PACIFIC JOURNAL OF MATHEMATICS
Vol. 116, No. 2, 1985

LIFTINGS OF SUPERCUSPIDAL REPRESENTATIONS OF Gl_2

JOSÉ E. PANTOJA

Let F be a p-field. Let E/F be a tamely ramified cyclic extension of odd degree. Denote by $\pi_{E/F}$ and Π respectively, the lift and the Shintani lift of an irreducible supercuspidal representation π of $\mathrm{Gl}_2(F)$. The comparison of these two lifts of π is made by breaking up the formula for the character of a supercuspidal representation into a sum over a certain set of double cosets. As a result, we show that the liftings $\pi_{E/F}$ and Π are equivalent.

Let F be a p-field; that is, the completion under the p-adic topology of either an algebraic number field or an algebraic function field. Let W_F be the absolute Weil group of F. Then it is a conjecture of Langlands that there should exist a "natural" map $\sigma \mapsto \pi(\sigma)$ between the set $A_d(F)$ of the continuous d-dimensional representations of W_F and a certain subset of the set $A(\mathrm{Gl}_d(F))$ of admissible irreducible representations of the general linear group $\mathrm{Gl}_d(F)$. (For the history and current status of this problems see for example [**J-L**], [**Sh**], [**K4**]; for generalizations see [**B**]).

Since the map $\sigma \mapsto \pi(\sigma)$ should be natural, we may expect, among other things, that the map which sends a d-dimensional representation σ of W_F to its restriction σ_E of W_E should correspond to a map which sends irreducible admissible representations of $\mathrm{Gl}_d(F)$ to irreducible admissible representations of $\mathrm{Gl}_d(E)$. Two candidates for this latter map have been proposed in different contexts by Shintani and Kutzko when $d = 2$ (see [**L**], [**K4**]). Shintani's map comes about from global considerations and is defined in fact as a map on characters in the case that the extension E/F is cyclic of prime degree. Kutzko's map is defined in terms of the representations and plays a central role in his proof of the correspondence in the case of $d = 2$ (see [**K4**]). However, it is defined only in case the extension is tamely ramified. In order to better understand the nature of Langland's proposed correspondence in dimensions greater than two it is thus of importance to compare these two maps. This will be our goal in what follows.

In §1 the requisite definitions and preliminaries are provided to describe the representation theory of Gl_2. In particular, the set of supercuspidal representations (those representations which should correspond to irreducible representations of W_F is constructed). In §2 Kutzko's map (the tame lift) is defined and several of its properties given. Section 3 is devoted to discussion of characters and a description of Shintani's lift. In §4 the main result of this work appears. Our approach here is to break up the formula for the character of a supercuspidal representation into a sum over a certain set of double cosets. We then show that these summands in the character formula for Kutzko's lift satisfy the condition to be Shintani's lift, at least on the set of elliptic elements. We should note here that we must use very different arguments depending on whether the double coset in question is the identity double coset or not. We then apply an orthogonality result of Langlands to conclude (Theorem 4.11.1) that the two lifts coincide whenever E/F is a prime cyclic tamely ramified extension of odd degree.

1.1. Let F be a p-field; that is, the completion under the p-adic topology of either an algebraic number field or an algebraic function field. Let O_F be its ring of integers, P_F the maximal ideal of O_F, $U_F = U_F^0$ the units of O_F, π_F a generator of P_F, and k_F the residue class field O_F/P_F. For an element x in F, we denote the valuation of x by $\nu_F(x)$. $\mathrm{Gl}_2(F)$ (respectively $\mathrm{Gl}_2(O_F)$) will denote the group of 2 by 2 invertible matrices with coefficients in F (respectively in O_F).

In what follows we will need certain subgroups of $\mathrm{Gl}_2(F)$ which are best realized as stability subgroups for certain natural actions of $\mathrm{Gl}_2(F)$. Our approach here is as in **[K4]**. Also for more details and proofs see **[Sp]**.

Let $V_F = F \otimes F$.

DEFINITION 1.1.1. A lattice flag in V_F is a sequence $L = \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$ of free rank 2 O_F-modules of V_F such that for all t, $P_F L_t = L_{e+t}$ where $e = 1$ or $e = 2$ and $\dim_{k_F} L_t/L_{t+1} = 1$. e will be called the ramification degree of the lattice.

DEFINITION 1.1.2. Two lattice flags L and L' are equivalent if there is an integer r such that for all t, $L_t = L'_{t+r}$. We denote the equivalence class of L by $[L]$.

There is a natural action of $\mathrm{Gl}_2(F)$ on both the set of lattices and the set of equivalence classes of lattices. These actions are transitive.

DEFINITION 1.1.3. Denote by $\mathscr{E}(L)$ the ring of endomorphisms g in $\mathrm{End}_F(V_F)$ for which for all t we have $gL_t \subset L_t$. Denote for any integer r, by $\mathscr{E}_r(L)$, the $\mathscr{E}(L)$-module (two-sided $\mathscr{E}(L)$ ideal if $r \geq 0$) consisting of all g for which for all t we have $gL_t \subset L_{t+r}$.

For this section we assume that $e = 2$, i.e., the flags are ramified.

We may choose an element Π_L in $\mathscr{E}_1(L)$ such that $\mathscr{E}_r(L) = \Pi_L^r\mathscr{E}(L) = \mathscr{E}(L)\Pi_L^r$.

DEFINITION 1.1.4. We denote by $K([L])$ the stabilizer in $\mathrm{Gl}_2(F)$ of $[L]$ and by $B(L)$ the stabilizer in $\mathrm{Gl}_2(F)$ of L.

Then $B(L)$ is the group of units of the ring $\mathscr{E}(L)$.

We obtain a natural filtration for $B(L)$ by setting $B_0(L) = B(L)$ and $B_r(L) = 1 + \mathscr{E}_r(L)$ for $r \geq 1$.

PROPOSITION 1.1.5. *$\mathscr{E}_r(L)$ depends only on the class $[L]$ to which L belongs. $K([L])$ is the normalizer of $B(L)$ in $\mathrm{Gl}_2(F)$; in fact $K([L]) = \langle\Pi_L\rangle B(L)$.*

DEFINITION 1.1.6. If ψ is a non-trivial (complex, continuous) character of F^+, the conductor of ψ is the largest ideal P_F^n contained in the kernel of ψ. In this case we write $n = f(\psi)$.

DEFINITION 1.1.7. If χ is a (complex, continuous, not necessarily unitary) character of $F^\times$ then we say that the conductor of χ is P_F^n if $U_F^n = 1 + P_F^n$ is the largest of the subgroups U_F^n contained in the kernel of χ. Again, we set $n = f(\chi)$.

A computation shows that $(x, y) \to \operatorname{tr} xy$ gives a non-degenerate pairing of $\mathscr{E}_n(L)/\mathscr{E}_{2n}(L)^\times\mathscr{E}_{1-2n}(L)/\mathscr{E}_{1-n}(L)$ into F/P_F. Also, $x \mapsto x - 1$ induces an isomorphism between $B_n(L)/B_{2n}(L)$ and the additive group of $\mathscr{E}_n(L)/\mathscr{E}_{2n}(L)$.

DEFINITION 1.1.8. Let ψ be a character of F^+ of conductor P_F and let b be an element of $\mathscr{E}_{1-2n}(L)/\mathscr{E}_{1-n}(L)$. Then we define the character ψ_b on $B_n(L)/B_{2n}(L)$ by $\psi_b(x) = \psi(\operatorname{tr} b(x-1))$.

We note that the map $b \mapsto \psi_b$ induces an isomorphism of the additive group of $\mathscr{E}_{1-2n}(L)/\mathscr{E}_{1-n}(L)$ with the complex dual $\widehat{B_n(L)/B_{2n}(L)}$ of $B_n(L)/B_{2n}(L)$ and that this isomorphism commutes with the natural action of $K([L])$ on both groups; i.e., $\psi_{xbx^{-1}} = \psi_b^x$ for x in $K(L)$ where ψ_b^x is defined by $\psi_b^x(y) = \psi_b(x^{-1}yx)$.

1.2. In this section we discuss the notion of generic elements. Our approach follows that of **[K5]**. The set of generic elements is introduced for two main reasons. First, as we will see, every irreducible supercuspidal representation will contain a subrepresentation ψ_b for generic b. Second, the set of generic elements will be a convenient set on which to compute characters.

Throughout this section we consider ramified lattices.

DEFINITION 1.2.1. An element x of $\mathscr{E}(L)$ is $\mathscr{E}(L)$-generic of level r if $F[x]/F$ is a totally ramified extension of degree 2, $\nu_{F[x]}(x) = r$ is odd, and $O_{F[x]} = F[x] \cap \mathscr{E}(L)$.

An element x of $B(L)$ is $B(L)$-generic of level r if $x - 1$ is $\mathscr{E}(L)$-generic of level r.

An element x of $K([L])$ is $K([L])$-generic of level r if for some d in $F^\times$, dx is $B(L)$-generic of level r. x is $K([L])$-generic of level $-\infty$ if x lies in $K([L]) - F^\times B(L)$.

We denote by $\mathscr{E}'(L)$, $B'(L)$, $K'([L])$, respectively, the sets of $\mathscr{E}(L)$, $B(L)$ and $K([L])$-generic elements. Also, we denote by $\mathscr{E}_r'(L)$, $B_r'(L)$, $K_r([L])$, respectively, the sets of $\mathscr{E}(L)$, $B(L)$ and $K([L])$-generic elements of level r.

DEFINITION 1.2.2. A subset S of a group G is a trivial intersection set in G or a T.I. set if it does not contain 1 and

(a) $S \subset N_G(S)$, the normalizer of S in G,

(b) if g is an element of G that does not lie in $N_G(S)$ then $g^{-1}Sg \cap S = \varnothing$.

Given an element x in $M_2(F)$ such that $F[x]/F$ is totally ramified and $\nu_{F[x]}(x)$ is odd there is a natural class of flags $\mathscr{L}_E$ associated to the field $E = F[x]$, this class having the property that if L is in $\mathscr{L}_E$ then x is $\mathscr{E}(L)$-generic. To construct this class we proceed as follows:

Given $v \neq 0$ in V_F, we map E into V_F by $g \mapsto gv$. This map is an F-isomorphism of vector spaces.

Define $L_i = P_E^i v$; the fact that E/F is ramified implies that $P_F P_E^r = P_E^{r+2}$, and we obtain in this way a lattice flag L on V_F.

We note here that $[L]$ is independent of v, for if w is in V_F, then by the previous isomorphism we know that there is an element y in E such that $w = yv$ and then $P_E^r w = P_E^r yv = yP_E^r v$. Thus, the lattices are equivalent. This class is the class $\mathscr{L}_E$ to which we alluded above.

The content of the next proposition is that, with x, E as above, $\mathscr{L}_E$ is the only class of lattices for which x is generic (cf. Corollary 1.2.4 below).

PROPOSITION 1.2.3. *Let L be a lattice flag in V_F; then $\mathscr{E}(L) \cap E = O_E$ if and only if L lies in $\mathscr{L}_E$.*

Proof. Suppose first that L is in $\mathscr{L}_E$. Then $L_\iota = P_E^\iota v$ and so $O_E L_\iota = O_E P_E^\iota v \subset P_E^\iota v = L_\iota$. This says that O_E is a subring of $\mathscr{E}(L) \cap E$. However, O_E is a maximal proper subring of E and so $O_E = \mathscr{E}(L) \cap E$.

Conversely we assume that $E \subset M_2(F)$, $\mathscr{E}(L) \cap E = O_E$. If $v \neq 0$ is an element of V_F then $V_F = Ev$. The action of E on V_F is given by $\alpha(\beta v) = (\alpha\beta)v$. Consider the isomorphism ϕ: $E \to V_F$ such that $\phi(\beta) = \beta v$ and define $\hat{\phi}$: $M_2(F) \to \mathrm{End}_F(E)$ given by $\hat{\phi}(g) = \phi^{-1}g\phi$; then $\hat{\phi}$ leaves E fixed. One checks that $\phi^{-1}(L)$ is a lattice flag in E and we have $\hat{\phi}^{-1}(\mathscr{E}(L) \cap E) = \hat{\phi}^{-1}(O_E)$. Thus $\mathscr{E}(\phi^{-1}(L)) \cap E = O_E$ so that $O_E\phi^{-1}(L_0) \subset \phi^{-1}(L_0)$ and $\phi^{-1}(L_0)$ is a fractional ideal of E. So $\phi^{-1}(L_0) = P_E^s$ for some s. By the same reasoning we have $\phi^{-1}(L_1) = P_E^t$ for some t; since $\dim_{k_F} L_0/L_1 = 1$ we have that $t = s + 1$. This implies that the lattice L in V_F is given by $\{P_E^s v\}$, so that L lies in $\mathscr{L}_E$.

COROLLARY 1.2.4. *If $F[x]/F$ is quadratic ramified and $v_{F[x]}(x)$ is odd then $\mathscr{L}_{F[x]}$ is the unique equivalence class of lattice flags for which x is generic.*

The next proposition gives a very useful and important property of generic elements.

PROPOSITION 1.2.5.

(a) *If g is an element of $K([L])$ then g lies in the normalizer $N_{\mathrm{Gl}_2(F)}(\mathscr{E}'(L))$ of $\mathscr{E}'(L)$ in $\mathrm{Gl}_2(F)$.*

(b) *If g is out of $K([L])$ then $g^{-1}\mathscr{E}'(L)g \cap \mathscr{E}'(L) = \varnothing$.*

Proof. Let x be an element of $\mathscr{E}'(L)$ and take any lattice flag L belonging to $\mathscr{L}_{F[x]}$. It follows that gxg^{-1} is generic with respect to the class $\mathscr{L}_{F[gxg^{-1}]}$ to which gL belongs. Now, if g does not lie in $K([L])$ it follows that gL is not equivalent to L and so $\mathscr{L}_{F[gxg^{-1}]} \neq \mathscr{L}_{F[x]}$. Thus, according to Corollary 1.2.4 gxg^{-1} is not generic with respect to $[L]$. On the other hand, if g is an element of $K([L])$, then gL is equivalent to L and then $\mathscr{L}_{F[gxg^{-1}]} = \mathscr{L}_{F[x]}$. Thus gxg^{-1} is generic with respect to $[L]$. We have proved (a) and (b).

COROLLARY 1.2.6. *$B'(L)$ is a T.I. set. Specifically*:

(a) *If g is an element of $K([L])$ then g lies in the normalizer $N_{\mathrm{Gl}_2(F)}(B'(L))$, of $B'(L)$ in $\mathrm{Gl}_2(F)$.*

(b) *If g is out of $K([L])$ then $g^{-1}B'(L)g \cap B'(L) = \varnothing$.*

We note that in fact, $K'([L])$ is also a T.I. set. This will be proved in Proposition 1.2.13 below.

Since $\mathrm{Gl}_2(F)$ acts transitively on the set of lattices flags, it is only necessary, for most applications, to work with one such flag. We may select a convenient lattice flag as follows.

DEFINITION 1.2.7. Let $L^0 = L^0(F)$ be the lattice flag in V_F defined by $L^0(F)_0 = O_F \oplus O_F$, $L^0(F)_1 = O_F \oplus P_F$.

Write $[A_{ij}]$ for the set of matrices $[a_{ij}]$ with a_{ij} in A_{ij}. As usual, if r is a real number, let $[r]$ denote the integer part of r.

