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$\mathscr{K}$-BORELIAN EMBEDDINGS AND IMAGES OF HAUSDORFF SPACES

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The following two questions are discussed in this article: (1) under what conditions is a Hausdorff space embeddable as a $\mathscr{K}$-Borelian subset of some Hausdorff space; and (2) under what conditions is the Hausdorff $1-1$ continuous image of a $\mathscr{K}$-Borelian subset of a Hausdorff space a $\mathscr{K}$-Borelian subset of some Hausdorff space containing it. We obtain necessary and sufficient conditions in answer to question (1) in the case of a $\mathscr{K}_{\sigma\delta}$ and necessary conditions for $\mathscr{K}$-Borelian subsets of each class α if the containing space is a perfectly σ-normal $\mathscr{K}_\sigma$. Question 2 does not always have a positive answer as is shown by an example of a Hausdorff $1-1$ continuous image of $\mathscr{K}$-Borelian subset of a compact Hausdorff space which is not $\mathscr{K}$-Borelian in any Hausdorff space containing it. In partial answer to question (2) necessary conditions on the domain and the range of the function are presented.

In §1 the necessary definitions are given. Many of the original definitions for completely regular spaces such as bianalytic and Borelian as well as the notion of a complete sequence of countable covers are due to Frolik and can be found in [3]. The class of σ-bianalytic spaces is an enlargement of the class of bianalytic spaces. In [8] the $\mathscr{K}_\sigma$-fiable spaces have already been introduced. For the reader's convenience, the definitions of these concepts have been given again.

The motivation of §2 is to extend to a larger class of spaces the well-known result that if $f(X)$ is a metrizable $1-1$ continuous image of a $\mathscr{K}$-Borelian subset of a compact metric space, then $f(X)$ is a $\mathscr{K}$-Borelian subset of any of its metrizable compactifications. An example is given of a Hausdorff $1-1$ continuous image of a $\mathscr{K}$-Borelian subset of a compact space which is not a $\mathscr{K}$-Borelian subset of any Hausdorff space in which it can be embedded. However, using some of the techniques that Frolik developed in [3] for completely regular spaces, necessary and sufficient conditions in order that a $1-1$ continuous image of σ-bianalytic space X be a $\mathscr{K}$-Borelian subset of some Hausdorff space are obtained. This result can be applied to the cases where X is a $\mathscr{K}$-Borelian subset of a Hausdorff space Y and Y is either a perfectly σ-normal $\mathscr{K}_\sigma$, or has property I ([10]) or is locally metrizable.

In §3 results are presented concerning the relationship between the Borel class of a $\mathscr{K}$-Borelian subset of a Hausdorff $\mathscr{K}_\sigma$ space and its complete sequence of countable covers as defined in [4]. Theorem 3.1, giving necessary and suffcient conditions for a space to be embeddable as a $\mathscr{K}_{\sigma\delta}$ subset of some Hausdorff space, is an extension of Frolik's result for completely regular spaces [4]. Theorem 3.3 gives necessary conditions for subsets of perfectly σ-normal $\mathscr{K}_\sigma$ Hausdorff spaces to be $\mathscr{K}$-Borelian of class α. The proofs depend on techniques developed by Frolik in [4].

1. Preliminaries. Throughout this article all spaces will be assumed to be Hausdorff and the notation of Frolik in [3] and [4] will be followed. If $X \subset Y$ then $\bar{X}^Y$ will denote the closure of X in Y.

DEFINITION 1.1. Y is said to be a *$\mathscr{K}_\sigma$-fication* of a Hausdorff space X provided that X can be embedded as a dense subspace of Y and Y is a Hausdorff $\mathscr{K}_\sigma$ space (that is, Y is equal to a countable union of its compact subsets). A Hausdorff space for which there exists such a $\mathscr{K}_\sigma$-fication is said to be *$\mathscr{K}_\sigma$-fiable*.

DEFINITION 1.2. X is said to be a perfectly σ-normal Hausdorff $\mathscr{K}_\sigma$ space if, in addition to being a Hausdorff $\mathscr{K}_\sigma$ space, every closed subset is a $\mathscr{G}_\delta$, that is, a countable intersection of open subsets.

