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INJECTIVE BANACH LATTICES WITH STRONG ORDER UNITS

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In this note it is shown that a Banach lattice with a strong order unit is injective (i.e. has the Hahn-Banach extension property for positive linear operators) if and only if E is a finite m-sum of spaces of the form $C(X, l_1^n)$, where X is compact and extremally disconnected and where l_1^n denotes $\mathbf{R}^n$ with the L-norm.

0. Introduction. In 1950–1952, a certain type of Banach space, called a P_1-space, appeared in the literature. A P_1-space is a Banach space G having the following extension property for linear maps:

Every bounded linear map $\varphi\colon F \to G$ defined on a linear subspace $F \subseteq E$ allows an extension $\psi\colon E \to G$ such that $\|\varphi\| = \|\psi\|$.

The classical Hahn-Banach theorem says that the one-dimensional space $\mathbf{R}$ is a P_1-space. From 1950–1952, D. B. Goodner [**Go 50**], L. A. Nachbin [**Na 50**] and J. L. Kelley [**Ke 52**] showed that a Banach space G is a P_1-space if and only if G is isometrically isomorphic to a space of the form $C(X)$, where X is an extremally disconnected compact topological space. One may say that P_1-spaces are obtained by "spreading the real line continuously across a compact space."

If one applies these ideas to Banach lattices, then of course one would wish to consider only positive linear maps φ and only linear sublattices $F \subseteq E$.

DEFINITION. A Banach lattice G is called *injective* provided that for every Banach lattice E, for every linear sublattice $F \subseteq E$ and for every bounded positive linear map $\varphi\colon F \to G$ there is a positive linear extension $\psi\colon E \to G$ such that $\|\varphi\| = \|\psi\|$.

In 1975 H. P. Lotz [**Lo 75**] proved that all Banach lattices of the form $C(X)$, X extremally disconnected and compact, are injective. But the class of injective Banach lattices exceeds the class of P_1-spaces: Lotz also showed that AL-spaces, i.e., spaces of the form $\mathfrak{L}_1(\mu)$, are injective. Also in 1975, D. I. Cartwright [**Ca 75**] gave, among other results, a characterization of finite-dimensional injective Banach lattices: They are exactly the m-sums of finite-dimensional AL-spaces. As it turned out, injective Banach lattices in general are obtained by "spreading AL-spaces continuously across an extremally disconnected compact space," i.e., by sections in certain bundles (see **[Ha 77]** and **[Gi 77]**). As bundle representations are sometimes viewed as "complicated," this representation may not seem to be satisfactory. However, in this note we shall see that the bundle representation may be reduced to a much nicer description of injective Banach lattices if we require in addition a strong order unit. We shall prove the following main result.

THEOREM. *Let G be a Banach lattice. Then the following statements are equivalent*:

(i) *G is injective and has a strong order unit.*

(ii) *G is isometrically isomorphic to $C(X_1, l_1^{n_1}) \oplus \cdots \oplus C(X_m, l_1^{n_m})$ where $m \in \mathbf{N}$ and*:

(a) *X_i is an extremally disconnected compact space*, $1 \leq i \leq m$.

(b) $l_1^{n_i} = (\mathbf{R}^{n_i}, \|\cdot\|)$; $\|(x_1, \ldots, x_n)\|_1 = |x_1| + \cdots + |x_n|$.

(c) *For $u_i \in C(X_i, l_1^{n_i})$ we have*

$$\|u_1 \oplus \cdots \oplus u_m\| = \max\{\|u_1\|, \ldots, \|u_m\|\}$$
$$= \sup\{\|u_i(x)\|_1 \mid 1 \leq i \leq n, x \in X_i\}.$$

For notations and results concerning Banach spaces and Banach lattices, we refer to **[Sch 71** and **74]**; fundamental results and definitions concerning bundles of Banach spaces and bundles of Banach lattices may be found in **[Ho 75]** and **[Gi 77** and **81]**. We shall only consider real Banach lattices. The word "compact" as used in this note contains Hausdorff separation. The symbol $\Gamma(p)$ always denotes the set of all continuous sections in a bundle $p: \mathfrak{E} \to X$.