We may take

$$\Pi_{L^0} = \begin{bmatrix} 0 & 1 \\ \pi_F & 0 \end{bmatrix}$$

and a computation shows that

$$\mathscr{C}(L^0) = \begin{bmatrix} O_F & O_F \\ P_F & O_F \end{bmatrix};$$

$$\mathscr{C}_n(L^0) = \begin{bmatrix} P_F^{[(n+1)/2]} & P_F^{[n/2]} \\ P_F^{[n/2]+1} & P_F^{[(n+1)/2]} \end{bmatrix};$$

$$B(L^0) = \begin{bmatrix} U_F & O_F \\ P_F & U_F \end{bmatrix}.$$

The following proposition provides additional characterizations of $\mathscr{C}(L^0)$-generic elements.

PROPOSITION 1.2.8. *Let x be an element of $\mathscr{C}(L^0)$; let $\nu_F(\det x) = r$ be an odd number. Then the following are equivalent*:

(a) *x lies in $\mathscr{C}_r(L^0)$*;

(b) *x lies in $\Pi_{L^0}^r B(L^0)$*;

(c) *x is generic of level r.*

Proof. The equivalence of (a) and (b) is clear.

Now we assume (a) and consider the element $y = \pi_F^{(1-r)/2}x$ of $\mathscr{C}_1(L^0)$. Let us denote by s the trace of y and by Δ the determinant of y; then y satisfies the Eisenstein equation $y^2 - sy + \Delta = 0$. Thus, $F[x] = F[y]$ is a quadratic and ramified extension of F. Since $\nu_{F[x]}(y) = \nu_F(N_{F[x]/F}(y))$ ($N_{F[x]/F}$ being the norm of the extension), it follows that y is a prime element for $F[x]$ and thus $O_{F[x]} = O_F[y]$. This gives $O_{F[x]} \subset \mathscr{E}(L^0) \cap F[x]$ and the maximality of the proper subring $O_{F[x]}$ yields the desired equality so that (c) holds.

Conversely, assume (c) and define y as above. Since $\nu_F(\det y) = 1$, y is a prime element in $F[x] = F[y]$. Thus, the equation $y^2 - sy + \Delta = 0$ is Eisenstein. Write $y = [y_{ij}]$. It follows that $\nu_F(y_{11}), \nu_F(y_{22}) \geq 1, \nu_F(y_{21}y_{12}) = 1$. From the fact that y belongs to $\mathscr{E}(L^0)$ we conclude that y lies in $\mathscr{E}_1(L^0)$ and so x is in $\mathscr{E}_r(L^0)$. We have then that (c) implies (a) and the proof is complete.

Next we state some lemmas which are going to be useful in proving that $K'([L])$ is a T.I. set

LEMMA 1.2.9. *If c is in F and A is in $M_2(F)$ then* $\det(c + A) = c^2 + c\operatorname{tr} A + \det A$.

Proof. Straightforward.

LEMMA 1.2.10. *If x lies in $K([L^0])$, then x is $K([L^0])$-generic of level $-\infty$ if and only if $\nu_F(\det x)$ is odd.*

Proof. $K([L^0])$ is the disjoint union of the sets $F^\times B(L^0)$ and $F^\times \Pi_{L^0} B(L^0)$. The first of these sets consists of the elements x of $K([L^0])$ such that $\nu_F(\det x)$ is even, and the second consists of elements such that $\nu_F(\det x)$ is odd. Thus the lemma follows from the fact that if x is in $K([L^0])$ then x lies in $F^\times B(L^0)$ if and only if $\nu_F(\det x)$ is even.

LEMMA 1.2.11. *Let x be an element of $F^\times B_n(L^0)$ with n odd. Then*

$$\sup_{c \text{ in } F^\times} \nu_F(\det(cx - 1)) \geq n.$$

We have equality if and only if x is $K([L^0])$-generic of level n.

Proof. x is of the form dk where d lies in $F^\times$ and k lies in $B_n(L^0)$. Then $d^{-1}x - 1$ belongs to $\mathscr{E}_n(L^0)$. So $\sup_{c \text{ in } F^\times} \nu_F(\det(cx - 1)) \geq n$. Furthermore if we write $d^{-1}x - 1 = [a_{ij}]$ we observe that $\nu_F(a_{11}a_{22}) \geq n + 1$ and $\nu_F(a_{12}a_{21}) \geq n$. Suppose now that $\sup_{c \text{ in } F^\times} \nu_F(\det(cx - 1)) = n$; then, by the above, it must be that $\nu_F(a_{12}a_{21}) = n$ and one checks that $d^{-1}x - 1$ belongs to $\Pi_L^n B(L^0)$. So x is $K_F([L^0])$-generic. Conversely, if x is $K_F([L^0])$-generic then

$$x = \begin{bmatrix} du_1 & d\pi_F^{(n+1)/2}u_3 \\ d\pi_F^{(n-1)/2}u_4 & du_2 \end{bmatrix}$$

where u_i is in $U_F (1 \leq i \leq 4)$ and d is in $F^\times$. Thus, if c lies in $F^\times$ then

$$\det(cx - 1) = (cdu_1 - 1)(cdu_2 - 1) - \pi_F^n c^2 d^2$$

and we observe that the highest possible valuation is n.

LEMMA 1.2.12. *If g does not lie in $K([L^0])$ then*

$$g^{-1}K_n'([L^0])g \cap K_n'([L^0]) = \varnothing.$$

Proof. We consider first the case $n = -\infty$. In this case we have that

$$K_{-\infty}' = \Pi_{L^0} F^\times B(L^0) = F^\times \Pi_{L^0} B(L^0) = F^\times \mathscr{E}_1'(L^0).$$

Thus, if $g^{-1}K_{-\infty}'([L^0])g$ intersects $K_{-\infty}'([L^0])$ then $F^\times g^{-1}\mathscr{E}_1'(L^0)g$ intersects $\mathscr{E}_1'(L^0)$ and so $v = cg^{-1}wg$ for some c in $F^\times$ and v and w in $\mathscr{E}_1'(L^0)$. From this $\nu_F(\det v) = \nu_F(c^2) + \nu_F(\det w)$, so $\nu_F(c^2) = 0$ and c is a unit. But then cw lies in $\mathscr{E}_1'(L^0)$. The element $v = g^{-1}cwg$ lies in both $\mathscr{E}_1'(L^0)$ and $g^{-1}\mathscr{E}_1'(L^0)g$, which contradicts Proposition 1.2.5.

For the case $n > 0$, $K_n'([L^0]) = F^\times B_n'(L^0)$. Assume that $g^{-1}F^\times B_n(L^0)g$ intersects $F^\times B_n'(L^0)$. It follows that $t = cg^{-1}kg$ where t and k lie in $B_n'(L^0)$ and c is an element of $F^\times$. Thus, $\nu_F(\det t) = \nu_F(c^2) + \nu_F(\det k)$. But both $\det t$ and $\det k$ are units of F and so c lies in U_F. The element $t - 1 = cg^{-1}kg - 1$ lies in $\mathscr{E}_n'(L^0)$, so that $\nu_F(\det(cg^{-1}kg - 1)) = n$. Write $cg^{-1}kg - 1 = (c - 1) + c(g^{-1}kg - 1)$. By Lemma 1.2.9 we have

$$\begin{aligned} n &= \nu_F\big(\det(cg^{-1}kg - 1)\big) \\ &= \nu_F\big((c-1)^2 + c(c-1)\mathrm{tr}(g^{-1}kg - 1) + c^2\det(g^{-1}kg - 1)\big). \end{aligned}$$

This last expression is less than n if $\nu_F(c - 1) \leq [n/2]$. So $\nu_F(c - 1) > [n/2]$ and then c is in $U_F^{[n/2]+1}$. Thus ck lies in $B_n'(L^0)$. We have then that $t = g^{-1}ckg$ lies in the intersection of $B_n'(L^0)$ with $g^{-1}B_n'(L^0)g$, which contradicts Corollary 1.2.6. This finishes the proof of the Lemma.

PROPOSITION 1.2.13. *$K'([L])$ is a T.I. set. Specifically*:

(a) *If g lies in $K([L])$ then g lies in the normalizer $N_{\mathrm{Gl}_2(F)}(K'([L]))$ of $K'([L])$ in $\mathrm{Gl}_2(F)$.*

(b) *If g lies out of $K([L])$ then $g^{-1}K'([L])g \cap K'([L]) = \varnothing$.*

Proof. Since $\mathrm{Gl}_2(F)$ acts transitively on the set of lattice flags, without loss of generality we may assume $L = L^0$.

Assume first that g lies in $K([L^0])$. We observe that $K'([L^0])$ is the union of the sets $K_{-\infty}'([L^0]) = F^\times \mathscr{E}_1'(L^0)$ and $K_n'([L^0]) = F^\times B_n'(L^0)$ where

n ranges over the set of the odd positive integers. Now, (a) follows using Proposition 1.2.5 and Corollary 1.2.6.

On the other hand if g lies out of $K([L^0])$, since the determinant is invariant under conjugation, it follows from Lemmas 1.2.10 and 1.2.11 that if x is a $K([L^0])$-generic element of level n then $g^{-1}xg$ lies in $K'[L^0]$ if and only if $g^{-1}xg$ is in $K'_n([L^0])$. Now by Lemma 1.2.12, (b) holds, and the proof is complete.

1.3. The set of unramified generic elements will be another convenient set on which to compute characters. Most of the definitions and propositions that follow are analogous to the ones in §§1.1–1.2. Thus, we will often omit details.

Throughout this section we consider unramified lattices flags, i.e., lattices such that $e = 1$ (see Definition 1.1.1).

DEFINITION 1.3.1. Denote by $\mathscr{E}^{unr}(L)$ the ring of endomorphisms g in $\mathrm{End}_F(V_F)$ for which for all t we have $gL_t \subset L_t$. For any integer r, denote by $\mathscr{E}_r^{unr}(L)$ the $\mathscr{E}^{unr}(L)$-module (two-sided ideal $\mathscr{E}^{unr}(L)$ ideal if $r \geq 0$) consisting of all g for which all t we have $gL_t \subset L_{t+r}$.

DEFINITION 1.3.2. Denote by $K^{unr}(L)$ the stabilizer in $\mathrm{Gl}_2(F)$ of $[L]$ and by $K_0^{unr}(L)$ the stabilizer in $\mathrm{Gl}_2(F)$ of L. Then $K_0^{unr}(L) = \mathscr{E}^{unr}(L)^{\times}$.

We obtain a filtration for $K_0^{unr}(L)$ by setting $K_r^{unr}(L) = 1 + \mathscr{E}_r^{unr}(L)$, $r \geq 0$.

PROPOSITION 1.3.3. $K^{unr}([L]) = F^{\times}K_0^{unr}(L)$. *Also* $K_r^{unr}(L)$ *is a normal subgroup of* $K^{unr}(L)$.

Now we select a convenient lattice.

DEFINITION 1.3.4. Let $L^0 = L^0(F)$ be the lattice in V_F defined by $L_0^0 = O_F \oplus O_F$.

It follows that

$$\mathscr{E}^{unr}(L^0) = M_2(O_F),$$
$$\mathscr{E}_r^{unr}(L^0) = P_F^r M_2(O_F),$$
$$K_0^{unr}(L^0) = \mathrm{Gl}_2(O_F).$$

DEFINITION 1.3.5. An element x of $\mathscr{E}^{unr}(L)$ is $\mathscr{E}^{unr}(L)$-generic if $F[x]/F$ is an unramified extension of degree two such that

$$O_{F[x]} = F[x] \cap \mathscr{E}^{unr}(L).$$

An element x of $K_0^{unr}(L)$ is $K_0^{unr}(L)$-generic if $x - 1$ is $\mathscr{E}^{unr}(L)$-unramified generic.

An element x of $K^{unr}([L])$ is $K^{unr}([L])$-generic if for some d in $F^\times$, dx is $K_0^{unr}(L)$-generic.

We denote by $\mathscr{E}^{unr\prime}(L)$, $K^{unr\prime}{}_0(L)$, $K^{unr\prime}([L])$, respectively, the sets of $\mathscr{E}^{unr}(L)$, $K_0^{unr}(L)$, $K^{unr}([L])$-unramified generic elements.

Given an element x in $M_2(F)$ such that $F[x]/F$ is quadratic unramified there is a natural class of lattices $\mathscr{L}_E$, associated to the field $E = F[x]$, such that if L is in $\mathscr{L}_E$, then x is $\mathscr{E}^{unr}(L)$-unramified generic. The construction of $\mathscr{L}_E$ is similar to the one made in the ramified case. As a consequence we have:

Proposition 1.3.6. *If $F[x]/F$ is quadratic unramified then $\mathscr{L}_{F[x]}$ is the unique equivalence class of lattices for which x is unramified generic.*

The set of unramified generic elements shares with the set of generic ones the T.I. property. Namely:

Proposition 1.3.7.

(a) *If g is an element of $K^{unr}([L])$ then q lies in the normalizer $N_{\mathrm{Gl}_2(F)}(\mathscr{E}^{unr\prime}(L))$ of $\mathscr{E}^{unr\prime}(L)$ in* $\mathrm{Gl}_2(F)$; *if g is out of $K^{unr}([L])$ then $g^{-1}\mathscr{E}^{unr\prime}(L)g \cap \mathscr{E}^{unr\prime}(L) = \varnothing$.*

(b) *$K_0^{unr\prime}(L)$ is a T.I. set with normalizer $K^{unr}([L])$.*

(c) *$K^{unr\prime}([L])$ is a T.I. set with normalizer $K^{unr}([L])$.*

The following propositions provide additional characterizations of $\mathscr{E}^{unr}(L_F^0)$-generic elements.

Definition 1.3.8. Denote by $\mathscr{E}_n^{unr\prime}(L^0)$ the set of unramified generic elements of $\mathscr{E}_n^{unr}(L^0)$ that do not lie on $P_F^n + \mathscr{E}_{n+1}^{unr}(L^0)$.

Lemma 1.3.9. $\bigcap_j F^\times K_j^{unr}(L^0) = F^\times$.

Proof. We need only prove that if x lies in $F^\times K_j^{unr}(L^0)$ (for all j) then x lies in $F^\times$. Let y be an element of $K^{unr}([L^0]) = F^\times \mathrm{Gl}_2(O_F)$. It follows from Proposition 1.3.3 that for all j, the commutator $yxy^{-1}x^{-1}$ lies in $K_j^{unr}(L^0)$. Thus $yxy^{-1}x^{-1}$ lies in the intersection of the $K_j^{unr}(L^0)$, an intersection which reduces to the identity since the $K_j^{unr}(L^0)$ form a fundamental system of neighborhoods of the identity. We get that x commutes with each element of $K^{unr}([L^0])$, i.e., x lies in the center $Z(K^{unr}([L^0])) = F^\times$.

PROPOSITION 1.3.10. *Let x be an unramified generic element of* $\mathrm{Gl}_2(F)$. *Then either*

(a) *there is an element c in $F^\times$ such that cx lies in* $\mathrm{Gl}_2(O_F)$ *but not in* $U_F K_1^{unr}(L^0)$, *or*

(b) *there is an element c in $F^\times$ and a unique r such that $cx - 1$ lies in* $\ell_r^{unr\prime}(L^0)$.

Proof. Since x is unramified generic, there is an element c in $F^\times$ such that cx lies in $\mathrm{Gl}_2(O_F)$. If cx is in $U_F K_1^{unr}$, then, by Lemma 1.3.9, there is a number r such that cx lies in $F^\times K_r^{unr}(L^0)$ but not in $F^\times K_{r+1}^{unr}(L^0)$ (observe that then for all d in F, dx is out of $F^\times K_{r+1}^{unr}(L^0)$). Finally by modifying c if necessary one may assume that cx lies in $K_r^{unr}(L^0)$ but not in $F^\times K_{r+1}^{unr}(L^0)$. this completes the proof of (a) and (b).

DEFINITION 1.3.11. The unramified generic element x has level 0 if part (a) of Proposition 1.3.10 holds. Otherwise x has level r.