It should be noted that this definition does not imply that X is normal; in fact, X need not be regular. For example, let A, B be two countable disjoint dense subsets of $[0, 1]$ in the usual topology. Let $X = A \cup B$ endowed with the following topology $\mathscr{T}: O \in \mathscr{T}$ if and only if $O = O' \cup (O'' \cap B)$ where O', O'' are open subsets of $[0, 1]$ in the usual topology. X is clearly a nonregular Hausdorff perfectly σ-normal $\mathscr{K}_\sigma$ space. Any perfectly normal $\mathscr{K}_\sigma$ space is clearly a perfectly σ-normal $\mathscr{K}_\sigma$ space.

DEFINITION 1.3. Let $\{X_i\}_{i=0}^\infty$ be a sequence of Hausdorff spaces that are mutally disjoint. Then $\sum_{i=0}^\infty X_i$ will represent the set $X = \bigcup_{i=0}^\infty X_i$ endowed with the following topology: O is open in X if and only if $O \cap X_i$ is open for all i.

DEFINITION 1.4. Let X be a Hausdorff space and let $\mathscr{K}(X)$ denote the class of compact subsets of X. Then $\mathscr{B}(\mathscr{K}(X))$, the class of $\mathscr{K}$-Borelian subsets of X, is defined to be the smallest class of subsets of X containing $\mathscr{K}(X)$ and closed under countable intersections and countable unions.

Let $\mathscr{Z}(X)$ denote the class of zero sets of continuous functions on X and $\mathscr{F}(X)$ the class of closed subsets of X. Let $\mathscr{B}(\mathscr{Z}(X))$, the class of Baire subsets of X, and $\mathscr{B}(\mathscr{F}(X))$, the class of $\mathscr{F}$-

Borelian subsets of X, be defined by analogy to the definition of $\mathscr{B}(\mathscr{K}(X))$.

Definitions of $\mathscr{K}$-Borelian or $\mathscr{F}$-Borelian subsets of a space X of class α for each ordinal α of second class (that is, for each α less than the first uncountable ordinal) are given in [6] or [1].

DEFINITION 1.5. A Hausdorff space X has property I if for all A, B in $\mathscr{K}(X)(A\backslash B)$ is in $\mathscr{K}_\sigma(X)$ (that is, the intersection of A and the complement of B is a $\mathscr{K}_\sigma$ subset of X).

The notion of a complete sequence of countable covers was introduced by Frolik in [4] and [3]. The following definitions are taken from there.

DEFINITION 1.6. A family of subsets $\{E_\gamma\}_{\gamma\in\mathscr{C}}$ of a topological space X is said to have *the finite intersection property* if every finite subfamily has nonvoid intersection.

DEFINITION 1.7. $\mu = \{\mathscr{H}_n\}_{n=1}^\infty$ is called *a sequence of countable covers of* X if, for each n,

(1) $\mathscr{H}_n$ consists of a sequence of subsets $\{H_{ni}\}_{i=1}^\infty$ of X; and

(2) $X \subset \bigcup_{i=1}^\infty H_{ni}$.

DEFINITION 1.8. A family $\mathscr{E} = \{E_\gamma\}_{\gamma\in\mathscr{C}}$ of subsets of X is a μ-*Cauchy family* if

(1) $\mathscr{E}$ has the finite intersection property; and

(2) for all n, there exists a $\gamma_n \in \mathscr{C}$ and a set $H_{ni_n} \in \mathscr{H}_n$ such that $E_{\gamma_n} \subset H_{ni_n}$.

DEFINITION 1.9. $\mu = \{\mathscr{H}_n\}_{n=1}^\infty$ is said to be *a complete sequence of countable covers of* X if, for each μ-Cauchy family $\mathscr{E} = \{E_\gamma\}_{\gamma\in\mathscr{C}}$, $\bigcap_{\gamma\in\mathscr{C}} \bar{E}_\gamma^X \neq \varnothing$.

