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APPROPRIATE CROSS-SECTIONALLY SIMPLE FOUR-CELLS ARE FLAT

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When X is a set in E^n, we let $X_t = X \cap H_t$—where H_t is the horizontal hyperplane in E^n of height t. In this note, we prove that a 4-cell B in E^4, such that each nonempty slice B_t is either a point or a 3-cell, is flat whenever, for all t, B_t is flat in H_t and Bd B_t is flat in Bd B.

1. Introduction and summary. Throughout, we let H_t denote the horizontal hyperplane in E^n at height t, and when X is a set in E^n, we let $X_t = X \cap H_t$. In [**10**], it is proved that an $(n-1)$-sphere S in E^n $(n > 5)$ such that each nonempty slice S_t is either an $(n-2)$-sphere or a point has a 1-ULC complement whenever, for all t, S_t is flat in both H_t and S; subsequently, in [**9**] and [**11**] (see also [**17**]), $(n-1)$-spheres in E^n $(n > 4)$ with 1-ULC complements were shown to be flat. The necessity of these conditions is discussed in [**10**] and [**12**]. Similarly, a 2-sphere in E^3 such that each nonempty slice is a point or a 1-sphere was earlier shown to be flat in [**13**] and [**14**] with each relying upon the 1-ULC taming theorem of [**3**]. In this note, we extend this work to the case $n = 4$ by solving a similar question for a 4-cell; specifically, we prove the following:

THEOREM. *A* 4-*cell B in E^4, such that each nonempty slice B_t is either a point or a* 3-*cell, is flat whenever, for all t, B_t is flat in H_t and* Bd B_t *is flat in* Bd B.

The proof relies upon a condition—first described to us by R. J. Daverman in 1976—under which an n-cell in E^n is flat; Lemma 1 presents it. We include a proof because no reference contains the result; when $n > 4$, it is superceded by the 1-ULC taming theorems of [**3**], [**9**], and [**11**]; yet when $n = 4$, it has utility. (Daverman has pointed out that its hypotheses are strong enough to make the argument in Chernavskii [**7**] work too.)

LEMMA 1. *Let B be a* 4-*cell in E^4. If for each $\varepsilon > 0$ there exists an ε-self-homeomorphism h of E^4 supported in the ε-neighborhood of $E^4 - B$ such that $h(\text{Bd } B) \cap B = \varnothing$, then B is flat.*

The proof of the theorem involves two other lemmas.

LEMMA 2. *Let B be a 4-cell in E^4, and T a 3-cell in B with* $\mathrm{Bd}\, T \subset \mathrm{Bd}\, B$ *and* $\mathrm{Int}\, T \subset \mathrm{Int}\, B$ *such that B is locally flat at each point not in* $\mathrm{Bd}\, T$, $\mathrm{Bd}\, T$ *is flat in* $\mathrm{Bd}\, B$, *and T flat in E^4. Then B is flat.*

LEMMA 3. *Let P be a 4-cell in $E^3 \times I$ such that P_0 and P_1 are points. Suppose P is locally flat at each point of* $\mathrm{Bd}\, P - (W \cup P_0 \cup P_1)$ *where W is a countable union of 2-spheres in* $\mathrm{Bd}\, P$ *and suppose that for each 2-sphere S in W, S is contained in a horizontal hyperplane H_q, S is flat in H_q,* $S = \mathrm{Fr}\, P_q$, *and S is flat in* $\mathrm{Bd}\, P$. *Then P is flat in E^4.*

Lemma 2 may be regarded as giving sufficient conditions for the union of two 3-cells (T and a closed complementary domain of $\mathrm{Bd}\, T$ in $\mathrm{Bd}\, B$) in E^4 along their boundary to be flat, and so is related to [**6**] and [**15**] (see also [**8**]).

