$ that $\sup\{D(Ox)\colon Ox$ is a neighborhood of $x\} = D(X)$ (see [2]). For a point $p \in X$ and any neighborhood $Op \ni p$ we obviously have $D(Op) \geqq \omega_0$. Consequently, for proving our lemma we have only to show $D(X) \leqq \omega_0$. Let us put $A_n = I^n$, $A_{\omega_0} = \{p\}$. Then the equality $X = \{A_\alpha\colon \alpha \leqq \omega_0\}$ is clearly a ω_0-D-representation X. □

Thus, condition (15) is satisfied. The space X has also some interesting property.

PROPOSITION 8.2. *For any n-dimensional space K*

$$\operatorname{ind} X \times K \leqq \omega_0 + [(n + 3)/2]\ . \tag{19}$$

Thus, although $\omega_0 = D(X) < \operatorname{ind} X = \omega_0 + 1$ *but for* $n \geqq 4$, *by Lemma* 8.5

$$D(X \times I^n) = \omega_0 + n > \omega_0 + [(n + 3)/2] \geqq \operatorname{ind}(X \times I^n)\ .$$

Proof. Let U_s be a $(1/s)$ neighborhood of the point p. Since $X\backslash U_s$ is clearly a discrete union of finite dimensional sets, we have $\operatorname{ind}(X\backslash U_s) \times I^n \leqq \omega_0$ for any $s = 1, 2, \cdots$. Consequently conditions (3), (4), (5) of Proposition 6.2 are satisfied and the inequality (19) holds. □

THEOREM 8.2. *Let X be a compactum and* $D(X) = \alpha + n$ *where* $\alpha \geqq \omega_0$ *is a limit number and* $n = 0, 1, 2, \cdots$, *then* $\operatorname{ind} X \leqq \alpha + [(n + 3)/2]$.

COROLLARY 8.2. *If for a compactum space X,* $D(X) = \beta + 4$, $\beta \geqq \omega_0$ *then* $\operatorname{ind} X < D(X)$.

Proof. Let $\beta = \alpha + k$, $\alpha = J(\beta)$ then, by Theorem 8.2 $\operatorname{ind} X \leqq \alpha + [(k + 7)/2] = \alpha + [(k + 1)/2] + 3 < \alpha + k + 4 = \beta + 4 = D(X)$.

To prove Theorem 8.2 we need some preliminary lemmas.

LEMMA 8.6. *Let X be a compactum and the equality* (a) *be a* β-*D*-*representation of X.* ($\beta \geqq \omega_0$), *then for any neighborhood* OA_γ *of the set* A_γ $D(X\backslash OA_\gamma) < J(\beta) = \gamma$.

Proof. By virtue of (e) the collection of sets U_δ, $\delta < J(\beta) = \gamma$ defined by equality (9) is an open covering of $(X \backslash A_\gamma)$. Since $X \backslash OA_\gamma$ is a compactum, there exist such numbers $\delta_1, \cdots, \delta_k$, $(\delta_i < \gamma)$ such that $X \backslash OA_\gamma = \cup \{U_{\delta_i}: i = 1, \cdots, k\}$. Let $\mu = \max\{\delta_i: i = 1, \cdots,\}$. Since $\bigcup_{\delta'} \subset U_\delta$ for $\delta > \delta'$, we have $X \backslash OA_\gamma \subset U_\mu$ and by virtue of Lemma 8.3 we have $D(X \backslash O_\gamma) \leqq D(U_\mu) < \gamma = J(\beta)$. □

Let (a) be a β-D-representation of a compactum X, then we define a mapping

$$\pi: X \longrightarrow X_\sharp \tag{20}$$

by identifying all points of a set A_γ. We put $\pi(A_\gamma) = \rho$.

LEMMA 8.7. *The equality*

$$X_\sharp = \{B_\gamma = \pi(A_\xi): \xi \leqq J(\beta)\} \tag{22}$$

is a $J(\beta)$ D-representation of the compactum $X_\sharp$ and $B_{J(\beta)} = \{p\}$. Moreover, π is a homeomorphism on the set $X \backslash A_{J(\beta)}$ and $\pi(X \backslash A_{J(\beta)}) = X_\sharp \backslash \{p\}$.

Lemma 8.7 is evidently follows from Definition 8.1 and the construction of mapping π.