1. The bundle representation. The starting point of our investigation is the following theorem.

1.1. THEOREM. *Let G be a Banach lattice. Then G is injective if and only if it is isometrically isomorphic to the Banach lattice $\Gamma(p)$ of all sections in a bundle $p: \mathfrak{E} \to X$ of AL-spaces such that*:

(a) *X is extremally disconnected and compact*;

(b) *if $\sigma \in \Gamma(p)$ then $x \mapsto \|\sigma(x)\|: X \to \mathbf{R}$ is continuous* (*i.e., $p: \mathfrak{E} \to X$ has continuous norm*);

(c) *if $\sigma: U \to \mathfrak{E}$ is a bounded continuous section defined on an open set $U \subseteq X$, then σ may be extended to a global continuous section $\bar{\sigma}: X \to \mathfrak{E}$.*

Maybe a few words concerning bundles are in order. The space of all sections in a bundle in our case alternatively can be described as follows:

Let X be a compact space. For every $x \in X$ let E_x be a Banach lattice. Then spaces of sections $\Gamma(p)$ may be characterized by:

(i) $\Gamma(p)$ is a closed linear sublattice of $\Sigma_{x\in X} E_x = \{\sigma \in \times_{x\in X} E_x\colon \sup\{\|\sigma(x)\|;\ x \in X\} < \infty\}$, equipped with the sup-norm;

(ii) if $x \in X$ and $\alpha \in E_x$ are given, then there is a $\sigma \in \Gamma(p)$ such that $\sigma(x) = \alpha$;

(iii) the mapping $x \mapsto \|\sigma(x)\|\colon X \to \mathbf{R}$ is upper semicontinuous for every $x \in X$;

(iv) given $f \in C(X)$ and $\sigma \in \Gamma(p)$, then $f \circ \sigma$ defined by $(f \circ \sigma)(x) = f(x) \circ \sigma(x)$ belongs to $\Gamma(p)$, too (i.e., $\Gamma(p)$ is a $C(X)$-module).

Hence, spaces of sections in bundles are nothing but upper semicontinuous function modules in the sense of F. Cunningham and N. M. Roy (see **[CR 74]**).

Of course, being a bundle of AL-spaces means that the Banach lattices E_x, $x \in X$, are all AL-spaces. It should be clear how the conditions (a) and (b) translate to upper semicontinuous function modules. The translation of (1.1.(c)) is less obvious.

Let $U \subseteq X$ be open. Let us call an element $\sigma \in \times_{x\in u} E_x$ a *bounded continuous section* provided that for every continuous function $f \in C(X)$ with support in U (i.e., for which $f(x) = 0$ for all $x \in X\backslash U$) the element $\sigma_f \in \times_{x\in X} E_x$ defined by

$$\sigma_f(x) = \begin{cases} f(x) \circ \sigma(x), & x \in U, \\ 0, & x \notin U, \end{cases}$$

belongs to $\Gamma(p)$. For compact spaces X, this definition coincides with the definition of local sections normally given by bundles.

Now condition (1.1.(c)) translates as expected. Thus, instead of talking about bundles of Banach lattices, the reader may wish to consider upper semicontinuous function modules, which should be possible without major problems.

2. Some results on bundles of Banach lattices.

In order to prove the theorem stated in the Introduction, we need four partial results, which maybe are interesting in themselves.

2.1. Proposition. *Let $p\colon \mathfrak{E} \to X$ be a bundle of Banach lattices, X compact, and assume that $\mathfrak{E}$ is Hausdorff. (This is especially the case if $x \mapsto \|\sigma(x)\|\colon X \to \mathbf{R}$ is continuous for every $\sigma \in \Gamma(p)$.) If $x \in X$ and if*

$0 < \alpha_1, \ldots, \alpha_n \in E_x$ are mutually orthogonal, then there is a neighborhood U of x and continuous sections $\sigma_1, \ldots, \sigma_n \in \Gamma(p)$ such that $\sigma_i(x) = \alpha_i$, $\sigma_i(y) \neq 0$ and $\sigma_i(y) \wedge \sigma_j(y) = 0$ for all $y \in U$ and all $i \neq j$.

