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SEMIFREE FINITE GROUP ACTIONS ON HOMOTOPY SPHERES

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Suppose G is a finite group acting semifreely (i.e., free off the fixed set) on a finite CW complex X in the homotopy type of S^n. When X^G is also homotopy equivalent to S^n (as e.g., in $S^n \times D^K$) necessary and sufficent conditions are given to determine the degree of the inclusion $X^G \to X$. It follows that for instance, if G is the group of quaternions (nonabelian of order 8), the only attainable degrees are those $\pm 1 \pmod 8$.

0. Introduction. Suppose that G is a finite group acting cellularly and semifreely (i.e., free off the fixed set) on a finite CW-complex X, in the homotopy type of S^n. Assume that X^G is also homotopy equivalent to S^n. The degree q of the inclusion $X^G \hookrightarrow X$ is defined and by Smith Theory is relatively prime to $|G|$ (see [1; Chap. III]). We give necessary and sufficient conditions for the converse to be true. First recall that when q is relatively prime to $|G|$, the trivial $\boldsymbol{Z}\,G$-module $\boldsymbol{Z}_q$ has projective dimension one and well-defines an element, $[\boldsymbol{Z}_q]$, in $\tilde{K}_0(\boldsymbol{Z}G)$, the reduced projective class group.

THEOREM. *Let G be a finite group acting cellularly and semifreely on a finite CW complex X. Assume that both X and X^G are in the homotopy type of S^n. Let q be the degree of $X^G \hookrightarrow X$. Then $[\boldsymbol{Z}_q] = 0$ in $\tilde{K}_0(\boldsymbol{Z}G)$. Conversely, suppose q is relatively prime to $|G|$ and $[\boldsymbol{Z}_q] = 0$ in $\tilde{K}_0(\boldsymbol{Z}G)$. Then there exists an action of G, as above, in which the degree of $X^G \hookrightarrow X$ is q.*

REMARKS. (i) From [2] it follows, e.g., that the quaternions, a nonabelian group of order 8, cannot act semifreely as above with degree of the inclusion $\pm 3 \pmod 8$.

(ii) When G is cyclic and q is prime to $|G|$, $[\boldsymbol{Z}_q]$ is always zero [4; §6] and therefore it is easy to see how to construct an effective, unrestricted action of the quaternions realizing a degree 3 inclusion. In [1; page 391], Bredon gave examples of semifree (smooth) cyclic group actions realizing any (appropriate) degree. In general, of course, one cannot hope to find smooth semifree actions of this type.

(iii) The invariant introduced here is in fact the same as the invariant, $\chi(f)$, introduced in [3] (when applied to the inclusion map).

1. Proof of the theorem. Suppose G is a finite group acting as described and let q denote the degree of $X^G \to X$. X/X^G has semifree G action with one fixed point, is $n-1$ connected and has $H_n(X/X^G) = \boldsymbol{Z}_q$. By an elementary construction (a nonequivariant version can be found in [5; Prop. 6.13]) one can replace X/X^G with a finite CW-complex Y, with semifree G-action, contractible fixed set, no m cells for $0 < m \leqq n-1$ and such that X/X^G is an equivariant strong deformation retract of Y. Let k denote the cellular dimension of Y. Then

$$0 \longrightarrow C_k(Y, Y^G) \longrightarrow \cdots \longrightarrow C_{n+1}(Y, Y^G) \longrightarrow C_n(Y, Y^G) \twoheadrightarrow \boldsymbol{Z}_q$$

is a finite free resolution of $\boldsymbol{Z}_q$ over $\boldsymbol{Z}G$. This implies that $[\boldsymbol{Z}_q] = 0$ in $\widetilde{K}_0(\boldsymbol{Z}G)$.

Now suppose q is relatively prime to $|G|$ and that $[\boldsymbol{Z}_q] = 0$ in $\widetilde{K}_0(\boldsymbol{Z}G)$. Let M_f be the mapping cylinder of a map $f: S^n \to S^n$ of degree q; $M_f = S^n \times I \cup S^n/(x, 0) \sim f(x)$. Denote by Y the semifree G-complex obtained by taking the disjoint union of $|G|$ copies of M_f, identified along the sphere $S^n \times \{1\}$. G acts on Y via the action induced from the free action on the disjoint union; the fixed set is S^n.

It follows from Mayer-Vietoris that if I_ε denotes the augmentation ideal, i.e., the kernel of $\boldsymbol{Z}G \overset{\varepsilon}{\twoheadrightarrow} \boldsymbol{Z}$ (where $\varepsilon(\Sigma n_i g_i) = \Sigma n_i$), then $H_n(Y) = \boldsymbol{Z}G/qI_\varepsilon$.

There is the exact sequence of $\boldsymbol{Z}G$-modules,

$$0 \longrightarrow I_\varepsilon/qI_\varepsilon \longrightarrow H_n(Y) \twoheadrightarrow \boldsymbol{Z}\,.$$

Since $I_\varepsilon/qI_\varepsilon$ has projective dimension one, it well-defines an element $[I_\varepsilon/qI_\varepsilon]$ in $\widetilde{K}_0(\boldsymbol{Z}G)$.

Now consider the exact sequence of $\boldsymbol{Z}G$-modules ($\boldsymbol{Z}_q$ has trivial G-action),

$$0 \longrightarrow \ker \varepsilon_q \longrightarrow \boldsymbol{Z}G \xrightarrow{\varepsilon_q} \boldsymbol{Z}_q$$

where ε_q is ε followed by reduction mod q). Since $\boldsymbol{Z}_q$ has projective dimension one, it well-defines an element in $\widetilde{K}_0(\boldsymbol{Z}G)$ which is $-[\ker \varepsilon_q]$. Ker ε_q and $I_\varepsilon/qI_\varepsilon$ are related by the exact sequence

$$0 \longrightarrow \boldsymbol{Z}G \xrightarrow{q} \ker \varepsilon_q \twoheadrightarrow I_\varepsilon/qI_\varepsilon\,.$$

The assumption that $[\boldsymbol{Z}_q] = 0$ forces $[I_\varepsilon/qI_\varepsilon] = 0$ and so $I_\varepsilon/qI_\varepsilon$ has a short exact projective resolution

$$0 \longrightarrow \widetilde{F} \longrightarrow F_0 \twoheadrightarrow I_\varepsilon/qI_\varepsilon$$

where F_0 is free and $\widetilde{F}$ is stably free. Adding free orbits of $n+1$

cells to Y to kill $I_\varepsilon/qI_\varepsilon$ (realizing $F_0 \twoheadrightarrow I_\varepsilon/qI_\varepsilon$) produces $\tilde{Y}$ with $H_{n+1}(\tilde{Y}) = \tilde{F}$, $H_n(\tilde{Y}) = \boldsymbol{Z}$. Using the surjective Hurewicz homomorphism ($\tilde{Y}$ is $n-1$ connected), $n+2$ cells may be added ($H_{n+1}(\tilde{Y})$ is made free by wedging, at a fixed point, with a suitable bouquet of spheres) to produce the complex X which is homotopy equivalent to S^n. By considering the homology sequence of the triple $(X, \tilde{Y}, Y^G)$ one can see that the degree $X^G \hookrightarrow X$ is q. $\square$

As in [4; §6], let (q, N) denote the ideal of $\boldsymbol{Z}G$ generated by q (relatively prime to $|G|$) and $N = \sum_{g \in G} g$. It is easy to see that $[\boldsymbol{Z}_q] = [(q, N)]$ in $\tilde{K}_0(\boldsymbol{Z}G)$. This explains Remark (i).

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