LEMMA 8.8. *Let U be an open set in a space X, $A = X \backslash U$. If $f: X \to K$, $g: X \to T$ are mappings such that $\dim(f^{-1}(x)) \cap U \leqq 0$, $\dim(g^{-1}(y) \cap A) \leqq 0$, $(y \in T, x \in K)$ then the mapping F: $X \to K \times T$ defined by the equality $F(x) = (f(x), g(x))$ is zero dimensional.*

Proof. Let $a = (x, y)$ be a point in $K \times T$, $x \in K$, $y \in T$. Then $F^{-1}(a) = f^{-1}(x) \cap g^{-1}(y)$. Therefore

$$\begin{aligned} F^{-1}(a) = f^{-1}(x) \cap g^{-1}(y) \cap (A \cup U) = ((f^{-1}(x) \cap U) \cap g^{-1}(y)) \\ \cup ((g^{-1}(y) \cap A) \cap f^{-1}(x)) \subset (f^{-1}(x) \cap U) \cup (g^{-1}(y) \cap A)\ . \end{aligned} \tag{21}$$

The set U is open, and consequently is F_σ set. Further, sets $f^{-1}(x)$, $g^{-1}(x)$, A are closed, consequently by the sum theorem for dimension dim we have $\dim(f^{-1}(x) \cap U) \cup (g^{-1}(y) \cap A) \leqq 0$. Our Lemma now follows from (21). □

LEMMA 8.9. *Let X be a compactum and* (a) (*Definition* 8.1) *be its β-D-representation. Then there exists a zero-dimensional mapping $F: X \to X_\sharp \times I^n$ where $n = K(\beta)$.*

Proof. By virtue of (d) (Definition 8.1) $\dim A_{J(\beta)} \leqq n$. Consequently, by Hurewicz's theorem [5] there exists a zero-dimensional mapping $\varphi: A_{J(\beta)} \to I^n$. Let $\psi: X \to I^n$ be any extension of mapping φ. We define the mapping $F: X \to X_\# \times I^n$ by the equality: $F(x) = \pi(x), \psi(x)$. Since by Lemma 8.7 π is a homeomorphism on the set $X \backslash A_{J(\beta)}$, then by virtue of Lemma 8.8 F is zero-dimensional mapping. □

Proof of Theorem 8.2. By virtue of Lemma 8.9 there exists a zero-dimensional mapping $F: X \to X_\# \times I^n$. Since X and $X_\# \times I^n$ are compacta, zero-dimensional mapping F does not lower dimension ind (see [20]) (Zarelua) and

$$\operatorname{ind} X \leqq \operatorname{ind} X_\# \times I^n . \tag{23}$$

By Lemma 8.7 the equality (22) is a $J(\beta)$-D-representation of X and $B_{J(\beta)}$ is a point p. Let $\{U_s: s = 1, 2, \cdots,\}$ be a collection of open in $X_\#$ sets such that

$$U_s \supset \bar{U}_{s+1}, \ U_1 = X \tag{24}$$

$$\lim_{s\to\infty} \operatorname{diam} U_s = 0 \ , \quad \bigcap_{s=1}^{\infty} U_s = \{p\} \ . \tag{25}$$

Then by Lemma 8.6

$$D(U_s \backslash \bar{U}_{s+1}) \leqq D(X_\# \backslash U_{s+1}) \leqq \gamma_1 < \gamma = J(\beta)$$

for some ordinal number $\gamma_1 < \gamma$. In [2] Henderson, Theorem 5 it was proved that $D(Z \times T) \leqq D(Z) \oplus D(T)$ where "$\oplus$" denotes the natural sum of ordinal numbers. In particular $D((U_s \backslash \bar{U}_{s+1}) \times I^n) \leqq D(U_s \backslash \bar{U}_{s+1}) \oplus D(I^n) = D(U_s \backslash \bar{U}_{s+1}) + n = \gamma_1 + n < J(\beta)$. The last inequality is true, because $J(\beta)$ is a limit number. Consequently, by Theorem 8.1

$$\operatorname{ind} (U_s \backslash \bar{U}_{s+1}) \times I^n \leqq \gamma_1 + n + 1 < J(\beta) \ . \tag{26}$$

By virtue of (24), (25), (26) and Proposition 6.2 $\operatorname{ind} (X_\# \times I^n) \leqq J(\beta) + [(n + 3)/2] = \alpha + [(n + 3)/2]$. Our theorem now follows from inequality (23). □

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