PROPOSITION 1.3.12. *Let x be an element of $P_F^r M_2(O_F)$ that does not lie in $P_F^r + P_F^{r+1} M_2(O_F)$. Then x is not unramified generic if and only if there is an element y in* $\mathrm{Gl}_2(O_F)$ *such that $y^{-1}xy$ is upper triangular modulo* $P_F^{r+1} M_2(F)$.

Proof. Without loss of generality we may assume $r = 0$, so that x lies in $M_2(O_F)$ but not in $O_F + P_F M_2(O_F)$. This means that $x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ where one of b, c or $a - d$ is a unit.

Let us denote by f_x the characteristic polynomial of x, i.e., $f_x = X^2 - (a + d)X + ad - bc$.

We claim that $O_F[x] = F[x] \cap M_2(O_F)$. For it is clear that $O_F[x]$ is included in $F[x] \cap M_2(O_F)$. On the other hand if $\alpha + \beta x$ lies in $F[x] \cap M_2(O_F)$, then by observing the conditions on the entries of x we conclude that β lies in O_F and so α lies in O_F.

Next we observe that x is unramified generic if and only if f_x is irreducible modulo P. In fact, if x is unramified generic then f_x is irreducible and $F[x]/F$ is quadratic unramified. Thus $k_{F[x]}/k_F$ is a quadratic extension. From the above we get $O_F[x] = O_{F[x]}$. It follows that $k_{F[x]} = k_F(x) \neq k_F$ and so f_x is irreducible modulo P_F.

Conversely, if f_x is irreducible modulo P then by [Se], $F[x]/F$ is unramified and $O_{F[x]} = O_F[x]$ so that x is unramified generic since $O_F[x] = F[x] \cap M_2(F)$ by the above.

Finally, since f_x is quadratic, f_x is reducible modulo P if and only if f_x, viewed over O_F/P_F, has eigenvalues in O_F/P_F if and only if, modulo P, x is similar to an upper triangular matrix. Our result now follows.

1.4. We recall now the definition of a supercuspidal representation. An admissible representation (ρ, V) of $\mathrm{Gl}_2(F)$ is supercuspidal if when restricted to M, the subgroup $[\begin{smallmatrix}1 & F\\ 0 & 1\end{smallmatrix}]$, it has no identity quotient, i.e., there is no proper M-subspace W of V such that the representation of M induced on V/W is the identity representation.

In constructing supercuspidal representations of $\mathrm{Gl}_2(F)$ we use the following fundamental fact: Let σ be an irreducible representation of $K([L])$ and suppose there is an n such that $\sigma/B_n(L)$ decomposes in orbits of ψ_b, where b is generic. Then σ induces an irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$. Furthermore, every irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$ is either equivalent to $(\mathrm{Ind}\,\sigma) \otimes \chi \circ \det$, where χ is a character of $F^\times$, or it may be induced irreducibly from a finite dimensional representation of $F^\times \mathrm{Gl}_2(O_F)$. The former representations are called ramified; the latter, unramified. (For details and proofs see **[K2]**.)

Properties of the unramified representations are well known. For more details see **[Ge]**. Therefore, we restrict our attention to the ramified supercuspidal representations.

Now we describe how to construct the ramified supercuspidal representations of $\mathrm{Gl}_2(F)$. We follow **[K4]**.

We consider the character ψ_b of $B_n(L)$ as defined in 1.1.7 with b a generic element. Let $H(\psi_b) = (F[b])^\times B_n(L)$. Then $H(\psi_b)$ is the stabilizer in $K([L])$ of the character ψ_b. Denote by θ a character of $(F[b])$ such that θ coincides with ψ_b on $F[b]^\times \cap B_n(L)$. We can now define a character ρ on $H(\psi_b)$ by $\rho(tx) = \theta(t)\psi_b(x)$.

DEFINITION 1.4.1. Let L, L' be lattice flags in V_F, let $n \geq 1$, and let ψ_b (resp. $\psi_{b'}$), $H(\psi_b)$ (resp. $H(\psi_{b'})$) and ρ (resp. ρ') be as above. Then we call the triples (L, ψ_b, ρ) and $(L', \psi_{b'}, \rho')$ equivalent if for some g in $\mathrm{Gl}_2(F)$ we have $gL = L'$, $\psi_{b'}(x) = \psi_b(g^{-1}xg)$ for x in $B_n(L')$, and $\rho'(x) = \rho(g^{-1}xg)$ for x in $H(\psi_{b'})$.

For a triple $(L; \psi_b, \rho)$ as above, let $\pi(L; \psi_b, \rho)$ be the representation of $\mathrm{Gl}_2(F)$ which is c-induced **[K3]** by ρ.

The following is Proposition 3.1.1 of **[K4]** (for more details see also **[K3]**, **[Cl]**).

PROPOSITION 1.4.2. *$\pi(L, \psi_b, \rho)$ is an irreducible admissible ramified supercuspidal representation of* $\mathrm{Gl}_2(F)$. *$\pi(L', \psi_{b'}, \rho')$ is equivalent to $\pi(L, \psi_b, \rho)$ if and only if $(L', \psi_{b'}, \rho')$ is equivalent to (L, ψ_b, ρ). Every irreducible admissible ramified supercuspidal representation of* $\mathrm{Gl}_2(F)$ *is* *equivalent to some* $\pi(L, \psi_b, \rho) \otimes \chi$, *where* χ *is a character of* $F^\times$ *which is either trivial or for which* $f(\chi) > n$.

2.1. We are going to define a lifting of lattice flags of V_F to V_E where E/F is a tamely ramified extension whose ramification degree $e(E/F)$ is odd. Using those liftings of flags we later define a tame lift for supercuspidal representations. This last notion is due to Kutzko and plays an important role in proving the Langlands conjecture for $\mathrm{Gl}_2(F)$.

DEFINITION 2.1.1. Let E/F be a tamely ramified extension whose ramification degree $e(E/F)$ is odd. We define the lattice $L^0_{E/F}$ (of ramification degree 2) in V_E by

$$\left(L^0_{E/F}\right)_0 = P_E^{(1-e(E/F))/2} \oplus O_E, \qquad \left(L^0_{E/F}\right)_1 = P_E^{(1-e(E/F))/2} \oplus P_E.$$

DEFINITION 2.1.2. If L is any lattice flag in V_F, then there is an element g of $\mathrm{Gl}_2(F)$ such that $L = gL_0$. We define the lift of L to V_E to be the V_E-lattice flag $L_{E/F} = gL^0_{E/F}$.

By Lemma 2.2.1 of **[K4]**, $L_{E/F}$ is well defined.

The following is Corollary 2.2.3 of **[K4]**. It is a two-dimensional analog of the properties of the trace (see **[Se]**).

PROPOSITION 2.1.3. $K([L_{E/F}]) \cap \mathrm{Gl}_2(F) = K([L])$,

$$\mathscr{E}_n\left(L_{E/F}\right) \cap \mathrm{End}_F(V_F) = \mathrm{Tr}_{E/F}\mathscr{E}_n\left(L_{E/F}\right) = \mathscr{E}_r\left(L_{E/F}\right)$$

with $r = 1 + [(n-1)/e(E/F)]$.

We notice here that

$$\mathscr{E}\left(L^0_{E/F}\right) = \begin{bmatrix} O_E & P_E^{(1-e(E/F))/2} \\ P_{EE}^{(1+e(E/F))/2} & O_E \end{bmatrix},$$

$$B\left(L^0_{E/F}\right) = \begin{bmatrix} U_E & P_E^{(1-e(E/F))/2} \\ P_E^{(1+e(E/F))/2} & U_E \end{bmatrix},$$

$$\mathscr{E}_n\left(L^0_{E/F}\right) = \begin{bmatrix} P_E^{[(n+1)/2]} & P_E^{[n/2]+(1-e(E/F))/2} \\ P_E^{[n/2]+(1e(E/F))/2} & P_E^{[(n+1)/2]} \end{bmatrix}.$$

We observe furthermore that

$$K\left(L^0_{E/F}\right) = \left\langle \begin{bmatrix} 0 & \pi_E^{(1-e(E/F))/2} \\ \pi_E^{(1-e(E/F))/2} & 0 \end{bmatrix} \right\rangle B\left(L^0_{E/F}\right)$$

and

$$K\left(L_{E/F}^0\right) = \begin{bmatrix} \pi_E^{(1-e(E/F))/2} & 0 \\ 0 & 1 \end{bmatrix} K\left(L_E^0\right) \begin{bmatrix} \pi_E^{(e(E/F)-1)/2} & 0 \\ 0 & 1 \end{bmatrix}.$$

In order to introduce the notion of tame lifting of a supercuspidal representation we will need the following. Let ψ_b, θ, ρ be as in 1.4.

DEFINITION 2.1.4. Set $n(E/F) = e(E/F)n - \frac{1}{2}(e(E/F) - 1)$ and define $\psi_{b,E/F}$ on $B_{n(E/F)}(L_{E/F})$ by

$$\psi_{b,E/F}(g) = \psi\left(\operatorname{tr} b \operatorname{Tr}_{E/F}(g-1)\right).$$

COROLLARY 2.1.5.

$$\mathscr{E}_{n(E/F)}\left(L_{E/F}^0\right) \cap \operatorname{End}_F(V_F) = \mathscr{E}_n\left(L_F^0\right),$$
$$B_{n(E/F)}\left(L_{E/F}^0\right) \cap \operatorname{Gl}_2(F) = B_n\left(L_F^0\right).$$

DEFINITION 2.1.6. Define the character $\theta_{E/F}$ on $E[b]^\times$ by $\theta_{E/F}(g) = \theta(N_{E[b]/F[b]}g)$ where $N_{E[b]/F[b]}$ is the norm map from $E[b]$ to $F[b]$.

DEFINITION 2.1.7. One obtains a character $\rho_{E/F}$ on $H(\psi_{b,E/F}) = E[b]^\times B_{n(E/F)}(L_{E/F})$ by setting $\rho_{E/F}(ga) = \theta_{E/F}(g)\psi_{b,E/F}(a)$ for g in $E[b]^\times, a$ in $B_{n(E/F)}(L_{E/F})$. We observe that by Lemma 2.3.4 of **[K4]** $\rho_{E/F}$ is independent of the choice of b. $\rho_{E/F}$ is then called the lift of ρ to $H(\psi_{b,E/F})$.

PROPOSITION 2.1.8. *Suppose* $\pi(L, \psi_b, \rho) \otimes \chi \cong \pi(L', \psi_{b'}, \rho') \otimes \chi'$. *Then*

$$\pi\left(L_{E/F}, \psi_{b,E/F}, \rho_{E/F}\right) \otimes \chi_{E/F} \cong \pi\left(L'_{E/F}, \psi_{b',E/F}, \rho'_{E/F}\right) \cong \chi'_{E/F},$$

where $\chi_{E/F} = \chi \circ N_{E/F}$.

Proof. See Proposition 3.1.4 of **[K4]**.

DEFINITION 2.1.9. Given an irreducible admissible ramified supercuspidal representation $\pi = \pi(L, \psi_b, \rho) \otimes \chi$ of $\operatorname{Gl}_2(F)$, we may, using Proposition 2.1.7, define the tame lift $\pi_{E/F}$ of π to $\operatorname{GL}_2(E)$ by

$$\pi_{E/F} = \pi\left(L_{E/F}, \psi_{b,E/F}, \rho_{E/F}\right) \otimes \chi_{E/F}.$$

2.2. If $\Gamma = \Gamma(E/F)$ is the Galois group of an odd prime cyclic extension of degree l such that $p \neq l$, then the natural action of Γ in E

provides us with an action of Γ on $\mathrm{Gl}_2(E)$. Thus, we may form the semidirect product $\tilde{\mathrm{G}}\mathrm{l}_2(E) = \Gamma \ltimes \mathrm{Gl}_2(E)$. Since both $H(\psi_{b,E/F})$, $K([L_{E/F}])$ are stable under Γ we may also consider

$$H(\psi_{b,E/F}) = \Gamma \ltimes H(\psi_{b,E/F}) \quad \text{and} \quad \tilde{K}([L_{E/F}]) = \Gamma \ltimes K([L_{E/F}]).$$

Since $\rho_{E/F}$ is fixed by Γ (is defined through norms and traces), we may extend $\rho_{E/F}$ to a character $\tilde{\rho}_{E/F}$ on $\tilde{H}(\psi_{b,E/F})$ in the obvious way, i.e., trivial on Γ. Furthermore, we may define:

DEFINITION 2.2.1.

$$\tilde{\pi}_{E/F} = \operatorname*{Ind}_{\tilde{H}(\psi_{b,E/F})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \rho_{E/F}.$$

PROPOSITION 2.2.2. *$\tilde{\pi}_{E/F}$ is an extension of $\pi_{E/F}$ to $\tilde{\mathrm{G}}\mathrm{l}_2(E)$.*

Proof. First we make use of Mackey's theorem (see **[K1]**) to see that in fact $\tilde{\pi}_{E/F}$ is irreducible, for if $I(\ ,\ )$ denotes the space of intertwining operators we have

$$\begin{aligned} I(\tilde{\pi}_{E/F}, \tilde{\pi}_{E/F}) &= I\Big(\operatorname*{Ind}_{\tilde{H}(\psi_{b,E/F})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \tilde{\rho}_{E/F}, \operatorname*{Ind}_{\tilde{H}(\psi_{b,E/F})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \tilde{\rho}_{E/F}\Big) \\ &= \oplus I(\tilde{\rho}_{E/F}, \tilde{\rho}^z_{E/F}), \qquad z \in \tilde{H}(\psi_{b,E/F}) \backslash \tilde{\mathrm{G}}\mathrm{l}_2(E)/\tilde{H}(\psi_{b,E/F}) \\ &= \oplus I(\rho_{E/F}, \rho^z_{E/F}), \qquad z \in H(\psi_{b,E/F}) \backslash \mathrm{Gl}_2(E)/H(\psi_{b,E/F}) \\ &= I(\pi_{E/F}, \pi_{E/F}). \end{aligned}$$

Now since Γ is cyclic and fixes $\pi_{E/F}$ it follows that any irreducible subrepresentation of $\operatorname{Ind}_{\mathrm{Gl}_2(E)\uparrow \mathrm{Gl}_2(E)} \pi_{E/F}$ is an extension of $\pi_{E/F}$ (see **[Cl]**). Thus, to complete our proof we need only show that

$$\dim I\Big(\tilde{\pi}_{E/F}, \operatorname*{Ind}_{\mathrm{Gl}_2(E)\uparrow \mathrm{Gl}_2(E)} \pi_{E/F}\Big) > 0.$$

But

$$\begin{aligned} &I\Big(\tilde{\pi}_{E/F}, \operatorname*{Ind}_{\mathrm{Gl}_2(E)\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \pi_{E/F}\Big) \\ &\qquad = I\Big(\operatorname*{Ind}_{\tilde{H}(\psi_{b,E/F})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \tilde{\rho}_{E/F}, \operatorname*{Ind}_{H(\psi_{b,E})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \rho_{E/F}\Big) \end{aligned}$$

and by Mackey's theorem

$$I\Big(\operatorname*{ind}_{\tilde{H}(\psi_{b,E/F})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \tilde{\rho}_{E/F}, \operatorname*{Ind}_{H(\psi_{b,E})\uparrow \tilde{\mathrm{G}}\mathrm{l}_2(E)} \rho_{E/F}\Big)$$

is isomorphic to a direct sum of spaces, one of which is $I(\tilde{\rho}_{E/F}, \rho_{E/F})$. Since this space has dimension one, we are done.