Proof. Pick any sequence of sections $\tau_1, \ldots, \tau_n \in \Gamma(p)$ such that $\tau_i(x) = \alpha_i$ for all i. We then define the sections σ_i by

$$\sigma_i = \tau_i - \tau_i \wedge (\tau_i \vee \cdots \vee \tau_{i-1} \vee \tau_{i+1} \vee \cdots \vee \tau_n).$$

Then, for $i \neq j$ we have

$$\begin{aligned}\sigma_i \wedge \sigma_j &= \left(\tau_i - \tau_i \wedge (\tau_1 \vee \cdots \vee \tau_{i-1} \vee \tau_{i+1} \vee \cdots \vee \tau_n)\right) \\ &\quad \wedge \left(\tau_j - \tau_j \wedge (\tau_i \vee \cdots \vee \tau_{j-1} \vee \tau_{j+1} \vee \cdots \vee \tau_n)\right) \\ &\leq (\tau_i - \tau_i \wedge \tau_j) \wedge (\tau_j - \tau_i \wedge \tau_j) \\ &= (\tau_i \wedge \tau_j) - (\tau_i \wedge \tau_j) = 0.\end{aligned}$$

On the other hand, $0 \leq \sigma_i$ as $\tau_i \geq \tau_i \wedge (\tau_1 \vee \cdots \vee \tau_{i-1} \vee \tau_{i+1} \vee \cdots \vee \tau_n)$ and therefore the σ_i are all positive and pairwise orthogonal. Moreover, $\sigma_i(x) = \alpha_i$. As the α_i are mutually orthogonal, we have

$$\begin{aligned}\sigma_i(x) &= \tau_i(x) - \tau_i(x) \wedge (\tau_1(x) \vee \cdots \vee \tau_{i-1}(x) \vee \tau_{i+1}(x) \vee \cdots \vee \tau_n(x)) \\ &= \alpha_i - \alpha_i \wedge (\alpha_1 \vee \cdots \vee \alpha_{i-1} \vee \alpha_{i+1} \vee \cdots \vee \alpha_n) \\ &= \alpha_i - 0 = \alpha_i.\end{aligned}$$

Finally, as $\mathfrak{E}$ is T_2 and as σ_i and 0 are continuous sections which do not agree at x, we can find a neighborhood U of x such that $\sigma_i(y) \neq 0$ for all $y \in U$, $1 \leq i \leq n$.

2.2. PROPOSITION. *Let $p\colon \mathfrak{E} \to X$ be a bundle of Banach lattices over a compact base space X. Assume that*

(b) *the mappings $x \mapsto \|\sigma(x)\|\colon X \to \mathbf{R}$ are continuous for all $\sigma \in \Gamma(p)$;*

(c) *if $\sigma\colon U \to \mathfrak{E}$, $U \subseteq X$ open, is a bounded continuous section, then σ may be extended to a global continuous section $\bar{\sigma}\colon X \to \mathfrak{E}$.*

Then the mapping dim: $X \to \mathbf{R} \cup \{\infty\}$; $x \mapsto \dim E_x$ *is continuous, where $E_x = p^{-1}(x)$ is the stalk over $x \in X$.*

Proof. We already know from (2.1) or **[Gi 81]** that the mapping dim is lower semicontinuous. Hence the sets of the form $\{x\colon \dim E_x \geq n\}$ are open and we have to show that they are closed, too. Thus, let