DEFINITION 1.10. A Hausdorff space X is said to have *a complete* sequence $\mu = \{\mathscr{H}_j\}_{j=1}^\infty$ *of countable* $\mathscr{F}$*-Borelian covers of at most class* α if

(1) μ is complete in the sense of Definition 1.9; and

(2) for all j, $\mathscr{H}_j = \{H_{ji}\}_{i=1}^\infty$ and each H_{ji} is a $\mathscr{F}$-Borelian subset of X of at most class α.

If X is a completely regular space, βX will denote the Stone-Čech compactification of X.

The following definition can be found in [3].

DEFINITION 1.11. $\boldsymbol{B}(X)$ will denote the class of *Borelian spaces* which are subsets of X. A space Y is said to be a *Borelian space* if there exists a *Borelian structure* in Y. A *Borelian structure* in Y is a complete sequence $\{\mathscr{H}_n\}_{n=1}^\infty$ of countable disjoint coverings of Y satisfying the following two conditions:

(1) $\mathscr{H}_{n+1}$ refines $\mathscr{H}_n$ for each n,

(2) if for each n M_n and N_n are elements of $\mathscr{H}_n$ and there exists a k such that $M_k \neq N_k$, then $\bigcap_{n=1}^\infty \bar{M}_n^Y \cap \bigcap_{n=1}^\infty \bar{N}_n^Y = \varnothing$.

DEFINITION 1.12. A Hausdorff space X is $\mathscr{K}$-analytic if it is the continuous image of a $\mathscr{K}_{\sigma\delta}$ subset of a compact space.

Let us define the following new class of spaces.

DEFINITION 1.13. A Hausdorff space X is said to be *σ-bianalytic* if there exists a sequence $\{X_i\}_{i=1}^\infty$ of subsets of X such that

(1) $X = \bigcup_{i=1}^\infty X_i$, and

(2) each X_i is bianalytic; that is, both X_i and $(\beta X_i \backslash X_i)$ are completely regular $\mathscr{K}$-analytic spaces for each i.

We should note that the union in Definition 1.13 need not be disjoint.

PROPOSITION 1.14. *Every bianalytic space is σ-bianalytic. However, there exist completely regular σ-bianalytic spaces that are not bianalytic.*

Proof. The first statement is obvious. If $X = N \cup \{x\}$ where N is the space of the positive integers with the discrete topology and $x \in (\beta N \backslash N)$, then Frolik showed in [5] that this space was not bianalytic.

PROPOSITION 1.15. *There exists a σ-bianalytic space which is not a $\mathscr{K}$-Borelian subset of any Hausdorff space.*

Proof. Let X denote the Hausdorff space [0, 1] with the following topology $\mathscr{T}$: $O \in \mathscr{T}$ if and only if $O = O' \cup (O'' \cap Q)$ where O', O' are open sets in [0, 1] with respect to the usual topology and Q is the set of rationals in [0, 1]. Since $X = I \cup Q$, where I is the set of irrationals in [0, 1], X is σ-bianalytic. In [7] it was shown that X could not be embedded in any Hausdorff $\mathscr{K}_\sigma$ space. Since any $\mathscr{K}$-Borelian subset of a Hausdorff space is necessarily contained in a $\mathscr{K}_\sigma$, X is not a $\mathscr{K}$-Borelian subset of any Hausdorff space.

However, the following is true.

PROPOSITION 1.16. *A $\mathscr{K}_\sigma$-fiable (respectively regular) σ-bianalytic space is a $\mathscr{K}$-Borelian subset of any of its $\mathscr{K}_\sigma$-fications (respectively compactifications).*

Proof. Let $X = \cup X_i$ where X_i is bianalytic and let $K = \bigcup_{i=1}^{\infty} K_i$ be a $\mathscr{K}_\sigma$-fication of X. Now in [5] Frolik showed that closed subspaces of bianalytic spaces are bianalytic and that any bianalytic space is a Baire subset and consequently a $\mathscr{K}$-Borelian subset of any of its compactifications. Let $X = \bigcup_{i,j=1}^{\infty}(X_i \cap K_j)$. Each $X_i \cap K_j$ is a $\mathscr{K}$-Borelian subset of K.

2. $1-1$ continuous images of $\mathscr{K}$-Borelian subsets of Hausdorff spaces.