3.1. Let Γ be the Galois group of a prime cyclic extension E/F of degree l. We remarked above that the action of Γ on E induces an action of Γ on $\mathrm{Gl}_2(E)$. For g in $\mathrm{Gl}_2(E)$ and τ in Γ let g^τ denote this action.

The following observation leads us to the definition of τ-conjugacy. Consider the elements (τ, x), (σ, y) of $\tilde{\mathrm{Gl}}_2(E)$. Then $(\sigma, y)^{-1}(\tau, x)(\sigma, y) = (\tau, (y^{-1}xy^\tau)^{\sigma^{-1}})$. In particular, for $y = (1, y)$ in $\tilde{\mathrm{Gl}}_2(E)$ we have $y^{-1}(\tau, x)y = (\tau, y^{-1}xy^\tau)$.

DEFINITION 3.1.1. The elements x, y of $\mathrm{Gl}_2(E)$ are τ-conjugate if there is an element g in $\mathrm{Gl}_2(E)$ such that $y = g^{-1}xg^\tau$.

Let us fix once for all a generator τ of Γ.

We define next a non-abelian norm map N: $\mathrm{Gl}_2(E) \to \mathrm{Gl}_2(E)$, which was first introduced by Saito.

DEFINITION 3.1.2. Let N: $\mathrm{Gl}_2(E) \to \mathrm{Gl}_2(E)$ be defined by $N(g) = gg^\tau \cdots g^{\tau^{l-1}}$.

We note that the map N is not multiplicative and, in fact, is not even well-defined, depending as it does on the choice of τ. However, one checks easily that if A/F is a quadratic extension of fields with A lying in $M_2(F)$ then the restriction of N to $(EA)^\times$ is just the abelian norm map $N_{EA/A}$.

PROPOSITION 3.1.3. *Suppose x and g lie in* $\mathrm{Gl}_2(E)$. *Then*:

(a) $N(g^{-1}xg^\tau) = g^{-1}N(x)g$;

(b) $N(x)^\tau = x^{-1}N(x)x$;

(c) $\det(N(x)) = N_{E/F}(\det x)$;

(d) $N(x)$ *is conjugate in* $\mathrm{Gl}_2(E)$ *to an element of* $\mathrm{Gl}_2(F)$;

(e) N *induces an injection form τ-conjugacy classes in* $\mathrm{Gl}_2(E)$ *to (ordinary) conjugacy classes in* $\mathrm{Gl}_2(F)$.

Proof. The first three statements are easy calculations. On the other hand (b) says that $N(x)$ has F-rational determinant and trace. Thus, the rational canonical form of $N(x)$ lies in $\mathrm{Gl}_2(F)$. This proves (d). Finally, (a) and (d) say that N induces a function from τ-conjugacy classes in $\mathrm{Gl}_2(E)$ to conjugacy classes in $\mathrm{Gl}_2(F)$. The injectivity is a result of [Sa].

It should be noticed now that x and y are τ-conjugate if and only if (τ, x) and (τ, y) are conjugate in $\tilde{\mathrm{Gl}}_2(E)$.

In the following lemma, we take $w_1, \ldots, w_{l-1}$ to be not necessarily commuting variables.

LEMMA 3.1.4.

$$
\begin{aligned}
w_0 w_1 \cdots w_{l-1} - 1 - \sum_{i=0}^{l-1} (w_i - 1) &\\
= (w_0 - 1) \cdots (w_{l-1} - 1) &\\
+ \sum_{j=1}^{l-2} \sum_{0 \le i_1 < \cdots < i_j \le l-1} (w_0 - 1)(w_1 - 1) \cdots \left(\widehat{w_{i_1} - 1}\right) &\\
\cdots \left(\widehat{w_{i_j} - 1}\right) \cdots (w_{l-1} - 1) &
\end{aligned}
$$

where $\wedge$ *indicates that the factor below it has been deleted.*

In particular,

$$
\begin{aligned}
N(w) - 1 - \mathrm{Tr}(w - 1) &\\
= N(w - 1) + \sum_{j=1}^{l-1} \sum_{0 \le i_1 < \cdots < i_j \le l-1} (w - 1)(w^{\tau} - 1) \cdots \left(\widehat{w^{\tau^{i_1}} - 1}\right) &\\
\cdots \left(\widehat{w^{\tau^{i_j}} - 1}\right) \cdots \left(w^{\tau^{l-1}} - 1\right). &
\end{aligned}
$$

Proof. We first prove that for all r-tuples $(k_1, \ldots, k_r)$, $2 \le r \le l - 1$, the number of times the element $w_{k_1} w_{k_2} \cdots w_{k_r}$ appears on the right side of the statement with positive sign equals the number of times the element appears on the right side of the statement with negative sign.

Now the summand $(w_0 - 1) \cdots (w_{l-1} - 1)$ provides us with one copy of the above element with sign $(-1)^{l-r}$. Also, only the terms such that $1 \le j \le l - r$ provide us with such copies. For each j we have $\binom{l-r}{j}$ times the element $w_{k_1} w_{k_2} \cdots w_{k_r}$, each with sign $(-1)^{l-r-j}$. But from the expansion of $0 = (1 - 1)^{l-r}$ we get the desired result in this case.

A similar computation shows that the linear terms and the constant term are the same in both sides of our equation. Our result now follows.

3.2. Let $C_c^\infty(\mathrm{Gl}_2(F))$ be the space of compactly supported and locally constant complex valued functions on $\mathrm{Gl}_2(F)$.

Let π be an irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$. Since π is admissible, we may extend this representation to a function π on $C_c^\infty(\mathrm{Gl}_2(F))$ by setting $\pi(f)v = \int_{\mathrm{Gl}_2(F)} f(x)\pi(x)v\,dx$ (where dx is an appropriate Haar measure on $\mathrm{Gl}_2(F)$). We note that $\int_{\mathrm{Gl}_2(F)} f(x)\pi(x)v\,dx$ may be defined without difficulty for f in $C_c^\infty(\mathrm{Gl}_2(F))$ and admissible π and that, in fact, this integral is a finite sum. For details, see **[H − C]**.

Furthermore, one may show that $\pi(f)$ has finite rank. Thus, $\operatorname{tr}\pi(f)$ exists and we get in this way a map $f \mapsto \operatorname{tr}\pi(f)$, which is a distribution on $\mathrm{Gl}_2(F)$, i.e., a linear functional on $C_c^\infty(\mathrm{Gl}_2(F))$.

It is a theorem of Harish-Chandra that there exists a locally integrable function χ_π, unique up to measure 0, such that $\operatorname{tr}\pi(f) = \int_{\mathrm{Gl}_2(F)} f(x)\chi_\pi(x)\,dx$, where dx is a Haar measure on $\mathrm{Gl}_2(F)$.

DEFINITION 3.2.1. χ_π is called the character of π.

If ρ and π are as in 1.5.1 then $\pi = \operatorname{Ind}_{H(\psi_b)\uparrow \mathrm{Gl}_2(F)} \rho$ and the Frobenius formula for induced characters holds.

$$\operatorname{tr}\pi(f) = \sum_{x \text{ in } \mathrm{Gl}_2(F)/H(\psi_b)} \int_{H(\psi_b)} f(xhx^{-1})\rho(h)\,dh \text{ (see [S])}.$$

One would, of course, like to apply Fubini's theorem to the above formula to obtain a Frobenius formula for χ_π. Although this application of Fubini's theorem is not permissible in general, there is a large set of elements (the elliptic elements) which have the property that Fubini's theorem may be applied to the above integral for functions supported on this set.

DEFINITION 3.2.2. An element x of $\mathrm{Gl}_2(F)$ is called elliptic in case x is irreducible as a matrix, i.e., the characteristic polynomial of x is irreducible.

An element (τ, x) of $\tilde{\mathrm{G}}\mathrm{l}_2(E)$ is elliptic if $N(x)$ is elliptic.

DEFINITION 3.2.3. Define $\dot{\rho}$ in $\mathrm{Gl}_2(F)$ by setting $\dot{\rho}(x) = \rho(x)$ if x lies in $H(\psi_b)$, and $\dot{\rho}(x) = 0$ otherwise.

By [S] if x is an elliptic element of $\mathrm{Gl}_2(F)$ then $\rho(y^{-1}xy) = 0$ for all but a finite number of y in $\mathrm{Gl}_2(F)/H(\psi_b)$ so that

$$\chi_\pi(x) = \sum_{y \in \mathrm{Gl}_2(F)/H(\psi_b)} \dot{\rho}(y^{-1}xy)$$

3.3. Following [L] (see also [G-L]), let π be an irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$. Let E/F be as in §2.2. Denote by Π an irreducible admissible representation of $\mathrm{Gl}_2(E)$ which is stable under Γ, i.e., which is equivalent to its conjugate by τ.

DEFINITION 3.3.1. Π is called a Shintani lift of π if there exists an extension $\tilde{\Pi}$ of Π to $\tilde{\mathrm{G}}\mathrm{l}_2(E)$ such that $\chi_{\tilde{\Pi}}(\tau, z) = \chi_\pi(N(z))$ for all z in $\mathrm{Gl}_2(E)$ such that Nz has distinct eigenvalues.

PROPOSITION 3.3.2. *Any π as above has, up to equivalence, a unique Shintani lift.*

4.1. By Lemmas 7.9 and 7.12 of [**L**], in order to show that $\pi_{E/F}$ is the Shintani lift of π, it is enough to prove the identity $\chi_{\tilde{\pi}_{E/F}}(\tau, x) = \chi_\pi(n(x))$ for those elements x for which $N(x)$ is elliptic (compare Proposition 3.3.3 of [**K4**]). The proof of this identity will be the goal of this chapter.

Set

$$z_{m,F} = \begin{bmatrix} 1 & 0 \\ 0 & \pi_F^m \end{bmatrix}.$$

Then the collection of $z_m = z_{m,F}$, as m ranges over the non-negative integers, is a complete set of representatives in $\mathrm{Gl}_2(F)$ of the double coset spaces $K([L_F^0]) \backslash \mathrm{Gl}_2(F)/K([L_F^0])$ and $F^\times \mathrm{Gl}_2(O_F) \backslash \mathrm{Gl}_2(F)/K([L_F^0])$ (see [**K2**]).

Let E/F be as in §2.2.

PROPOSITION 4.1.1. *The set $\{z_{m,E}\}_{m=0}^\infty$ is a complete set of representatives of $K([L_{E/F}^0]) \backslash \mathrm{Gl}_2(E)/K([L_{E/F}^0])$.*

Proof. We consider the case E/F ramified, the unramified case being clear.

Consider the element

$$w = \begin{bmatrix} \pi_E^{(1-l)/2} & 0 \\ 0 & 1 \end{bmatrix} \text{ of } \mathrm{Gl}_2(E).$$

Then we have

$$\begin{aligned} \mathrm{Gl}_2(E) = w\,\mathrm{Gl}_2(E)w^{-1} &= \bigcup_{m=0}^{\infty} wK([L_E^0])z_m K([L_E^0])w^{-1} \\ &= \bigcup_{m=0}^{\infty} wK([L_E^0])w^{-1}wz_m w^{-1}K([L_E^0])w^{-1}. \end{aligned}$$

But a computation shows that $wz_m w^{-1} = z_m$. Also as we remarked after Proposition 2.1.3, we have $wK([L_E^0])w^{-1} = K([L_{E/F}^0])$. Thus,

$$\mathrm{Gl}_2(E) = \bigcup_{m=0}^{\infty} K([L_{E/F}^0])z_m K([L_{E/F}^0]).$$

Finally, it is clear that the sets $K([L_{E/F}^0])z_m K([L_{E/F}^0])$ are disjoint.

PROPOSITION 4.1.2.
(a) *The set* $\{z_{m,E}\}_{m=0}^{\infty}$ *is a complete set of representatives of*

$$\tilde{K}\left(\left[L^0_{E/F}\right]\right)\backslash \tilde{\mathrm{G}}\mathrm{l}_2(E)/\tilde{K}\left(\left[L^0_{E/F}\right]\right).$$

(b) *There is a natural projection between the sets* $\tilde{\mathrm{G}}\mathrm{l}_2(E)/\tilde{H}(\psi_{b,E/F})$ *and* $\mathrm{Gl}_2(E)/H(\psi_{b,E/F})$.

Proof. If (τ, g) is an element of $\tilde{\mathrm{G}}\mathrm{l}_2(E)$ then g is of the form $k_1 z_m k_2$ for some $m \geq 0$ and for k_1 and k_2 in $K([L^0_{E/F}])$. Thus $(\tau, g) = (1, k_1)(1, z_m)(\tau, k_2)$ lies in $\bigcup_{m=0}^{\infty} \tilde{K}([L^0_{E/F}])z_m\tilde{K}([L^0_{E/F}])$. From this (a) follows.

As for (b) we observe that for all i, $(\tau^i, 1)$ lies in $\tilde{H}(\psi_{b,E/F})$. Thus (τ^i, g) and $(1, g)$ have the same image in $\tilde{\mathrm{G}}\mathrm{l}_2(E)/\tilde{H}(\psi_{b,E/F})$.

PROPOSITION 4.1.3. *The set* $\{z_{m,E}\}_{m=-(e(E/F)-1)/2}^{\infty}$ *is a complete set of representatives of* $E^\times \mathrm{Gl}_2(O_E)\backslash \mathrm{Gl}_2(E)/K([L^0_{E/F}])$.

Proof. We prove the proposition when E/F is ramified, the unramified case being clear.

If w is as in Proposition 4.1.1, we recall that $wK([L^0_E])w^{-1} = K([L^0_{E/F}])$.

Now,

$$\begin{aligned}\mathrm{Gl}_2(E) = \mathrm{Gl}_2(E)w^{-1} &= \bigcup_{m=0}^{\infty} E^\times \mathrm{Gl}_2(O_E)z_m K\left(\left[L^0_E\right]\right)w^{-1}\\ &= \bigcup_{m=0}^{\infty} E^\times \mathrm{Gl}_2(O_E)z_m w^{-1}wK\left(\left[L^0_E\right]\right)w^{-1}\\ &= \bigcup_{m=0}^{\infty} E^\times \mathrm{Gl}_2(O_E)z_m w^{-1}K\left(\left[L^0_{E/F}\right]\right).\end{aligned}$$

But since

$$\begin{aligned}&E^\times \mathrm{Gl}_2(O_E)z_m w^{-1}K\left(\left[L^0_{E/F}\right]\right)\\ &\quad= E^\times \mathrm{Gl}_2(O_E)\begin{bmatrix}\pi_E^{(l-1)/2} & 0\\ 0 & \pi_E^m\end{bmatrix}K\left(\left[L^0_{E/F}\right]\right)\\ &\quad= E^\times \mathrm{Gl}_2(O_E)\pi_E^{(l-1)/2}\begin{bmatrix}1 & 0\\ 0 & \pi_E^{m-(l-1)/2}\end{bmatrix}K\left(\left[L^0_{E/F}\right]\right)\\ &\quad= E^\times \mathrm{Gl}_2(O_E)z_{m(l-1)/2}K\left(\left[L^0_{E/F}\right]\right),\end{aligned}$$

we get then from above that

$$\mathrm{Gl}_2(E) = \bigcup_{m=-(l-1)/2}^{\infty} E^{\times}\,\mathrm{Gl}_2(O_E) z_m K\left(\left[L^0_{E/F}\right]\right).$$

Finally, we check that the sets are disjoint. Let $a_1 h_1 z_{m_1} k_1 = a_2 h_2 z_{m_2} k_2$ where a_1 and a_2 lie in E, h_1 and h_2 are elements of $\mathrm{Gl}_2(O_E)$ and k_1 and k_2 lie in $K([L^0_{E/F}])$. Since for $i = 1, 2$ we have

$$z_{m_i} w = \pi_E^{(1-l)/2} z_{m_i + (l-1)/2},$$

we get

$$a_1 h_1 z_{m_1 + (l-1)/2} w^{-1} k_1 w = a_2 h_2 \pi_E^{(1-l)/2} z_{m_2 + (l-1)/2} w^{-1} k_2 w.$$

Thus,

$$m_1 + \frac{l-1}{2} = m_2 + \frac{l-1}{2} \quad \text{and so} \quad m_1 = m_2.$$

This completes the proof.