$$U_n = \{x\colon \dim E_x \geq n\}.$$

Then for every $x_0 \in U_n$ there are $0 < \alpha_1, \ldots, \alpha_n \in E_{x_0}$ which are pairwise orthogonal with norm 1. We therefore can find a closed neighborhood $V \subseteq U_n$ of x_0 and continuous sections $\sigma_i \in \Gamma(p)$ with $\sigma_i \wedge \sigma_j = 0$ for $i \neq j$ and $\sigma_i(x_0) = \alpha_i$. As $x \mapsto \|\sigma_i(x)\|$ is continuous and $\|\sigma_i(x_0)\| = \|\alpha_i\| = 1$, we may assume $\|\sigma_i(x)\| \geq \frac{1}{2}$ for all $x \in V$. Therefore the mapping $x \mapsto \|\sigma_i(x)\|^{-1}$: $V \to \mathbf{R}$ is well defined and continuous. Extend this mapping to a continuous function f_i: $X \to \mathbf{R}$. Then $f_i(x_0) = 1$ and hence $(f_i \circ \sigma_i)(x_0) = \alpha_i$. We now define new sections τ_i by

$$\tau_i := f_i \circ \sigma_i, \qquad 1 \leq i \leq n.$$

Clearly $\|\tau_i(y)\| = 1$ for all $y \in V$. Hence we have proved:

(*) Every $x_0 \in U_n$ has a neighborhood V such that there exist positive pairwise orthogonal continuous sections $\tau_1, \ldots, \tau_n \in \Gamma(p)$ satisfying $\|\tau_i(y)\| = 1$ for all $y \in V$ and all $1 \leq i \leq n$.

We now let

$$\mathcal{F} = \{(V, \sigma_1, \ldots, \sigma_n) \colon V \subseteq U_n,\ V \text{ open},\ \sigma_i \colon V \to \mathcal{E} \text{ is a continuous section},$$
$$\sigma_i \wedge \sigma_j = 0 \text{ for } i \neq j \text{ and } \|\sigma_i(y)\| = 1 \text{ for } y \in V\}.$$

We order $\mathcal{F}$ by

$$(V, \sigma_1, \ldots, \sigma_n) \leq (W, \tau_1, \ldots, \tau_n) \quad \text{iff} \quad V \subseteq W \text{ and } \tau_{i|_V} = \sigma_i.$$

Apply Zorn's lemma to find a maximal element $(U, \sigma_1, \ldots, \sigma_n)$ of $\mathcal{F}$. We claim

(1) $U_n \subseteq \overline{U}$, as otherwise we would have $U_n \backslash \overline{U} \neq \varnothing$. Pick $x_0 \in U_n \backslash \overline{U}$ and apply (*) to obtain an open set $W \subseteq U_n \backslash \overline{U}$ and continuous sections ρ_i: $W \to \mathcal{E}$ which are positive, pairwise orthogonal and satisfy $\|\rho_i(y)\| = 1$ for $y \in W$. Let $U' = U \cup W$ and define

$$\sigma_i'(x) = \begin{cases} \sigma_i(x), & x \in U, \\ \rho_i(x), & x \in W. \end{cases}$$

We obtain $(U, \sigma_1, \ldots, \sigma_n) < (U', \sigma_1', \ldots, \sigma_n')$, contradicting the maximality of $(U, \sigma_1, \ldots, \sigma_n)$.

(2) $\overline{U} \subseteq U_n$. By property (c) we can find extensions $\bar{\sigma}_i$: $\overline{U} \to \mathcal{E}$ of σ_i. Now (b) implies $\bar{\sigma}_i(y) \wedge \bar{\sigma}_j(y) = 0$ and $\|\bar{\sigma}_i(y)\| = 1$ for all $i \neq j$, $y \in \overline{U}$. Especially, for every $y \in \overline{U}$ the elements $\bar{\sigma}_1(y), \ldots, \bar{\sigma}_n(y) \in E_y$ are linearly independent, showing dim $E_y \geq n$, i.e., $\overline{U} \subseteq U_n$.

Now (1) and (2) mean $\overline{U} = U_n$, i.e., U_n is closed.