2. Proofs of the lemmas.

Proof of Lemma 1. Let $D = \mathrm{Bd}\, B$, $e\colon D \times I \to B$ be a collar on D in B, and let $\{s_i\}$ be a decreasing sequence of numbers from $\mathrm{Int}\, I$ which converges to 0. Use the hypotheses to find a sequence of numbers ε_i and a sequence of ε_i-self-homeomorphisms h_i $(i = 1, 2, \ldots)$ or E^4 such that $\varepsilon_i < \mathrm{dist}(e(D \times \{0\}),\ e(D \times \{s_i\}))$, $\varepsilon_{i+1} < \mathrm{dist}(D, h_i(D))$, h_i leaves $e(D \times \{s_j\})$ fixed for all $j \leq i$, and $h_i(D) \cap B = \varnothing$. Then $\varepsilon_i \to 0$, $h_i(D) \cap h_j(D) = \varnothing$ for $i \neq j$, and $h_i | D$ converges uniformly to the identity. Let $q_i \in (0, 1)$ be so close to 0 that $q_i < s_i$ and

$$\mathrm{dist}\{h_{i+1}e(d, 0), h_{i+1}e(d, q_i)\} < \tfrac{1}{4}\, \mathrm{dist}\{h_{i+1}(D), h_j(D)\}$$

for all $j \neq i + 1$, and d in D. Observe that the spheres $h_i(D)$ and $h_i e(D \times \{q_i\})$ are all pairwise disjoint and "concentric".

Now use the product structure of $h_{i+1}e(D \times I)$ to find ε_i-self-homeomorphisms F_i of E^4 such that

$$F_i h_{i+1} e(d, s_i) = h_{i+1} e(d, q_i) \quad \text{for all } d \text{ in } D. \tag{1}$$

and

$$F_i h_i e(d, q_{i-1}) = h_i e(d, q_{i-1}) \quad \text{for all } d \text{ in } D. \tag{2}$$

Then $F_i h_i e$ embeds $D \times [q_{i-1}, s_i]$ as the annulus between $h_i e(D \times \{q_{i-1}\})$ and $h_{i+1} e(D \times \{q_i\})$.

Let g_i: $D \times [1/(i+1), 1/i] \to D \times [q_{i-1}, s_i]$ be a homeomorphism which preserves first coordinates and takes $D \times \{1/i\}$ to $D \times \{q_{i-1}\}$. Now define G: $D \times I \to E^4 - \text{Int } B$ by

(3) $$G(d, 0) = d \quad \text{for all } d \text{ in } D$$

and

(4) $$G(d, t) = F_i h_i e g_i(d, t) \quad \text{when } 1/(i+1) \leq t \leq 1/i \text{ and } d \in D.$$

First observe that G is continuous on $D \times (0, 1]$ because each composition $F_i h_i e g_i$ is continuous on $D \times [1/(i+1), 1/i)]$ and because (1) and (2) force these maps to agree whenever they have common domain; that is,

(5) $$F_{i+1} h_{i+1} e(d, q_i) = F_i h_{i+1} e(d, s_i) = F_i h_i e(d, s_i).$$

Next observe that G is continuous on $D \times I$ because

$$\text{dist}(F_i h_i e g_i(d, q), e(d, 0)) \to 0 \quad \text{as } i \to \infty.$$

Finally, G is 1-1 because the images $F_i h_t e g_i(D \times (1/(i+1), 1/i))$ are pairwise disjoint—they lie between different pairs of "concentric" spheres. G is a collar on B, so B is flat [**2**]. □

Proof of Lemma 2. Assume the hypotheses. Let G be the decomposition of $\text{Bd } B \times I$ into points and arcs of the form $\{x\} \times I$ with $x \in \text{Bd } T$, let π: $\text{Bd } B \times I \to \text{Bd } B \times I/G$ be the decomposition map, and let e: $\text{Bd } B \times I/G \to B$ be a collar of $\text{Bd } B$ in B pinched at $\text{Bd } T$ such that $\text{diam } e\pi(\{x\} \times I) \leq \frac{1}{2}\varepsilon$ for all $x \in \text{Bd } B$ and such that $e\pi(\text{Bd } B \times I) \cap T = \text{Bd } T$. Let K_1 and K_2 denote the closed complementary domains of $\text{Bd } T$ in $e\pi(\text{Bd } B \times \{\frac{1}{2}\})$. Since B is a 4-cell and since $\text{Bd } T$ is flatly embedded in $\text{Bd } B$, $e\pi(\text{Bd } B \times \{\frac{1}{2}\})$ bounds a 4-cell with $\text{Bd } T$ flatly embedded in its boundary; therefore there exists a homeomorphism h of E^4 fixed on $\text{Bd } B$ such that $h(K_1) = K_2$. Set $T_1 = h(T)$ and $T_2 = h^{-1}(T)$; then $\text{Bd } T_i = \text{Bd } T$, $\text{Int } T \subset \text{Int}(e\pi(\text{Bd } B \times I))$, and each T_i is flat. Also the union of $e\pi(\text{Bd } B \times [0, 1))$ and the compact set bounded by $T_1 \cup T_2$ is B.