We note that since $E^{\times} \subset K([L^0_{E/F}])$ then

$$E^{\times}\,\mathrm{Gl}_2(O_E) z_m K\left(\left[L^0_{E/F}\right]\right) = \mathrm{Gl}_2(O_E) z_m K\left(\left[L^0_{E/F}\right]\right).$$

4.2. Let b be an elliptic element of $\mathrm{Gl}_2(F)$. Denote by s the trace of b and by Δ its determinant. Then b is a zero of the irreducible polynomial $X^2 - sX + \Delta$.

We embed $F[b]^{\times}$ into $\mathrm{Gl}_2(F)$ using the right regular representation A with respect to the basis $\{1, b\}$, i.e., if γ is an element of $F[b]^{\times}$, so that $\gamma = x + yb$ for some x, y in F, then we set

$$A(x + yb) = \begin{bmatrix} x & y \\ -\Delta y & x + sy \end{bmatrix}.$$

DEFINITION 4.2.1. Let Q_F be the set of matrices $\left[\begin{smallmatrix} F^{\times} & F \\ 0 & 1 \end{smallmatrix}\right]$ and let $Q_{n,F}$ be the subgroup of Q_F of matrices

$$\begin{bmatrix} U_F^{[(n+1)/2]} & P_F^{[n/2]} \\ 0 & 1 \end{bmatrix} = B_n(L) \cap Q_F.$$

If E/F is a prime cyclic extension of degree l we define the subgroup $Q_{n,E/F}$ of $Q_{E/F} = Q_E$ by

$$Q_{n,E/F} = \begin{bmatrix} U_E^{[(n+1)/2]} & P_E^{[n/2]+(1-e(E/F))/2} \\ 0 & 1 \end{bmatrix},$$

i.e., $Q_{n,E/F} = Q_E \cap B_n([L^0_{E/F}])$.

PROPOSITION 4.2.2. *$\mathrm{Gl}_2(F) = Q_F A(F[b]^\times) = A(F[b]^\times)Q_F$. Also, $Q_F \cap A(F[b]^\times) = \{1\}$.*

COROLLARY 4.2.3. *$\mathrm{Gl}_2(F) = Q_F F[b]^\times = F[b]^\times Q_F$ and $Q_F \cap F[b]^\times = \{1\}$.*

PROPOSITION 4.2.4. *$H(\psi_{b,E,E/F}) \cap Q_E = Q_{n(E/F),E/F}$.*

Proof. The result follows from the following lemmas:

LEMMA 4.2.5. *Let b be a $\mathscr{E}(L)$-generic element. Then $K([L]) = F[b]^\times(Q_F \cap B(L))$.*

Proof. Let x be an element of $K([L])$. By Corollary 4.2.3 there is an element γ in $F[b]^\times$ and an element q in Q_F such that $x = \gamma q$. Since $F[b]^\times$ is a subgroup of $K([L])$ we have, in fact, that q lies in

$$K([L]) \cap Q_F = \big((K([L]) - F^\times B(L)) \cap Q_F\big) \cup \big(F^\times B(L) \cap Q_F\big)$$
$$= F^\times B(L) \cap Q_F$$

since all elements of Q_F are split (i.e., reducible as matrices) and so can never be $K([L])$-generic. Now, $F^\times B(L) \cap Q_F = B(L) \cap Q_F$, since elements of Q_F have one eigenvalue equal to one.

LEMMA 4.2.6. *If b is a $\mathscr{E}(L)$-generic element, then $B_n(L) = U^n_{F[b]}Q_{n,F}$.*

Proof. Let x be an element of $B_n(L)$. By Lemma 4.2.5, $x = \gamma q$ where γ lies in $F[b]^\times$ and q lies in $Q_F \cap B(L)$. Since q and x lie in $B(L)$, γ lies in $B(L) \cap F[b]^\times = U_{F[b]}$. It is easy to see that if γ lie in F, then γ lies in $U^n_{F[b]}$. It follows that q lies in $B_n(L)$ and the lemma holds in this case.

We assume then that γ does not lie in F, i.e., γ is generic. We observe as above that on the other hand q is not generic.

There are indices r, s such that $\gamma - 1$ lies in $\mathscr{E}_r(L)$ but not in $\mathscr{E}_{r+1}(L)$ and $q - 1$ lies in $\mathscr{E}_s(L)$ but not in $\mathscr{E}_{s+1}(L)$.

Suppose that $r < s$. Since q lies in $B(L)$ we have that $(\gamma - 1)qL_t = (\gamma - 1)L_t \subset L_{t+r}$. Also, given that $r < s$, we have that $(q - 1)L_t \subset L_{t+r}$, so

$$(x - 1)L_t = ((\gamma - 1)q + q - 1)L_t \subset L_{t+r}.$$

This is for all t. On the other hand there is an index t such that $(\gamma - 1)qL_t \not\subset L_{t+r+1}$. Since $r < s$ we also have $(q - 1)L_t \subset L_{t+r+1}$. Thus,

$x - 1$ lies in $\ell_r(L)$ but not in $\ell_{r+1}(L)$, from which $r \geq n$, whence γ and q lie in $\ell_n(L)$. This completes the proof when $r < s$. We omit the proof when $r > s$, since an argument similar to the one above can be applied.

Finally, we consider the case $r = s$. Since $q - 1$ is not generic, there is, for some t, an element y lying in L_t but not in L_{t+1} such that $(q-1)y$ lies in L_{t+r+1}. It follows that $(x-1)y = ((\gamma - 1)q + q - 1)y$ lies in L_{t+r} but not in L_{t+r+1}, so $x - 1$ lies in $\ell_r(L)$ but not $\ell_{r+1}(L)$. We then have $r \geq n$ and the result follows.

4.3. It was proved in **[K4]** that $K([L_F^0]) \subset K([L_{E/F}^0])$. Let $z_m = z_{m,F}$ be as in 4.1. Write $Z_F = \{z_m\}_{m=0}^{\infty}$. Then Z_F is a complete set of representatives of double cosets of $K([L_F^0]) \setminus \mathrm{Gl}_2(F)/K([L_F^0])$.

Let E/F be a prime cyclic extension of degree l and let π_F be a prime element of F. If E/F is unramified then π_F remains prime in E and we may take $\pi_E = \pi_F$. If $Z_{E/F} = \{z_{m,E}\}_{m=0}^{\infty}$, we have then that

$$Z_F = Z_{E/F} \cap \mathrm{Gl}_2(F).$$

If E/F is ramified, we observe that, given $m \geq 0$, there exist a and r such that $m = la + r$. Let

$$z_{m,E/F} = \begin{bmatrix} 1 & 0 \\ 0 & \pi_F^a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \pi_E^r \end{bmatrix},$$

where π_F and π_E are, respectively, prime elements in O_F and O_E. Set $z_m = z_{m,E/F}$ and $Z_{E/F} = \{z_m\}_{m=0}^{\infty}$. It follows that $Z_F = Z_{E/F} \cap \mathrm{Gl}_2(F)$.

It should be observed that $Z_{E/F}$ is a complete set of representatives of double cosets of $K([L_E^0]) \setminus \mathrm{Gl}_2(E)/K([L_E^0])$.

Next, we state some results that are going to be needed later.

LEMMA 4.3.1. *If z lies in $Z_{E/F}$ and $K([L_{E/F}^0])zK([L_{E/F}^0]) \cap \mathrm{Gl}_2(F) \neq \varnothing$, then z lies in Z_F.*

Proof. Let y be an element of $K([L_{E/F}^0])zK([L_{E/F}^0]) \cap \mathrm{Gl}_2(F)$. Then there is z_1 in Z_F such that y lies in $K([L_F^0])z_1K([L_F^0])$, which is a subset of $K([L_{E/F}^0])z_1K([L_{E/F}^0])$. We conclude that

$$K\left(\left[L_{E/F}^0\right]\right)zK\left(\left[L_{E/F}^0\right]\right) \cap K\left(\left[L_{E/F}^0\right]\right)z_1K\left(\left[L_{E/F}^0\right]\right)$$

is not empty. Thus, $z = z_1$ and the lemma holds.

LEMMA 4.3.2. *Suppose z_m lies in $Z_{E/F}$ and*

$$Q_F \cap \left(K\left(\left[L_{E/F}^0\right]\right)z_mK\left(\left[L_{E/F}^0\right]\right)\right) \neq \varnothing.$$

Then z_m lies in Z_F and

$$Q_F \cap \left(K\left(\left[L^0_{E/F}\right]\right) z_m K\left(\left[L^0_{E/F}\right]\right)\right) = Q_F \cap \left(K\left(\left[L^0_F\right]\right) z_m K\left(\left[L^0_F\right]\right)\right).$$

Proof. The first assertion follows from Lemma 4.3.1. As for the second,

$$\begin{aligned} Q_F &\cap K\left(\left[L^0_{E/F}\right]\right) z_m K\left(\left[L^0_{E/F}\right]\right) \\ &= \left(\bigcup_{z \in Z_F} Q_F \cap K\left(\left[L^0_F\right]\right) z K\left(\left[L^0_F\right]\right)\right) \cap K\left(\left[L^0_{E/F}\right]\right) z_m K\left(\left[L^0_{E/F}\right]\right) \\ &= \bigcup_{z \in Z_F} Q_F \cap K\left(\left[L^0_F\right]\right) z K\left(\left[L^0_F\right]\right) \cap K\left(\left[L^0_{E/F}\right]\right) z_m K\left(\left[L^0_{E/F}\right]\right) \\ &= Q_F \cap K\left(\left[L^0_F\right]\right) z_m K\left(\left[L^0_F\right]\right). \end{aligned}$$

Similar results may be obtained for the decomposition of $\mathrm{Gl}_2(E)$ determined by the double cosets

$$\mathrm{Gl}_2(O_E) z_m K\left(\left[L^0_{E/F}\right]\right), \qquad m \geq -(e(E/F) - 1)/2.$$

Namely, if $Z_{E/F} = \{z_{m,E}\}_{m=-(e(E/F)-1)/2}^{\infty}$ for the next two lemmas, then

LEMMA 4.3.3. *If z lies in $Z_{E/F}$ and $(\mathrm{Gl}_2(O_E) z K([L^0_{E/F}])) \cap \mathrm{Gl}_2(F)$ is not empty then z lies in Z_F.*

LEMMA 4.3.4. *If z_m lies in $Z_{E/F}$ and $(\mathrm{Gl}_2(O_E) z_m K([L^0_{E/F}])) \cap Q_F$ is not empty then z_m lies in Z_F and $Q_F \cap (\mathrm{Gl}_2(O_E) z_m K([L^0_{E/F}])) = Q_F \cap (\mathrm{Gl}_2(O_F) z_m K([L^0_{E/F}]))$.*

LEMMA 4.3.5. *Let $n(E/F)$ be as in Definition 2.1.4. Then $Q_{n,F} = Q_F \cap Q_{n(E/F),E/F}$.*

Proof. If E/F is ramified, let s_1 be the least multiple of l which is greater than or equal to $[(n(E/F) + 1)/2]$ and let s_2 be the least multiple of l which is greater than or equal to $[n(E/F)/2] + (1 - l)/2$. Then

$$\begin{aligned} Q_F \cap Q_{n(E/F),E/F} &= \begin{bmatrix} U_E^{(n(E/F)+1)/2} \cap F & P_E^{n(E/F)/2+(1-l)/2} \cap F \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} U_E^{s_1} \cap F & P_E^{s_2} \cap F \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

One checks that $s_1 = [(n + 1)/2]l$ and $s_2 = [n/2]l$. From this the lemma follows for E/F ramified. The remaining case follows at once.

Let E/F and τ be, respectively, as in 2.2 and 3.1.

LEMMA 4.3.6. *If w lies in E and $w^\tau - w$ lies in P_E^s then there exist elements a in P_E^s and r in F such that $w = a + r$.*

Proof. We consider first the case that E/F is unramified. In this case $w = \sum_{i=m}^{\infty} c_i \pi_F^i$ where $\nu_E(w) = m$ and c_i are $(q_E - 1)$th-roots of unity where $q_E = [O_E : P_E]$. We then get $w^\tau - w = \sum_{i=m}^{\infty}(c_i^\tau - c_i)\pi_F^i$. Write $w = \sum_{i=m}^{s-1} c_i \pi_F^i + \sum_{i=s}^{\infty} c_i \pi_F^i$ and define $r = \sum_{i=m}^{s-1} c_i \pi_F^i$, $a = \sum_{i=s}^{\infty} c_i \pi_F^i$. Since $\nu_E(w^\tau - w) \geq s$ we have $c_i^\tau - c_i = 0$ for $i < s$. It follows that $w = r + a$, where r lies in F and a lies in P_E^s.

We assume now that E/F is ramified. We have $w = \sum_{i=0}^{l-1} c_i \pi_E^i$ for some c_i in O_F, $i = 1, \ldots, l-1$. Since the extension is tamely ramified, $\pi_E^\tau = cu\pi_E$ where c lies in $U_F - U_F^1$ and u lies in U_E^1. If we apply the norm $N_{E/F}$ to $\pi_E = cu\pi_E$ we get $1 = c^l N(u)$, from which c has order l in U_F/U_F^1. But then $\nu_E(\pi_E^{i^\tau} - \pi_E^i) = \nu_E(\pi_E^i)$. Also note that for $i \neq j$ we have $\nu_E(c_i \pi_E^i) \neq \nu_E(c_j \pi_E^j)$ (because i and j are different modulo l). Thus,

$$\nu_E\left(\sum_{i=1}^{l-1} c_i\left(\pi_E^{i^\tau} - \pi_E^i\right)\right) = \nu_E\left(\sum_{i=1}^{l-1} c_i \pi_E^i\right).$$

Define, then, $r = c_0$ and $a = \sum_{i=1}^{l-1} c_i \pi_E^i$. It follows that $w = r + a$, where r lies in F and $\nu_E(a) = \nu_E(w^\tau - w)$, i.e., a lies in P_E^s. This completes the proof.

LEMMA 4.3.7. *If w lies in $E^\times$ and w^τ/w lies in U_E^s for some $s > 0$ then there exist elements a in U_E^s and r in F such that $w = ra$.*

Proof. We claim first that w lies in $F^\times U_E$, for, if $w = \pi_E^t u$ with u in U_E, then

$$\frac{w^\tau}{w} = \left(\frac{\pi_E^\tau}{\pi_E}\right)^t \frac{u^\tau}{u}.$$

Since w^τ/w lies in U_E^s and E/F is a tamely ramified extension, we see that t is a multiple of $e(E/F)$. Thus, w lies in $F^\times U^E$.

Write then $w = r_1 a_1$ for some r_1 in F and a_1 in U_E. We have $w^\tau/w = a_1^\tau/a_1 = 1 + (a_1^\tau - a_1)/a_1$ lies in U_E^s. It follows that $a_1^\tau - a_1$ is in P_E^s. Now, Lemma 4.3.6 implies that $a_1 = a_2 + r_2$ for some a_2 in P_E^s and r_2 in F. It should be noticed that r_2 is a unit. Then define $r = r_1 r_2$ and $a = 1 + a_2/r_2$. Whence a lies in U_E^s, r lies in F and $w = ra$.