THEOREM 2.1. *There exists a Hausdorff space X which is the $1-1$ continuous image of a $\mathscr{K}$-Borelian subset A of a compact space Y, such that X is not a $\mathscr{K}$-Borelian subset of any Hausdorff space in which it can be embedded.*

Proof. Let X be the Hausdorff space $([0, 1], \mathscr{T})$ defined in Proposition 1.15. We have shown that X is not a $\mathscr{K}$-Borelian subset of any Hausdorff space in which it can be embedded. It remains to show that X is the $1-1$ continuous image of a $\mathscr{K}$-Borelian subset A of a compact space Y. Let $A = \sum_{i=0}^{\infty} A_i$; where $A_0 =$ the irrationals in $[0, 1]$ and $A_i = \{q_i\}$ for each $i \geqq 1$, $A_i \cap A_j = \emptyset (i \neq j)$ and $\bigcup_{i=1}^{\infty}\{q_i\} = Q =$ the rationals in $[0, 1]$. A is obviously a $\mathscr{K}_{\sigma\delta}$ subset of the one point compactification Y of the space $\sum_{i=0}^{\infty} B_i$; where $B_0 = [0, 1]$ endowed with the usual topology and $B_i = A_i$ for all $i \geqq 1$. X is clearly a $1-1$ continuous image of A.

THEOREM 2.2. *A $\mathscr{K}_\sigma$-fiable $1-1$ continuous image Y of a σ-bianalytic space X is a $\mathscr{K}$-Borelian subset of any of its $\mathscr{K}_\sigma$-fications. In particular, a regular $1-1$ continuous image of a σ-bianalytic space is a $\mathscr{K}$-Borelian subset of any of its $\mathscr{K}_\sigma$-fications or compactifications.*

Proof. Let P be some Hausdorff $\mathscr{K}_\sigma$-fication of Y. Then $P = \bigcup_{n=1}^{\infty} K_n$, where K_n is a compact subset of P for each n.

Since X is σ-bianalytic, $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is bianalytic.

For each i and n, let

$$Y_{i,n} = f(X_i) \cap (K_n \cap Y) .$$

Now each $Y_{i,n}$ is completely regular and $Y = \bigcup_{i,n=1}^{\infty} Y_{i,n}$. Since closed subspaces of bianalytic spaces are bianalytic, then each $Y_{i,n}$ is a $1-1$ continuous image of a bianalytic space. Therefore, from Theorem 13 [3] it follows that $Y_{i,n} \in \boldsymbol{B}(P)$ for all i, n.

It follows from Theorem 11 [3] that $Y_{i,n}$ is an element of the family consisting of countable intersections of countable disjoint unions of sets of the form $(F \cap B)$, where F is a closed set in P and $B \in \mathscr{B}(\mathscr{K}(P))$. That is, for each i and n,

$$Y_{i,n} = \bigcap_k \bigcup_j (F_{kj}^{in} \cap B_{kj}^{in}) \quad \text{where}$$

F_{kj}^{in} is a closed set in P and consequently a $\mathscr{K}_\sigma$,

B_{kj}^{in} is a Baire subset of P and consequently a $\mathscr{K}$-Borelian subset of P.

Therefore $Y = \bigcup_{i,n=1}^{\infty} Y_{i,n}$ is a $\mathscr{K}$-Borelian subset of P.

The previous theorem can be restated as follows.

THEOREM 2.3. *Let Y be a $1-1$ continuous of image of a σ-bianalytic space X. Then Y is a $\mathscr{K}$-Borelian subset of some Hausdorff space if and only if*

(1) *Y is Hausdorff;*

(2) *there exists a sequence $\{R_n\}_{n=1}^{\infty}$ of subsets Y such that*

(i) $Y = \bigcup_{n=1}^{\infty} R_n$,

and for each n,

(ii) *$R_n \subset R_{n+1}$, and*

(iii) *R_n is closed and strongly regular with respect to Y; that is, R_n is a closed subspace of Y and for each closed set $C \subset R_n$ and for each point $y \in (R_n \backslash C)$ there exist sets O, U open in Y such that $y \in O$, $C \subset U$ and $O \cap U = \varnothing$.*

In fact, if conditions (1) *and* (2) *are satisfied, Y is a $\mathscr{K}$-Borelian subset of any of its Hausdorff $\mathscr{K}_\sigma$-fications.*

Proof. Necessity of (1) *and* (2).