Now, according to [**15**], $T_1 \cup T_2$ bounds a flat 4-cell W; hence there exists a $\frac{1}{2}\varepsilon$-self-homeomorphism f of E^4 supported in the ε-neighborhood of $E^4 - W$ such that $f(\text{Bd } W) \cap W = \varnothing$, which means that f is supported in the ε-neighborhood of $E^4 - B$ and

$$f(\text{Bd } B) \subset (E^4 - B) \cup (\text{Bd } B - \text{Bd } T) \cup \text{Int}(e\pi(\text{Bd } B \times I)).$$

Hence, using the pinched collar and the fact that B is locally flat at points not in $\text{Bd } T$, we can produce another $\frac{1}{2}\varepsilon$-self-homeomorphism g of E^4

supported in $\operatorname{Int}(e\pi(\operatorname{Bd} B \times I)) \cup (\operatorname{Bd} B - \operatorname{Bd} T) \cup (E^4 - B)$ such that $gf(\operatorname{Bd} B) \subset E^4 - B$. Lemma 1, with $h = gf$, now shows B is flat. $\square$

Proof of Lemma 3. Assume the hypotheses. Let W' be the set of t in $(0, 1)$ such that P is wild at some point of $\operatorname{Bd} P_t$. Let W^* be the closure of W' in I. Then $W^* \subset W' \cup \{0, 1\}$, so W^* is closed and countable.

We want to show that W^* equals the empty set; suppose it does not. Then by the Baire Category Theorem there exists an isolated point q in W^*. In fact q is in W'. Now by using a pinched collar find a 4-cell $R \subset P$ such that $\operatorname{Bd} R \cap \operatorname{Bd} P$ is a neighborhood in $\operatorname{Bd} P$ of $\operatorname{Bd} P \cap H_q$, such that R is locally flat modulo $\operatorname{Bd} P \cap H_q$, and such that $\operatorname{Bd} P \cap H_q = \operatorname{Bd}(R_q)$. By hypotheses, $\operatorname{Bd} P \cap H_q$ is flat in H_q and $\operatorname{Bd} P$; therefore it is flat in $\operatorname{Bd} R$ too. So according to Lemma 2, R is flat. Hence P is locally flat at each point of $\operatorname{Bd} P - (W - \operatorname{Bd} P \cap H_q)$. It follows that q is not in W', which is a contradiction. Therefore W^* and W' are empty. Hence P is locally flat at each point of $\operatorname{Bd} P - (P_0 \cup P_1)$. It follows from [**4**] that B is flat. $\square$

3. Proof of the theorem. Assume the hypotheses, and assume that $B \subset E^3 \times I \subset E^4$ with B_0 and B_1 singleton sets. Let $J = [-1, 1]$. We want to apply Lemma 1; so let $\varepsilon > 0$ be given. Since B_t is flat in H_t, there exists for each $t \in (0, 1)$ a homeomorphism h_t of $S^2 \times E^1$ onto H_t such that $h_t | S^2 \times J$ is a bicollar on $\operatorname{Bd} B_t$ with $h_t(S^2 \times \{1\}) \subset H_t - B_t$. As in [**10**], there exists a countable set $D \subset I$ such that $s \in I - D$ implies the existence of monotone sequences $\{s(i)\}$ and $\{t(i)\}$ in I converging to t from above and below, respectively, such that $\{h_{s(i)}\}$ and $\{h_{t(i)}\}$ converge to h_t.

Fix t in $I - D$, and let $p\colon E^4 \to E^3$ denote projection. The local contractibility of the homeomorphism group of E^3 [**5**] at the point ph_t shows that for each $\gamma > 0$ there exist an integer k and an isotopy $\{\phi_q\}$ of E^3 such that $\operatorname{dist}(\phi_q(x), ph_t(x)) < \gamma$ for all $q \in I$ and $x \in E^3$, $\phi_1 = ph_{s(k)}$, and $\phi_0 = ph_{t(k)}$. When γ is small enough, an embedding $f_t\colon (S^2 \times J) \times I \to E^4$ may be defined by the rule

$$f_t((a, b), c) = (\phi_c(a, b), c \cdot s(k) + (1 - c) \cdot t(k)),$$

possessing the following six properties:

$$f_t | (S^2 \times J) \times \{1\} = h_{s(k)}; \qquad f_t | (S^2 \times J) \times \{0\} = h_{t(k)};$$

$$f_t\big((S^2 \times \{1\}) \times I\big) \subset E^4 - B; \qquad f_t\big((S^2 \times \{-1\}) \times I\big) \subset \operatorname{Int} B;$$

$$\operatorname{diam} f_t((\{s\} \times J) \times \{q\}) < \tfrac{1}{4}\varepsilon \quad \text{for all } s \in S^2, q \in I;$$

and each set $f_t((S^2 \times J) \times \{q\})$, $q \in I$, is contained in a horizontal hyperplane.

Now let $Q = S^2 \times J \times I$. There exists a countable collection $\{F_i\}$ of these embeddings (each F_i equals some f_t) such that the union $\bigcup_{i=1}^{\infty} F_i(Q) \cup \bigcup_{d \in D} H_d$ is a neighborhood of Bd B in $E^3 \times I$. Let K be the set of $q \in I$ for which $H_q \cap F_i(\text{Int}\, Q) = \varnothing$ for all i. K is countable because D and $\{F_i\}$ are, and K is closed because $\bigcup F_i(\text{Int}\, Q)$ is open.

Let W be the union of the sets $(\text{Bd}\, B)_t$, $t \in K$; then W is a closed subset of Bd B. Hence, as in the proof of Lemma 2, one may use a pinched collar to find a map e: Bd $B \times I \to B$ such that $e(x, 0) = x$ for $x \in \text{Bd}\, B$; $e(x, t) = x$ for $x \in W \cup B_0 \cup B_1$, $t \in I$; $\text{diam}(e(\{x\} \times I)) < \frac{1}{2}\varepsilon$ for $x \in \text{Bd}\, B$; $e \mid (\text{Bd}\, B - W) \times I$ is an embedding; and when $t \in K$, $e(\text{Bd}\, B \times I) \cap E_t \subset W$. Let P be the 4-cell bounded by $e(\text{Bd}\, B \times \{q\})$ where q is so close to D that Bd P is contained in the $\frac{1}{4}\varepsilon$-neighborhood of Bd B. Also, assume without loss of generality that $\text{Bd}\, P \subset \text{Bd}\, B \cup (\bigcup F_i(\text{Int}\, Q))$.

P satisfies the hypotheses of Lemma 3 and is therefore flat in E^4. Hence there exists a $\frac{1}{2}\varepsilon$-self-homeomorphism g of E^4, supported in the ε-neighborhood of Bd B such that $g(\text{Bd}\, P) \cap P = \varnothing$. It follows that

$$g(\text{Bd}\, B) \subset (E^4 - B) \cup \left(\bigcup F_i(\text{Int}\, Q)\right).$$

So, because $g(\text{Bd}\, B) \cap B$ is compact and contained in $\bigcup F_i(\text{Int}\, Q)$, there exists a finite subcollection $F_1, F_2, \ldots, F_N$, say, of the F_i such that $g(\text{Bd}\, B) \cap B \subset \bigcup_{j=i+1}^{N} F_i(\text{Int}\, Q)$. We assume this subcollection is minimal; consequently, no point of E^4 lies in more than two of the sets $F_i(\text{Int}\, Q)$, $i = 1, 2, \ldots, N$.

Now, for each $i = 1, 2, \ldots, N$, let h_i be a $\frac{1}{4}\varepsilon$-self-homeomorphism of E^4 supported in $F_i(\text{Int}\, Q)$, preserving fourth coordinates of E^4, and satisfying

$$h_i h_{i-1} \cdots h_i g(\text{Bd}\, B) \subset (E^4 - B) \cup \left(\bigcup_{j=i+1}^{N} F_i(\text{Int}\, Q) \right).$$

Each h_i is easily found as the composition of F_i and a homeomorphism of Q $(= S^2 \times J \times I)$ onto itself which leaves Bd Q fixed and only changes J coordinates. Observe that $h_N \cdots h_1 g(\text{Bd}\, B) \cap B = \varnothing$.

Then because no point is moved by more than two of the h_i's, $h \equiv h_N \cdots h_1 g$ is an ε-self-homeomorphism of E^4. Clearly h is supported in the ε-neighborhood of B, so Lemma 1 shows B is flat. $\square$

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