PROPOSITION 4.3.8. *Let y be an element of Q_E. If $y^{-1}y^\tau$ lies in $Q_{n,E/F}$ then there is an element r in Q_F and an element a in $Q_{n,E/F}$ such that $y = ra$.*

Proof. Write

$$y = \begin{bmatrix} y_1 & y_2 \\ 0 & 1 \end{bmatrix}.$$

Since $y^{-1}y^{\tau}$ lies in $Q_{n,E/F}$ we have that y_1^{τ}/y_1 lies in $U_E^{[(n+1)/2]}$ and $y_1^{\tau}(y_2^{\tau} - y_2)$ lies in $P_n^{[n/2]+(1-e(E/F))/2}$. Lemma 4.3.7 says that there are elements a_1 in $U_E^{[(n+1)/2]}$ and r_1 in F such that $y_1 = r_1 a_1$. Since

$$y_1^{-1}(y_2^{\tau} - y_2) = a_1^{-1}\left(\left(y_2 r_1^{-1}\right)^{\tau} - y_2 r_1^{-1}\right)$$

lies in $P_E^{[n/2]+(1-e(E/F))/2}$, it follows that $(y_2 r_1^{-1})^{\tau} - y_2 r_1^{-1}$ is in $P_2^{[n/2]+(1-e(E/F))/2}$. Lemma 4.3.6 implies the existence of an element a_2 lying in $P_E^{[n/2]+(1-e(E/F))/2}$ and an element r_2 in F such that $y_2 r_1^{-1} = a_2 + r_2$. Define

$$r = \begin{bmatrix} r_1 & r_2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} a_1 & a_2 \\ 0 & 1 \end{bmatrix}.$$

Then one checks that $y = ra$, from which the proposition follows.

4.4. In this section we study the connection between generic elements of $\mathrm{Gl}_2(F)$ with respect to the standard lattice and generic elements of $\mathrm{Gl}_2(E)$ (with respect to the class of the corresponding lifting lattice).

PROPOSITION 4.4.1. *If z is $K([L_F^0])$-generic of level $2m + 1$, where $m \geq 0$ or $m = -\infty$, then z is $K([L_{E/F}^0])$-generic of level $(2m + 1)e(E/F)$.*

Proof. We consider first the case when z has level $-\infty$. Since $K'_{-\infty}(L_F^0) = F^{\times}\mathscr{E}_1'(L_F^0)$, we see that z is generic of level $-\infty$ if and only if $\nu_F(\det z)$ is odd. But $\nu_E(\det z) = e(E/F)\nu_F(\det z)$ and $e(E/F)$ is odd, so that $\nu_E(\det z)$ is odd and z is $K([L_{E/F}^0])$-generic of level $-\infty$.

Suppose now that z has level $2m + 1$, $m \geq 0$. Then there is an element d in $F^{\times}$ such that dz is $B(L_F^0)$-generic of level $2m + 1$, i.e., $dz - 1$ is $\mathscr{E}(L_F^0)$-generic of level $2m + 1$. So $dz - 1$ lies in $\mathscr{E}_{2m+1}(L_F^0)$, which is contained in $\mathscr{E}_{e(E/F)(2m+1)}(L_{E/F}^0)$.

$$\nu_E(\det(dz - 1)) = e(E/F)\nu_F(\det(dz - 1)) = e(E/F)(2m + 1).$$

Since the elements g of $\mathscr{E}_{e(E/F)(2m+1)}(L_{E/F}^0)$ are such that $\nu_E(\det g) \geq e(E/F)(2m + 1)$, it follows that, in fact, $dz - 1$ is $\mathscr{E}(L_{E/F}^0)$-generic of level $e(E/F)(2m + 1)$. The result now follows.

PROPOSITION 4.4.2. *If x is $K^{unr}([L_F^0])$-generic, then x is elliptic over E and x is $K^{unr}([L_{E/F}^0])$-generic.*

Proof. Without loss of generality we may assume that x has level 0. Since x is unramified generic, it follows from Proposition 1.3.12 that f_x, the characteristic polynomial of x, is irreducible modulo P_F. Also, f_x has distinct roots in some extension field.

Now, either f_x has distinct roots modulo P_E or f_x is irreducible modulo P_E. The first is not possible because if f_x has distinct roots modulo P_E, Hensel's lemma implies that f_x is reducible in E. Byt $[E : F]$ is odd and $\deg f_x = 2$. Thus f_x is irreducible over E and in particular x is elliptic. Also, by **[Se]**, $O_{E[x]} = O_E[x]$. It follows that x is $K^{unr}([L^0_{E/F}])$-generic (see proof of Proposition 1.3.12).

4.5. We introduce in this section the notion of τ-generic element. These elements are going to be useful in dealing with calculations of characters that involve τ-conjugations.

Let E/F be as in §2.2. All lattices in this section are assumed to be ramified.

DEFINITION 4.5.1. AN element x of $\mathrm{Gl}_2(E)$ is called τ-$\mathscr{E}(L_E)$ (respectively, τ-$B(L_E)$, τ-$K([L_E])$) generic if $N(x)$ if $\mathscr{E}(L_E)$ (respectively, $B(L_E)$, $K([L_E])$) generic. Denote by $\mathscr{E}'_\tau(L_E)$, $B'_\tau(L_E)$ and $K'_\tau([L_E])$ the respective sets of τ-generic elements.

DEFINITION 4.5.2. The τ-normalizer of a subset S of $\mathrm{Gl}_2(E)$ is the set $N^\tau_{\mathrm{Gl}_2(E)}(S)$ of elements g in $\mathrm{Gl}_2(E)$ such that $g^{-1}Sg^\tau = S$.

DEFINITION 4.5.3. A subset S of $\mathrm{Gl}_2(E)$ is a τ-trivial intersection set in $\mathrm{Gl}_2(E)$, or a τ-T.I. set, if it does not contain 1 and

(a) $S \subset N^\tau_{\mathrm{Gl}_2(E)}(S)$,

(b) if g is an element of $\mathrm{Gl}_2(E)$ that does not lie in $N^\tau_{\mathrm{Gl}_2(E)}$ then $g^{-1}Sg^\tau \cap S = \varnothing$.

We observe that in fact $N^\tau_{\mathrm{Gl}_2(E)}(S)$ is a subgroup of $\mathrm{Gl}_2(E)$.

PROPOSITION 4.5.4.

(a) *If g lies in $K([L_E])$ then g lies in $N^\tau_{\mathrm{Gl}_2(E)}(\mathscr{E}'_\tau(L_E))$. On the other hand if g lies out of $K([L_E])$ then $g^{-1}\mathscr{E}'_\tau(L_E)g^\tau \cap \mathscr{E}'_\tau(L_E) = \varnothing$.*

(b) *$B'_\tau(L_E)$ and $K'_\tau([L_E])$ are τ-T.I. sets with τ-normalizer $K([L_E])$.*

Proof. We prove the statement concerning $K'_\tau([L_E])$, the others being similar. So, let g be an element of $K([L_E])$. Then by Proposition 1.2.13 g lies in $N_{\mathrm{Gl}_2(E)}(K'([L_E]))$. From this $g^{-1}K'_\tau([L_E])g^\tau \subset K'_\tau([L_E])$, because if x lies in $K'_\tau([L_E])$ then $N(g^{-1}xg^\tau) = g^{-1}N(x)g$ lies in $K'([L_E])$. Given that g^{-1} also lies in $K([L_E])$, the same argument shows that $gK'_\tau([L_E])g^{-\tau} \subset K'_\tau([L_E])$, i.e., $g^{-1}K'_\tau([L_E])g^\tau \supset K'_\tau([L_E])$.

Now, assume that g lies out of $K([L_E])$. If there is an element x that lies in both $g^{-1}K'_\tau([L_E])g^\tau$ and $K'_\tau([L_E])$, then $N(x)$ and $gN(x)g^{-1}$ are generic. This says that $K'([L_E]) \cap g^{-1}K'([L_E])g$ is nonempty for g out of $K([L_E])$, which contradicts Proposition 1.2.13.

4.6. We now introduce the notion of τ-unramified generic element.

Let E/F be as in §2.2. The lattices we consider in this section are unramified.

DEFINITION 4.6.1. An element x of $\mathrm{Gl}_2(E)$ is called τ-$\mathscr{E}^{unr}(L_E)$ (respectively, τ-$K_0^{unr}(L_E)$, τ-$K^{unr}([L_E])$) generic if $N(x)$ is $\mathscr{E}^{unr}(L_E)$ (respectively, $K_0^{unr}(L_E)$, $K^{unr}([L_E])$) generic. Denote by $\mathscr{E}_\tau^{unr'}(L_E)$, $K_{0\tau}^{unr'}(L_E)$ and $K_\tau^{unr'}([L_E])$ the respective sets of unramified generic elements.

PROPOSITION 4.6.2.

(a) *If g lies in $K^{unr}([L_E])$ then g lies in $N^\tau_{\mathrm{Gl}_2(E)}(\mathscr{E}_\tau^{unr'}(L_E))$. On the other hand, if g lies out of $K^{unr}([L_E])$ then $g^{-1}\mathscr{E}_\tau^{unr'}(L_E)g^\tau \cap \mathscr{E}_\tau^{unr'}(L_E) = \varnothing$.*

(b) *$K_{0\tau}^{unr'}(L_E)$ and $K_\tau^{unr'}([L_E])$ are τ-T.I. sets with normalizer $K^{unr}([L_E])$.*

Proof. The proof is analogous to the one given in Proposition 4.5.4 and we omit it.

4.7. Let E/F and $\tilde{\pi}_{E/F}$ be as in §2.2.

It is a consequence of Propositions 4.8.4, 4.8.16, 4.8.23 and 4.9.9 below and our remark following 3.2.4 that if (τ, x) is an elliptic element of $\tilde{\mathrm{G}}\mathrm{l}_2(E)$ then

$$\chi_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{(\gamma, y)\in \tilde{\mathrm{G}}\mathrm{l}_2(E)/H(\psi_{b,E/F})} \dot{\tilde{\rho}}_{E/F}\big((\gamma, y)^{-1}(\tau, x)(\gamma, y)\big);$$

this last expression is, from Proposition 4.1.2(b) and the definition of $\tilde{\rho}_{E/F}$, equal to

$$\sum_{y\in \mathrm{Gl}_2(E)/H(\psi_{b,E/F})} \dot{\tilde{\rho}}_{E/F}\big(y^{-1}(\tau, x)y\big)$$

$$= \sum_{y\in \mathrm{Gl}_2(E)/H(\psi_{b,E/F})} \dot{\tilde{\rho}}_{E/F}\big(\tau, y^{-1}xy^\tau\big)$$

$$= \sum_{y\in \mathrm{Gl}_2(E)/H(\psi_{b,E/F})} \dot{\rho}_{E/F}\big(y^{-1}xy^\tau\big).$$

4.8. In this section the comparison for τ-generic elements is made.

DEFINITION 4.8.1. Set

$$\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \in K([L^0_{E/F}]) z_m K([L^0_{E/F}])/H(\psi_{b,E/F})} \dot{\rho}_{E/F}(y^{-1} x y^{\tau}),$$

where z_m is as in §4.3.

DEFINITION 4.8.2. Let

$$\tilde{\sigma}_{E/F} = \operatorname*{Ind}_{\tilde{H}(\psi_{b,E/F}) \uparrow \tilde{K}([L^0_{E/F}])} \tilde{\rho}_{E/F}, \qquad \sigma = \operatorname*{Ind}_{H(\psi_b) \uparrow K([L^0_F])} \rho.$$

We observe that $\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \chi_{\tilde{\sigma}_{E/F}}(\tau, x)$. We limit ourselves then to studying the sums $\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x)$.

The proof of next lemma is similar to the one given for Lemma 1.2.11 and we omit it.

LEMMA 4.8.3. *Let x be an element of $E^\times B_n(L^0_{E/F})$ with n odd. Then* $\sup_{c \text{ in } E^\times} \nu_E(\det(cx - 1)) \geq n$. *We have equality if and only if x is $K([L^0_{E/F}])$-generic of level n.*

PROPOSITION 4.8.4. *Let x be an element of $K([L^0_{E/F}])$ that does not lie in $N^{-1}(E^\times B_{n(E/F)}(L^0_{E/F}))$ (N denoting the non-abelian norm map defined in* 3.1.2). *If x is τ-$K([L^0_{E/F}])$ generic then for $m > 0$, $\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = 0$.*

Proof. Let y be an element of $K([L^0_{E/F}]) z_m K([L^0_{E/F}])$ and suppose $m > 0$. Then y does not lie in $K([L^0_{E/F}])$. But given that x lies in $K'_\tau([L^0_{E/F}])$ and $K'_\tau([L^0_{E/F}])$ is a τ-T.I. set, we have that $y^{-1} x y^{\tau}$ is not τ-$K([L^0_{E/F}])$ generic.

On the other hand the set $H(\psi_{b,E/F}) - N^{-1}(E^\times B_{n(E/F)}(L^0_{E/F}))$ consists only of τ-$K([L^0_{E/F}])$ generic elements, because if g lies in $H(\psi_{b,E/F}) - N^{-1}(E^\times B_{n(E/F)}(L^0_{E/F}))$ then $N(g)$ lies in

$$H(\psi_{b,E/F}) - E^\times B_{n(E/F)}(L^0_{E/F})$$

and the elements of this last set are generic. It follows that $y^{-1} x y^{\tau}$ does not lie in $H(\psi_{b,E/F}) - N^{-1}(E^\times B_{n(E/F)}(L^0_{E/F}))$.

We claim now that $y^{-1} x y^{\tau}$ does not lie in $N^{-1}(E^\times B_{n(E/F)}(L^0_{E/F}))$, because if it does, then $y^{-1} N(x) y$ lies in $E^\times B_{n(E/F)}(L^0_{E/F})$ and by Lemma 4.8.3 $\sup_{c \text{ in } E^\times} \nu_E(\det(cN(x)) - 1) \geq n$. But by hypothesis $N(x)$ is a generic element of $K([L^0_{E/F}])$ that does not lie in $E^\times B_{n(E/F)}(L^0_{E/F})$ and then $\sup_{c \text{ in } E^\times} \nu_E \det(cN(x) - 1) < n$, a contradiction. This proves the claim and we have as a consequence that $y^{-1}xy^\tau$ does not lie in $H(\psi_{b,E/F})$, so that $\dot{\rho}_{E/F}(y^{-1}xy^\tau) = 0$ whence $\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = 0$.

In view of the proposition above we now analyze τ-generic elements lying in $N^{-1}(E^\times B_{n(E/F)}(L^0_{E/F}))$. In particular, the next proposition will allow us to assume, without loss of generality, that for such elements, $N(x)$ can be taken to be F-rational, i.e., an element of $\mathrm{Gl}_2(F)$.

PROPOSITION 4.8.5. *Let x be a τ-$K([L^0_{E/F}])$ generic element. Then there is an element k in $K([L^0_{E/F}])$ such that $k^{-1}N(x)k$ is F-rational.*

Proof. By Proposition 3.1.3(d), there is an element y in $\mathrm{Gl}_2(E)$ such that $y^{-1}N(x)y$ is F-rational. We observe that $y^{-1}N(x)y$ satisfies the same irreducible polynomial as $N(x)$, i.e., an Eisenstein polynomial, from which $F[y^{-1}N(x)y]/F$ is quadratic ramified.

Now Corollary 1.2.4 provides us with a unique equivalence class of lattices $[L]$ on V_F such that $y^{-1}N(x)y$ is $[L]$-generic. On the other hand the transitivity of the action of $\mathrm{Gl}_2(F)$ on lattice flags provides us of an element r in $\mathrm{Gl}_2(F)$ such that $K([L^0_F]) = r^{-1}K([L])r$. It follows that $r^{-1}y^{-1}N(x)yr$ is $K([L^0_F])$-generic.