2.3. PROPOSITION. *Let $p\colon \mathfrak{E} \to X$ be a bundle of finite-dimensional Banach lattices over a compact base space X and assume* (b) *and* (c) *hold. Then there are finitely many (possibly empty) open and closed subsets $U_1, \ldots, U_n \subseteq X$ which cover X and have the property that E_y is k-dimensional for every $y \in U_k$, where $E_y = p^{-1}(y)$ is the stalk over y. Moreover, the Banach lattice $\Gamma(p)$ of all continuous sections of p is, up to an equivalent norm, isomorphic to the Banach lattice $C(U_1, \mathbf{R}) \oplus C(U_2, \mathbf{R}^2) \oplus \cdots \oplus C(U_n, \mathbf{R}^n)$. In addition, if all stalks are AL-spaces and if we equip $\mathbf{R}^k$ with the norm $\|\cdot\|_1$ given by $\|(x_1, \ldots, x_k)\| = |x_1| + \cdots + |x_k|$, then this isomorphism is in fact an isometry.*

Proof. Firstly, note that for every open and closed subset $U \subseteq X$ the Banach lattice $\Gamma(p)$ is isometrically isomorphic with $\Gamma(p_{|U}) \oplus \Gamma(p_{|X \setminus U})$, where $\Gamma(p_{|U})$ denotes the Banach lattice of all continuous sections $\sigma\colon U \to \mathfrak{E}$. Hence it is enough to consider the case where $X = U_n$ and $U_1 = \cdots = U_{n-1} = \varnothing$. In this case let $(U, \sigma_1, \ldots, \sigma_n)$ be a maximal element of $\mathfrak{F}$, where $\mathfrak{F}$ is defined as in the proof of (2.2). Then actually $U = X$, as we saw in the proof of (2.2). Thus we can find continuous sections $\sigma_1, \ldots, \sigma_n \in \Gamma(p)$ which are mutually orthogonal and satisfy $\|\sigma_i(y)\| = 1$ for all $y \in X$. Define a map

$$T\colon C(X, \mathbf{R}^n) \to \Gamma(p),$$
$$\vec{f} \mapsto f_1\sigma_1 + \cdots + f_n\sigma_n$$

where the f_i are the coordinate functions of $\vec{f}$. Then T is linear. Moreover, an easy calculation shows

$$\begin{aligned}
|F(\vec{f})|(x) &= |f_1\sigma_1 + \cdots + f_n\sigma_n|(x) \\
&= |f_1(x) \circ \sigma_1(x) + \cdots + f_n(x)\sigma_n(x)| \\
&= |f_1(x) \circ \sigma_1(x)| + \cdots + |f_n(x) \circ \sigma_n(x)|, \\
&\qquad \text{as the } \sigma_i \text{ are mutually orthogonal,} \\
&= |f_1(x)| \circ \sigma_1(x) + \cdots + |f_n(x)| \circ \sigma_n(x), \quad \text{as } \sigma_i(x) \geq 0, \\
&= T(|\vec{f}|)(x).
\end{aligned}$$

Thus T is a lattice homomorphism.

Next, equip $\mathbf{R}^n$ with the l_1-norm $\|\cdot\|_1$. Then T is a contraction. Let $\vec{f} \in C(X, \mathbf{R}^n)$. Then we have

$$\|\vec{f}\| = \sup_{x \in X} |f_1(x)| + \cdots + |f_n(x)|$$

and

$$\begin{aligned}\|T(\vec{f})\| &= \|\,|T(\vec{f})|\,\| = \|T(|\vec{f}|)\| \\ &= \sup_{x\in X} \|\,|f_1(x)|\sigma_1(x) + \cdots + |f_n(x)|\sigma_n(x)\| \\ &\leq \sup_{x\in X} \|\,|f_1(x)|\sigma_1(x)\| + \cdots + \|\,|f_n(x)|\sigma_n(x)\| \\ &= \sup_{x\in X} |f_1(x)|\,\|\sigma_1(x)\| + \cdots + |f_n(x)|\,\|\sigma_n(x)\| \\ &= \sup_{x\in X} |f_1(x)| + \cdots + |f_n(x)|\,.\end{aligned}$$