Let us suppose that Y is a $\mathscr{K}$-Borelian subset of some Hausdorff space. Then Y is Lindelof and Y is a subset of some Hausdorff $\mathscr{K}_\sigma$ space. It follows from Theorem 3.1 [8] that conditions (1) and (2) are satisfied.

COROLLARY 2.4. *A $\mathscr{K}_\sigma$-fiable* (*respectively regular*) *$1-1$ continuous image of a $\mathscr{K}$-Borelian subset of a perfectly σ-normal $\mathscr{K}_\sigma$ space is a $\mathscr{K}$-Borelian subset of any of its $\mathscr{K}_\sigma$-fications* (*respectively compactifications*).

The following result in the case where the image was regular was first proved by Sion in [9].

COROLLARY 2.5. *Let X be a Hausdorff space which has property*

I. Then a $\mathscr{K}_\sigma$-fiable (respectively regular) $1-1$ continuous image of a $\mathscr{K}$-Borelian subset of X is a $\mathscr{K}$-Borelian subset of any of its $\mathscr{K}_\sigma$-fications (respectively compactifications).

Proof. If Y is a $\mathscr{K}$-Borelian subset of X then Y is contained in a $\mathscr{K}_\sigma$ subset Z of X. In [10] it was shown that a space Z with property I satisfies the following condition: for all $A \in \mathscr{K}(Z)$ and B open in Z, $A \cap B \in \mathscr{K}_\sigma(Z)$. Since Z is also a $\mathscr{K}_\sigma$ then property I implies that it is perfectly σ-normal $\mathscr{K}_\sigma$.

COROLLARY 2.6. *A regular $1-1$ continuous image X of a $\mathscr{K}$-Borelian subset of a metrizable $\mathscr{K}_\sigma$ space is a $\mathscr{K}$-Borelian subset of any compactification of X or of any Hausdorff $\mathscr{K}_\sigma$ space containing X.*

Theorem 1.10 can now be applied to regular locally metrizable spaces.

COROLLARY 2.7. *Let X be a regular locally metrizable space and let A be a $\mathscr{K}$-Borelian subset of X. A $\mathscr{K}_\sigma$-fiable (respectively regular) $1-1$ continuous image of A is a $\mathscr{K}$-Borelian subset of any of its $\mathscr{K}_\sigma$-fications (respectively compactifications).*

Proof. If A is a $\mathscr{K}$-Borelian subset of X, then A is contained in a $\mathscr{K}_\sigma$ of X. This $\mathscr{K}_\sigma$ is Lindelof and therefore paracompact [2]. It follows from [11] that this $\mathscr{K}_\sigma$ is metrizable. The result now follows from Corollary 2.6.

3. $\mathscr{K}$-Borelian subsets of Hausdorff $\mathscr{K}_\sigma$ spaces and their complete sequences of countable covers. It would be interesting to characterize for each α the Hausdorff spaces that can be embedded as $\mathscr{K}$-Borelian subsets of class α of some Hausdorff $\mathscr{K}_\sigma$ space. Theorem 3.1 answers this question in the particular case of a $\mathscr{K}_{\sigma\delta}$. This is a generalization of Frolik's result in [4] for completely regular spaces.

Theorem 3.3 gives necessary conditions for a subset of a perfectly σ-normal $\mathscr{K}_\sigma$ space to be a $\mathscr{K}$-Borelian subset of class α. This theorem is proved by a method similar to that used for Theorem 9 of [4].

Both these theorems depend upon the notion of a complete sequence of countable covers defined by Frolik in [4].

THEOREM 3.1. *Let X be a Hausdorff space. Then X can be embedded in some Hausdorff space Y as a $\mathscr{K}_{\sigma\delta}$ subset if and only* *if the following two conditions are satisfied.*

(1) *There exists a sequence* $\{X_n\}_{n=1}^{\infty}$ *of subspaces of* X *such that*

(i) $X = \bigcup_{n=1}^{\infty} X_n$,

and for each n,

(ii) $X_n \subset X_{n+1}$,

(iii) X_n *is closed and strongly regular with respect to* X.