By Proposition 4.4.1, we have that, in fact, $(yr)^{-1}N(x)yr$ is $K([L^0_{E/F}])$-generic. Since, by hypothesis, $N(x)$ is also $K([L^0_{E/F}])$-generic, the T.I. property implies that $k = yr$ lies in $K([L^0_{E/F}])$. This completes the proof.

As a consequence of the proposition above, we may assume without loss of generality that for x τ-$K([L^0_{E/F}])$ generic, $N(x)$ is an element of $\mathrm{Gl}_2(F)$.

PROPOSITION 4.8.6. *Let x be a τ-$B(L^0_{E/F})$ generic element of $N^{-1}(B_{n(E/F)}(L^0_{E/F}))$ such that $N(x)$ lies in $\mathrm{Gl}_2(F)$. Then there is a $B(L^0_{E/F})$-generic element z of the same level as $N(x)$ such that $N(z) = N(x)$.*

Proof. We observe that the restriction of N to $E[N(x)]$ is the norm of the field extension $E[N(x)]/F[N(x)]$ (see the remark after Definition 3.1.2).

We consider first the case when E/F is ramified. Then $E[N(x)]/F[N(x)]$ is a ramified, tamely ramified extension of degree l. It follows that $N(x)$ is generic of level sl for some s. Since the units of the field $E[N(x)]$ are given by $U^t_{E[N(x)]} = B_t(L^0_{E/F}) \cap E[N(x)]$, we have, in fact, that $N(x)$ lies in $U^{sl}_{E[N(x)]}$ but not in $U^{sl+1}_{E[N(x)]}$. On the other hand it is known that N induces an epimorphism $N(U^{sl}_{E[N(x)]}/U^{sl+1}_{E[N(x)]}) = U^{s}_{F[N(x)]}/U^{s+1}_{F[N(x)]}$ (see [Se]) and since $U^{sl}_{E[N(x)]} \cap F[N(x)] = U^{s}_{F[N(x)]}$, it follows that $N(x)$ as an element of $U^{s}_{F[N(x)]}/U^{s+1}_{F[N(x)]}$ is not the identity. We are able then to pick z in $U^{sl}_{E[N(x)]}/U^{sl+1}_{E[N(x)]}$ such that $N(z) = N(x)$. Finally, we observe that the non-zero elements of $U^{sl}_{E[N(x)]}/U^{sl+1}_{E[N(x)]}$ are in fact generic elements of level sl. This completes the proof in the ramified case.

We assume now that E/F is unramified. In this case the norm N satisfies $N(U^{t}_{E[N(x)]}) = U^{t}_{F[N(x)]}$ for all $t \geq 1$ ([Se]). An argument similar to the one above allows us to pick z generic of level the same as the level of $N(x)$ and such that $N(z) = N(x)$. From this the result follows.

From the above, we see that by τ-conjugating if necessary, we may assume for all our purposes that if x is τ-$B(L^0_{E/F})$ generic and lies in $N^{-1}(B_{n(E/F)}(L^0_{E/F}))$ then x is $B(L^0_{E/F})$-generic, $N(x)$ is F-rational and level x = level $N(x)$.

The next proposition tells us that, in fact, it is enough to consider elements as above, rather than elements lying in $N^{-1}(E^{\times}B_{n(E/F)}(L^0_{E/F}))$.

PROPOSITION 4.8.7. *Let x be a τ-$K([L^0_{E/F}])$ generic element lying in $N^{-1}(E^{\times}B_{n(E/F)}(L^0_{E/F}))$ and such that has F-rational norm. Then there is an element α in $E^{\times}$ and a τ-$B(L^0_{E/F})$ generic element g such that x is τ-conjugate with αg.*

Note. It is here that our assumption concerning the parity of l is used.

Proof. We have that $N(x)$ lies in $\mathrm{Gl}_2(F)$ and has the form $N(x) = \beta h$ where β lies in $E^{\times}$ and h is $B(L^0_{E/F})$-generic. So

$$N(x) = \begin{bmatrix} \beta u & \beta a \\ \beta c & \beta v \end{bmatrix}$$

with u and v units. Set $r = \beta u$. Then r lies in F and $r^{-1}N(x) = r^{-1}\beta h$ lies in $\mathrm{Gl}_2(F)$. We observe that $r^{-1}\beta h = u^{-1}h$ and if we set $h_1 = u^{-1}h$, it follows that h_1 is a $B(L^0_{E/F})$-generic element of the same level as h and we have $N(x) = \beta h = rh_1$. We may assume then without loss of generality that $N(x) = \beta h$, where β lies in F and h is $B(L^0_{E/F})$-generic element lying in $\mathrm{Gl}_2(F)$.

Now, given that h lies in $\mathrm{Gl}_2(F)$ and is a unit of $E[N(x)]$ we have, by the properties of the norm of $E[N(x)]/F[N(x)]$, that there is a $B(L^0_{E/F})$-generic element g such that $N(g) = h$ (see proof of Proposition 4.8.6).

At this point, we know that β is an $E[N(x)]/F[N(x)]$-norm and we want to prove now that β is an E/F-norm. To this end, set $A = F[N(x)]$, then

$$\begin{tikzcd}
 & AE = E[N(x)] \arrow[dl, dash] \arrow[dr, dash] & \\
E \arrow[dr, dash, "l"'] & & A = F[N(x)] \arrow[dl, dash, "2"] \\
 & F &
\end{tikzcd}$$

and β lies in $F^{\times} \cap N_{AE/A}AE$. By class field theory it is known that $[F^{\times} : N_{E/F}E^{\times}] = l$. Also, we have the inclusions $N_{E/F}E^{\times} \subset F^{\times} \cap N_{AE/A}AE \subset F^{\times}$ and since l is prime we need only prove that the last inclusion is proper in order to prove that β lies in $N_{E/F}E^{\times}$.

Assume first that E/F is unramified. Then the prime element π_F of F has valuation two in A (because A/F is ramified). But then it cannot be in $N_{AE/A}AE$, because in this last set the elements have valuations which are multiples of l.

Suppose now that E/F is ramified. Then there exists a set C_{q-1} of $(q - 1)$th (distinct even modulo P) roots of unity in O_F $(q = [E : F])$ such that $U_F = C_{q-1}U_F^1$. Let d be a generator of C_{q-1}. Since A/F is ramified we have that $U_A = C_{q-1}U_A^1$. By **[Se]**, the element d of F is not a norm, so does not lie in $N_{AE/A}AE$.

We have proved that there is an element α in E such that $N(\alpha) = \beta$. It follows that $N(x) = N(\alpha)N(g) = N(\alpha g)$. By Proposition 3.1.3(e) the result follows.

PROPOSITION 4.8.8. *Let x be a $B(L^0_{E/F})$-generic element of level s and let y be an element of $K([L^0_{E/F}])z_m K([L^0_{E/F}])$. Then $y^{-1}xy$ lies in $B_n(L^0_{E/F})$ if and only if $m \leq (s-1)/2 - [n/2]$.*

Proof. Write $y = k_1 z_m k_2$ where k_1, k_2 lie in $K([L^0_{E/F}])$. Then we have

$$k_1^{-1}xk_1 = 1 + \pi_E^{(s-1)/2}\begin{bmatrix} \pi_E a & \pi_E^{(1-e(E/F))/2}v \\ \pi_E^{(1+e(E/F))/2}u & \pi_E c \end{bmatrix}$$

where u, v lie in U_E and a, c lie in O_E.

It follows that $y^{-1}xy$ lies in $B_n(L^0_{E/F})$ if and only if $z_m^{-1}(k_1^{-1}xk_1)z_m$ lies in $B_n(L^0_{E/F})$, ($B_n(L^0_{E/F})$ is normal in $K([L^0_{E/F}])$) if and only if

$$1 + \pi_E^{(s-1)/2}\begin{bmatrix} \pi_E a & \pi_E^{m+(1-e(E/F))/2}v \\ \pi_E^{-m+(1+e(E/F))/2}u & \pi_E c \end{bmatrix}$$

lies in

$$1 + \begin{bmatrix} P_E^{[(n+1)/2]} & P_E^{[n/2]+(1-e(E/F))/2} \\ P_E^{[n/2]+(1+e(E/F))/2} & P_E^{[(n+1)/2]} \end{bmatrix} = B_n\left(L_{E/F}^0\right)$$

if and only if $-m + (s-1)/2 \geq [n/2]$ if and only if $m \leq (s-1)/2 - [n/2]$.

DEFINITION 4.8.9. An element x of $N^{-1}(B_{n(E/F)}(L_{E/F}^0))$ such that x is $B(L_{E/F}^0)$-generic, τ-$B(L_{E/F}^0)$ generic, $N(x)$ is F-rational and level x = level $N(x)$ will be called reduced.

COROLLARY 4.8.10. *Let x be reduced and let y be an element of $K([L_{E/F}^0])z_m K([L_{E/F}^0])$. Then $y^{-1}N(x)y$ lies in $B_{n(E/F)}(L_{E/F}^0)$ if and only if $y^{-1}xy$ lies in $B_{n(E/F)}(L_{E/F}^0)$.*

PROPOSITION 4.8.11. *Let x be reduced. Then*

$$\chi_{\tilde{\pi}_{E/F}}^{(m)}(\tau, x) = \sum_{y \in Q_E \cap K([L_{E/F}^0])z_m K([L_{E/F}^0])/Q_{n(E/F),E/F}} \dot{\psi}_{b,E/F}(y^{-1}xy^{\tau}).$$

Proof. We first observe that there is a natural bijection between the sets $K([L_{E/F}^0])z_m K([L_{E/F}^0])/H(\psi_{b,E/F})$ and

$$Q_E \cap K\left(\left[L_{E/F}^0\right]\right)z_m K\left(\left[L_{E/F}^0\right]\right)/Q_{n(E/F),E/F}.$$

Thus, we have

$$\chi_{\tilde{\pi}_{E/F}}^{(m)}(\tau, x) = \sum_{y \in Q_E \cap K([L_{E/F}^0])z_m K([L_{E/F}^0])/Q_{n(E/F),E/F}} \dot{\rho}_{E/F}(y^{-1}xy^{\tau}).$$

We have to prove that if $y^{-1}xy^{\tau}$ lies in $H(\psi_{b,E/F})$ then it lies in $B_{n(E/F)}(L_{E/F}^0)$. Let, then, $y^{-1}xy^{\tau}$ be an element of $H(\psi_{b,E/F})$, where y lies in $Q_E \cap K([L_{E/F}^0])z_m K([L_{E/F}^0])$. It follows that $y^{-1}xy^{\tau} = tk$, where t lies in $E\,[b]^{\times}$ and k is in $B_{n(E/F)}(L_{E/F}^0)$. Since $B_{n(E/F)}(L_{E/F}^0)$ is normal in $K([L_{E/F}^0])$, we get, after we apply the norm to $y^{-1}xy^{\tau} = tk$, that $y^{-1}N(x)y = N(t)h$ for some h in $B_{n(E/F)}(L_{E/F}^0)$.

If y lies in $K([L_{E/F}^0])$ then $y^{-1}N(x)y$ lies in $B_{n(E/F)}(L_{E/F}^0)$, because $N(x)$ is in $B_{n(E/F)}(L_{E/F}^0)$. If y does not lie in $K([L_{E/F}^0])$, then $y^{-1}N(x)y$ is a non-generic element lying in $H(\psi_{b,E/F})$. It follows that $y^{-1}N(x)y$ lies in $E^{\times}B_{n(E/F)}(L_{E/F}^0)$ since $H(\psi_{b,E/F}) - E^{\times}B_{n(E/F)}(L_{E/F}^0)$ consists entirely of generic elements. We prove now that under these circumstances $y^{-1}N(x)y$ lies in fact in $B_{n(E/F)}(L_{E/F}^0)$. Let

$$\nu = \nu_E\left(\det\left(y^{-1}N(x)y - 1\right)\right) = \nu_E\left(\det(N(x) - 1)\right);$$

ν is in fact an odd number greater than or equal to $n(E/F)$ since ν is the level of $N(x)$. Write

$$y^{-1}N(x)y = \begin{bmatrix} \alpha u & \alpha d \\ \alpha c & \alpha v \end{bmatrix}$$

with α in $E^{\times}$; u, v in $U_E^{[(\nu+1)/2]}$; c in $P_E^{[\nu/2]+(1+e(E/F))/2}$, d in $P_E^{[\nu/2]+(1-e(E/F))/2}$. We now observe that since

$$\nu = \nu_E\big((\alpha u - 1)(\alpha v - 1) - \alpha^2 cd\big)$$

is odd, α cannot have positive valuation (otherwise $\nu = 0$), α cannot have negative valuation (otherwise $\nu = 2\nu_E(\alpha)$) and α cannot be a unit in $U_E - U_E^{[(\nu+1)/2]}$ (otherwise $\nu_E(\alpha u - 1)$, $\nu_E(\alpha v - 1) < \nu$ and since $\nu_E(\alpha cd) \geq \nu$, $\det(N(x) - 1)$ would have valuation less than ν). So α lies in $U_E^{[(\nu+1)/2]}$ and then $y^{-1}N(x)y$ lies in $B_\nu(L^0_{E/F})$, which is a subset of $B_{n(E/F)}(L^0_{E/F})$, i.e., $y^{-1}N(x)y$ lies in $B_{n(E/F)}(L^0_{E/F})$.

At this point we have then that, in any case, $y^{-1}N(x)y$ lies in $B_{n(E/F)}(L^0_{E/F})$. By Corollary 4.8.10, we have that $y^{-1}xy$ lies in $B_{n(E/F)}(L^0_{E/F})$.

We observe that $y^{-1}xy^\tau = (y^{-1}xy)(y^{-1}y^\tau)$, from which $y^{-1}y^\tau = (y^{-1}xy)^{-1}(y^{-1}xy^\tau)$ lies in $Q_E \cap H(\psi_{b,E/F})$. Thus, $y^{-1}y^\tau$ lies in $Q_{n(E/F),E/F}$ (see Proposition 4.2.4). It follows that $y^{-1}xy^\tau$ lies in $B_{n(E/F)}(L^0_{E/F})$. This completes the proof.

PROPOSITION 4.8.12. *Let x be reduced. Then*

$$\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \in Q_F \cap K([L^0_F])z_{m,F}K([L^0_F])/Q_{n,F}} \dot{\psi}_{b,E/F}(y^{-1}xy).$$

Proof. We observed in Proposition 4.8.11 that if y lies in

$$Q_E \cap K\big(\big[L^0_{E/F}\big]\big)z_m K\big(\big[L^0_{E/F}\big]\big)/Q_{n(E/F),E/F}$$

then $y^{-1}xy^\tau = (y^{-1}xy)(y^{-1}y^\tau)$ and $y^{-1}y^\tau$ lies in $Q_{n(E/F),E/F}$. By Proposition 4.3.8 there is an element a in $Q_{n(E/F),E/F}$ and r in Q_F such that $y = ra$. Thus, without loss of generality we may take $y = r$, i.e., we may choose y lying in $\mathrm{Gl}_2(F)$, whence $y^{-1}xy^\tau = y^{-1}xy$. Thus y is F-rational modulo $Q_{n(E/F),E/F}$ so that by Lemmas 4.3.2 and 4.3.5 z_m lies in Z_F and we may replace the sum over $Q_E \cap K([L^0_{E/F}])z_m K([L^0_{E/F}])/Q_{n(E/F),E/F}$ in the formula in Proposition 4.8.11 by a sum over

$$Q_F \cap K\big(\big[L^0_F\big]\big)z_m K\big(\big[L^0_F\big]\big)/Q_{n,F}.$$

We get

$$\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \in Q_F \cap K([L^0_F])z_m K([L^0_F])/Q_{n,F}} \dot{\psi}_{b,E/F}(y^{-1}xy).$$

LEMMA 4.8.13. *Let w be an element of $B_{n(E/F)}(L^0_{E/F})$ such that $N(w)$ lies in $\mathrm{Gl}_2(F)$. Then $N(w)$ lies in $B_n(L^0_F)$ and $\psi_{b,E/F}(w) = \psi_b(N(w))$.*

Proof. $N(w)$ lies in $B_{n(E/F)}(L^0_{E/F}) \cap \mathrm{Gl}_2(F)$ so in $B_n(L^0_F)$ by Corollary 2.1.5.