Note that the only inequality occurring in the computation becomes equality, providing that every stalk E_x is an AL-space. Moreover, for every $\vec{f} \in C(X, \mathbf{R}^n)$ we have

$$\begin{aligned}\|T(\vec{f})\| &= \sup_{x\in X} \|\,|f_1(x)|\sigma_1(x) + \cdots + |f_n(x)|\sigma_n(x)\| \\ &\geq \sup_{x\in X} \|\,|f_i(x)|\sigma_i(x)\| \quad (\text{as } 0 \leq a \leq b \text{ implies } \|a\| \leq \|b\| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{in every Banach lattice}) \\ &= \sup_{x\in X} |f_i(x)|\,\|\sigma_i(x)\| \\ &= \sup_{x\in X} |f_i(x)| = \|f_i\|\end{aligned}$$

showing that

$$\|T(\vec{f})\| \geq \max\{\|f_1\|, \ldots, \|f_n\|\} \geq (1/n)\|\vec{f}\|.$$

This last inequality yields the injectivity of T and shows that the norm on $C(X, \mathbf{R}^n)$ and the norm on $\Gamma(p)$ restricted to the image of T are equivalent. Especially $C(X, \mathbf{R}^n)$ being a Banach space, the image of T is closed in $\Gamma(p)$. It remains to show that the image of T is also dense in $\Gamma(p)$. Clearly, T is a $C(X)$-module homomorphism and therefore the image of T is a $C(X)$-submodule of $\Gamma(p)$. Moreover, as the $\sigma_1(x), \ldots, \sigma_n(x)$ form a base of E_x, we have $E_x = \{T(\vec{f})(x) \colon \vec{f} \in C(X, \mathbf{R}^n)\}$. Hence, the Stone-Weierstrass theorem for bundles (see **[Ho 75]**) shows that the image of T is dense in $\Gamma(p)$.

3. The proof of the theorem. In this last section we shall give a proof of the theorem stated in the Introduction. Let us begin with an injective Banach lattice G having a strong order unit u. Then every quotient of G has strong order unit, too. Represent G as the space of all sections in a bundle of AL-spaces. As the stalks of a bundle with compact base space are always quotients of the space of all sections, we obtain that all the stalks of the bundle used in (1.1) are AL-spaces with strong order units. Now an AL-space with strong order unit (in fact, every Banach lattice with strong order unit) is, up to an equivalent norm, isomorphic to a Banach lattice of the form $C(Y)$, Y compact. For an AL-space this can only be true if it is finite dimensional. Hence all the stalks of the bundle used in (1.1) have to be finite dimensional. Now an application of (2.3) provides us with a proof of (i) $\Rightarrow$ (ii).

To verify "(ii) $\Rightarrow$ (i)" we first recall from Cartwright's paper **[Ca 75]** that a sum of injective Banach lattices is again injective. Since a finite sum of Banach lattices with a strong order unit also has a strong order unit, it is enough to consider $C(X, l_1^n)$, where X is extremely disconnected. Clearly, the function $\mathbf{1}$: $X \to \mathbf{R}^n$; $x \mapsto (1,\ldots,1)$ is a strong order unit for $C(X, l_1^n)$. It remains to show that $C(X, l_1^n)$ is injective. We shall apply (1.1) to do so. Let $\mathfrak{E} = X \times l_1^n$ and let p: $\mathfrak{E} \to X$ be the first projection. Then p is a bundle and $\Gamma(p) = C(X, l_1^n)$. Moreover, if $U \subseteq X$ is open, then σ: $U \to \mathfrak{E}$ is a bounded continuous section if and only if there are bounded continuous functions $f_1,\ldots,f_n$: $U \to l_1^n$ such that $\sigma(x) = (f_1(x),\ldots,f_n(x))$ for every $x \in U$. As X is extremely disconnected, each of the f_i has a continuous extension g_i: $X \to l_1^n$. Clearly $\bar{\sigma} \in \Gamma(p)$ defined by $\bar{\sigma}(x) = (g_1(x),\ldots,g_n(x))$ extends σ. Now (1.1) tells us that $\Gamma(p) = C(X, l_1^n)$ is injective.

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