(2) *There exists a complete sequence* $\mu = \{\mathscr{H}_j\}_{j=1}^{\infty}$ *of countable closed covers of* X *(that is, each member of each cover is closed in* X*).*

Proof. Let us suppose that X can be embedded in some Hausdorff space as a $\mathscr{K}_{\sigma\delta}$ subset. Then X can necessarily be embedded in some Hausdorff $\mathscr{K}_\sigma$ space and condition (1) now follows from Proposition 1, [7]. Condition (2) follows from Proposition 4, [4].

Now, conversely, let us suppose that conditions (1) and (2) are satisfied. Condition (2) implies that for each $n\{\mathscr{H}_j \cap H_n\}_{j=1}^{\infty}$ is a complete sequence of countable closed covers of X_n. Now, since X_n is regular and has a complete sequence of countable closed covers, it follows from Proposition 3, [4] that X_n is Lindelof. Therefore $X = \bigcup_{n=1}^{\infty} X_n$ is Lindelof and we have by Theorem 3.1 in [8] that X is embeddable in a Hausdorff $\mathscr{K}_\sigma$ space $Y = (\bigcup_{n=1}^{\infty} K_n, \tau)$ where, for each n, $K_n = \beta X_n$ and $K_n \subset K_{n+1}$. (The topology τ was described in [8].)

Let us consider the following subspace of Y:

$$A = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (\overline{H_{ji} \cap X_n})^{K_n} ,$$

where each $\mathscr{H}_j$ in μ consists of $\{H_{ji}\}_{i=1}^{\infty}$; the X_n satisfy condition (1); $K_n = \beta X_n$ for each n; and $(\overline{\quad\quad})^{K_n}$ represents the closure of $(\ \ )$ in K_n.

Now A is obviously a $\mathscr{K}_{\sigma\delta}$ subset of Y and A contains X. Let us suppose that there exists $y \in (A \backslash X)$. Since $y \in Y$, then either $y \in K_1$ or else there exists an n_0 such that $y \in (K_{n_0} \backslash K_m)$ for all $m < n_0$. (If $y \in K_1$, we shall let $n_0 = 1$.) Now let $\mathscr{D}$ be the family of all closed neighborhoods of y in K_{n_0}; let $\mathscr{E}$ be the family of all closed subsets F of X such that $y \in \bar{F}^{K_n}$ (the closure of F in K_n) for some $n \geqq n_0$. Clearly, $\mathscr{D} \cap X = \{D \cap X : D \in \mathscr{D}\}$ is a subfamily of $\mathscr{E}$.

We claim that $\mathscr{E}$ is a μ-Cauchy family. To show that $\mathscr{E}$ has the finite intersection property, let F_1, F_2 be members of $\mathscr{E}$. Therefore, $y \in (\bar{F}_1^{K_m} \cap \bar{F}_2^{K_n} \cap K_{n_0})$ for some $m \geqq n_0$ and $n \geqq n_0$. Therefore, as it was shown in [8], $y \in ((\overline{F_1 \cap X_{n_0}})^{K_{n_0}} \cap (\overline{F_2 \cap X_{n_0}})^{K_{n_0}})$. From Lemma 2.1, [8] we have that X_{n_0} is strongly normal with respect to X and consequently X_{n_0} is a normal subspace of X. Since disjoint closed subsets of a normal space have disjoint closures in their Stone-Cech compactifications, it follows that $F_1 \cap F_2 \neq \emptyset$. Thus $\mathscr{E}$ has the finite intersection property.

Since $y \in A$, then for all j there exists an i_j and an n_j such that $(X_{n_j} \cap H_{ji_j}) \in \mathscr{E}$. Thus $\mathscr{E}$ is a μ-Cauchy family. Since μ is complete $\bigcap_{E \in \mathscr{E}} \bar{E}^X \neq \emptyset$. This is impossible because $\bigcap_{E \in \mathscr{E}} \bar{E}^X \subset \bigcap_{E \in \mathscr{E}} \bar{E}^Y \subset \bigcap_{D \in \mathscr{D}} \bar{D}^Y = \bigcap_{D \in \mathscr{D}} \bar{D}^{K_{n_0}} = \{y\}$, and $y \in (A \backslash X)$. This contradiction proves that $A = X$.

As a corollary we have the following result due to Frolik (Theorem 7, [4]).

COROLLARY 3.2. *A completely regular space X is a $\mathscr{K}_{\sigma\delta}$ subset of some Hausdorff $\mathscr{K}_\sigma$ space if and only if X has a complete sequence of countable closed covers. In particular, a completely regular space X is a $\mathscr{K}_{\sigma\delta}$ subset of its Stone-Cech compactification βX if and only if X has a complete sequence of countable closed covers.*

Proof. Let $X_n = X$ for each n in condition (1) of Theorem 3.5 and the proof follows immediately.

THEOREM 3.3. *Let X be a perfectly σ-normal $\mathscr{K}_\sigma$ Hausdorff space. If Y is a $\mathscr{K}$-Borelian subset of X of class α, then Y has a complete sequence of countable $\mathscr{F}$-Borelian covers of at most class α.*

Proof. We shall proceed by transfinite induction. For the case $\alpha = 2$ the result follows from Corollary 3.6.

From the assumption the result is true for all $\beta < \alpha$, we must prove the result is true for α in the following three cases.

Case (1). $\alpha = \alpha_0$ where α_0 is a limit ordinal.

Since Y is a $\mathscr{K}$-Borelian subset of X of class α_0 then $Y = \bigcap_{n=1}^\infty Y_n$ where each Y_n is a $\mathscr{K}$-Borelian subset of X of class $\beta_n < \alpha_0$. Thus each Y_n has a complete sequence $\mu_n = \{\mathscr{H}_j^n\}_{j=1}^\infty$ of countable $\mathscr{F}$-Borelian covers of class at most β_n. Let us consider the sequence $\mu = \{\mathscr{H}_j^n \cap Y\}_{n,j=1}^\infty$. Now μ is a sequence of countable $\mathscr{F}$-Borelian covers of Y of at most class α_0 and μ is complete from the proof of Theorem 9, [4].

Case (2). $\alpha = \alpha_0 + m$ (where α_0 is a limit ordinal and m is odd).

Since Y is a $\mathscr{K}$-Borelian subset of X of class $\alpha = \alpha_0 + m$, then $Y = \bigcup_{n=1}^\infty Y_n$ where each Y_n is a $\mathscr{K}$-Borelian subset of X of class $(\alpha - 1)$ and each Y_n has a complete sequence $\mu_n = \{\mathscr{H}_j^n\}_{j=1}^\infty$ of countable $\mathscr{F}$-Borelian covers of at most class $(\alpha - 1)$.

Let us define for each n the following sequence $\{T_n\}_{n=1}^\infty$ of subsets of X: $T_0 = \emptyset$,

$$T_n = \bigcup_{k=1}^{n} Y_k \quad \text{for all} \quad n \geqq 1 .$$

Let us consider the following sequence $\mu = \{\mathscr{M}_j\}_{j=1}^{\infty}$ of countable covers of Y where for each j:

$$\mathscr{M}_j = \{H_{ji}^n \cap (Y_n \backslash T_{n-1})\}_{n,i=1}^{\infty} , \quad \text{where for each } n \text{ and } j ,$$
$$\mathscr{H}_j^n = \{H_{ji}^n\}_{i=1}^{\infty} .$$

Since X is perfectly σ-normal $\mathscr{K}_\sigma$ we can easily verify that, for all j, each element of $\mathscr{M}_j$ is a $\mathscr{F}$-Borelian subset of Y of at most class α.

To show that μ is complete, let $\mathscr{E}$ be a maximal μ-Cauchy family. Clearly there exists an n such that $\mathscr{E}$ is a μ_n-Cauchy family. By Proposition 2, [4], $\varnothing \neq \bigcap_{E \in \mathscr{E}} \bar{E}^X \subset Y_n \subset Y$. Therefore, $\bigcap_{E \in \mathscr{E}} \bar{E}^Y \neq \varnothing$ and μ is complete.

Case (3). $\alpha = \alpha_0 + m$ (where α_0 is a limit ordinal and m is even). We can use the same method of proof as for Case (1).

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