As for the second assertion, we have that $\psi_b(N(w)) = \psi_{b,E/F}(w)$ if and only if $\psi(\mathrm{tr}\, b(N(w) - 1)) = \psi(\mathrm{tr}\, bTr(w-1))$ if and only if $\psi(\mathrm{tr}\, b(N(w) - 1 - \mathrm{Tr}(w-1))) = 1$.

Set $a = N(w) - 1 - \mathrm{Tr}(w-1)$. Then by Lemma 3.1.4

$$a = N(w-1) + \sum_{j=1}^{l-2} \sum_{0 \le i_1 < \cdots < i_j \le l-1} (w-1)(w^{\tau}-1)\cdots\left(w^{\tau^{\hat{i_1}}}-1\right) \cdots \left(w^{\tau^{\hat{i_j}}}-1\right)\cdots\left(w^{\tau^{l-1}}-1\right).$$

We observe that a lies in $\ell_{2n(E/F)}(L^0_{E/F}) \cap \mathrm{End}_F(V_F)$ which is, by Proposition 2.1.3, equal to $\ell_{2n}(L^0_F)$. Thus, $\mathrm{tr}\, ba$ lies in P_F, because b lies in $\ell_{1-2n}(L^0_F)$, and then $\psi(\mathrm{tr}\, ba) = 1$. Now the lemma follows.

LEMMA 4.8.14. *Let x be an element of $H(\psi_{b,E/F})$ such that $N(x)$ lies in $\mathrm{Gl}_2(F)$. Then $N(x)$ lies in $H(\psi_b)$ and $\rho_{E/F}(x) = \rho(N(x))$.*

Proof. Write $x = tk$ with t in $E[b]^\times$ and k in $B_{n(E/F)}(L^0_{E/F})$. Then

$$N(x) = N(t)(t^{\tau^{-(l-1)}} t^{\tau^{-(l-2)}} t^{\tau^{-1}} k t^{\tau} \cdots t^{\tau^{l-1}})(t^{\tau^{-(l-1)}} \cdots t^{\tau^{-2}} k^{\tau} t^{\tau^2} \cdots t^{\tau^{l-1}}) \cdots (t^{\tau^{-(l-1)}} k^{\tau^{l-2}} t^{\tau^{l-1}}) k^{\tau^{l-1}}.$$

Since $N(t)$ lies in $\mathrm{Gl}_2(F)$ we have $a = (N(t))^{-1}N(x)$ lies in $\mathrm{Gl}_2(F) \cap B_{n(E/F)}(L^0_{E/F}) = B_n(L^0_F)$, so that $N(x)$ lies in $H(\psi_b) = F[b]^\times B_n(L^0_F)$.

On the other hand, by Lemma 2.3.3 of **[K4]** and Lemma 3.1.4, we get that $a - 1 - \mathrm{Tr}(k-1)$ lies in $B_n(L^0_F)$ and

$$\begin{aligned} a &- 1 - \mathrm{Tr}(k-1) \\ &= \sum_{i=0}^{l-1} (t^{\tau^{-(l-1)}} t^{\tau^{-(l-2)}} \cdots t^{\tau^{-(i+1)}} k^{\tau^i} t^{\tau^{i+1}} \cdots t^{\tau^{(l-1)}}) - \mathrm{Tr}(k-1) \end{aligned}$$

(modulo $\ell_{2n(E/F)}(L^0_{E/F})$). It follows that $\psi(\mathrm{tr}\, b(a - 1 - \mathrm{Tr}(k-1))) = 1$, from which $\psi_b(a) = \psi_{b,E/F}(k)$. Finally,

$$\begin{aligned} \rho(N(x)) &= \rho(N(t)a) = \theta(N(t))\psi_b(a) \\ &= \theta_{E/F}(t)\psi_{b,E/F}(k) = \rho_{E/F}(tk) = \rho_{E/F}(x). \end{aligned}$$

LEMMA 4.8.15. *Let x be reduced. Then*

$$\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \in Q_F \cap K([L^0_F]) z_m K([L^0_F])/Q_{n,F}} \dot{\psi}_b(y^{-1}N(x)y).$$

Proof. We observe that if y lies in $Q_F \cap K([L^0_F])z_m K([L^0_F])/Q_{n,F}$ then $N(y^{-1}xy) = y^{-1}N(x)y$. Now, the proposition follows from Proposition 4.8.12 and Lemma 4.8.13.

PROPOSITION 4.8.16. *Let x be reduced. Then*

$$\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = \chi^{(m)}_{\pi}(N(x)).$$

Proof. We observe that the correspondence $yQ_{n,F} \mapsto yH(\psi_b)$ establishes a bijection between

$$Q_F \cap K\left(\left[L^0_F\right]\right)z_m K\left(\left[L^0_F\right]\right)/Q_{n,F} \quad \text{and} \quad K\left(\left[L^0_F\right]\right)z_m K\left(\left[L^0_F\right]\right)/H(\psi_b).$$

Now, the result follows from Lemma 4.8.15.

In order to complete the comparison in the ramified elliptic case, it is thus only necessary to compare $\chi_{\tilde{\sigma}_{E/F}}$ with χ_σ. In order to do this we need some preliminaries.

PROPOSITION 4.8.17. *Let z be $K([L^0_{E/F}])$-generic of level $-\infty$. Then $E[z]^\times \subset E[b]^\times B_{n(E/F)}(L^0_{E/F})$ if and only if z lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$.*

Proof. If $E[z]^\times \subset E[b]^\times B_{n(E/F)}(L^0_{E/F})$ then it is clear that z lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$.

Conversely, if z lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$, let w be in $E[z]^\times$. Then $w = \alpha + \beta z$ for some α, β in E. If $\alpha = 0$ we clearly have that w lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$. We then consider the case when $\alpha \neq 0$. We have $w = \alpha(1 + \alpha^{-1}\beta z)$, so w lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$ if and only if $1 + \alpha^{-1}\beta z$ lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$. We observe now that $\alpha^{-1}\beta z$ is generic (because z is generic), lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$, and does not lie in $E^\times B_{n(E/F)}(L^0_{E/F})$. Thus, we need only show that if z lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F}) - E^\times B(L^0_{E/F})$ (so that z is generic) then $1 + z$ lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$.

Note that

$$\begin{aligned} K\left(\left[L^0_{E/F}\right]\right) - E^\times B\left(L^0_{E/F}\right) &= \Pi_{L^0_{E/F}} E^\times B\left(L^0_{E/F}\right) \\ &= \Pi_{L^0_{E/F}} \langle \pi_E \rangle B\left(L^0_{E/F}\right) = \bigcup_{n \text{ odd}} \mathscr{E}'_n\left(L^0_{E/F}\right). \end{aligned}$$

Suppose first that z lies in $\mathscr{E}'_m(L^0_{E/F})$ with $m > 0$. We have that $z = tk$ with t in $E[b]^\times$, k in $B_{n(E/F)}(L^0_{E/F})$. We observe then that t lies in $\mathscr{E}'_m(L^0_{E/F})$ and

$$1 + z = 1 + tk = (1 + t)(1 + t)^{-1}(1 + tk).$$

But $1 + t$ lies in $E[t]^\times = E[b]^\times$, so that is enough to prove that $(1 + t)^{-1}(1 + tk)$ lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$. Write

$$\begin{aligned}(1 + t)^{-1}(1 + tk) &= (1 + t)^{-1}(1 + t + t(k - 1))\\ &= 1 + (1 + t)^{-1}t(k - 1).\end{aligned}$$

It follows that $(1 + t)^{-1}(1 + tk)$ lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$ (because $t(k - 1)$ lies in $\mathscr{E}_{n(E/F)+m}(L^0_{E/F})$ and $1 + t$ lies in $B(L^0_{E/F})$).

We suppose finally that z lies in $\mathscr{E}'_m(L^0_{E/F})$ with $m < 0$. Thus, z^{-1} lies in $\mathscr{E}'_{-m}(L^0_{E/F})$. Write $1 + z = z(1 + z^{-1})$. Then, by the above, $(1 + z^{-1})$ lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$. It follows that $1 + z$ lies in $E[b]^\times B_{n(E/F)}(L_{E/F})$. This completes the proof.

Corollary 4.8.18. *Let x be $K([L^0_{E/F}])$ generic of level $-\infty$ and τ-$K([L^0_{E/F}])$ generic of level $-\infty$ and such that $N(x)$ lies in $H(\psi_b)$. Then x lies in $H(\psi_{b,E/F})$.*

Proof. Since $H(\psi_b) \subset H(\psi_{b,E/F})$ we have that $N(x)$ lies in $H(\psi_{b,E/F}) = E[b]^\times B_{n(E/F)}(L^0_{E/F})$ and $E[N(x)] \subset E[b]^\times B_{n(E/F)}(L^0_{E/F})$. Since $x^{-1}N(x)x = N(x)$ and $E[N(x)]$ is the centralizer of $N(x)$, we have $E[x]^\times = E[N(x)]^\times$. By Proposition 4.8.17 our result follows.

Proposition 4.8.19. *Let z be a $B(L^0_{E/F})$-generic element of level $r < n(E/F)$ (so that z does not lie in $E^\times B_{n(E/F)}(L^0_{E/F})$). Then $E[z]^\times \subset E[b]^\times B_{n(E/F)-r}(L^0_{E/F})$ if and only if z lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$.*

Proof. We assume first that z lies in $E[b]^\times B_{n(E/F)}(L^0_{E/F})$. Write $w = z - 1$ and $z = tk$ with t in $E[b]^\times$ and k in $B_{n(E/F)}(L^0_{E/F})$. Then w is $\mathscr{E}(L^0_{E/F})$-generic and

$$w = tk - 1 = (t - 1)\big(1 + (t - 1)^{-1}t(k - 1)\big).$$

Since $t - 1$ lies in $E[b]^\times \cap \mathscr{E}_r(L^0_{E/F})$ and $k - 1$ lies in $\mathscr{E}_{n(E/F)}(L^0_{E/F})$, it follows that w lies in $E[b]^\times B_{n(E/F)-r}(L^0_{E/F})$. By Proposition 4.8.17 we then have $E[w]^\times = E[z]^\times \subset E[b]^\times B_{n(E/F)-r}(L^0_{E/F})$.

Conversely, assume that $E[z]^\times \subset E[b]^\times B_{n(E/F)-r}(L^0_{E/F})$. Then

$$w = z - 1$$

lies in $E[b]^{\times}B_{n(E/F)-r}(L^0_{E/F})$. Write $w = tk$ with k in $B_{n(E/F)-r}(L^0_{E/F})$ and t in $E[b]^{\times}$, $\ell(L^0_{E/F})$-generic of level r. Thus,

$$z = 1 + w = (1+t)\big(1 + (1+t)^{-1}t(k-1)\big)$$

lies in $E[b]^{\times}B_{n(E/F)}(L^0_{E/F})$.

COROLLARY 4.8.20. *If x is $B(L^0_{E/F})$-generic, τ-$B(L^0_{E/F})$ generic of level r (i.e., both x and $N(x)$ have level r) and $N(x)$ lies in $H(\psi_b)$. Then x lies in $H(\psi_{b,E/F})$.*

Proof. We have $E[N(x)]^{\times} \subset E[b]^{\times}B_{n(E/F)-r}(L^0_{E/F})$, because $N(x)$ lies in $H(\psi_{b,E/F})$. Since $E[x]^{\times} = E[N(x)]^{\times}$ it follows that $E[x]^{\times} \subset E[b]^{\times}B_{n(E/F)-r}(L^0_{E/F})$. Now by Proposition 4.8.19 the result follows.

DEFINITION 4.8.21. Let $C_{K([L^0_F])}(x)$ be the set of conjugates $y^{-1}xy$ of x as y ranges over $K([L^0_F])$. Let $C^{\tau}_{K([L^0_{E/F}])}(x)$ be the set of τ-conjugates $y^{-1}xy^{\tau}$ of x as y ranges over $K([L^0_{E/F}])$.

Suppose x is an element out of $N^{-1}(E^{\times}B_{n(E/F)}(L^0_{E/F}))$, which is generic, τ-generic and such that $N(x)$ is F-rational. Then, either

(a) $C^{\tau}_{K([L^0_{E/F}])}(x) \cap H(\psi_{b,E/F}) = \varnothing$ (it follows, by Corollaries 4.8.18 and 4.8.20 that in this case $C_{K([L^0_F])}(N(x)) \cap H(\psi_b) = \varnothing$; thus, $\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \chi^{(0)}(N(x)) = 0$), or

(b) there is an element y in $K([L^0_{E/F}])$ such that $w = y^{-1}xy^{\tau}$ lies in $H(\psi_{b,E/F})$, so that $\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, w)$.

From this observation we may assume without loss of generality that if x is out of $N^{-1}(E^{\times}B_{n(E/F)}(L^0_{E/F}))$ and is generic, τ-generic, $N(x)$ is F-rational, then x lies in $H(\psi_{b,E/F})$.

PROPOSITION 4.8.22.

(a) *Let z be a $K([L^0_{E/F}])$-generic element of level $-\infty$ of $H(\psi_{b,E/F})$ and let y be in $K([L^0_{E/F}])$. Then $y^{-1}zy$ lies in $H(\psi_{b,E/F})$ if and only if y lies in the normalizer $N_{K([L^0_{E/F}])}(E[b]^{\times}B_{n(E/F)}(L^0_{E/F}))$ of $E[b]^{\times}B_{n(E/F)}(L^0_{E/F})$ in $K([L^0_{E/F}])$.*

(b) *Let z be a $B(L^0_{E/F})$-generic element of level $r < n(E/F)$ in $H(\psi_{b,E/F})$ and let y be in $K([L^0_{E/F}])$. Then $y^{-1}zy$ lies in $H(\psi_{b,E/F})$ if and only if y lies in $N_{K([L^0_{E/F}])}(E[b]^{\times}B_{n(E/F)-r}(L^0_{E/F}))$.*

Proof. In order to prove (a), we observe that $z = tk$ where t is in $E[b]^{\times}$ and k is in $B_{n(E/F)}(L^0_{E/F})$. Then t lies in $E[z]^{\times}B_{n(E/F)}(L^0_{E/F})$ and

$E[b]^{\times}= E[t]^{\times}$. By Proposition 4.8.17

$$E[b]^{\times}= E[t]^{\times}\subset E[z]^{\times}B_{n(E/F)}\big(L^0_{E/F}\big).$$

Also, by the same proposition, $E[z]^{\times}\subset E[b]^{\times}B_{n(E/F)}(L^0_{E/F})$. It follows that

$$E[z]^{\times}B_{n(E/F)}\big(L^0_{E/F}\big) = E[b]^{\times}B_{n(E/F)}\big(L^0_{E/F}\big).$$

Thus, if $y^{-1}zy$ lies in $H(\psi_{b,E/F}) = E[z]^{\times}B_{n(E/F)}(L^0_{E/F})$, then y lies in $N_{K([L^0_{E/F}])}(E[z]^{\times}B_{n(E/F)}(L^0_{E/F}))$ and conversely.

We now prove (b). Let us assume that $y^{-1}zy$ lies in $H(\psi_{b,E/F})$. Since $y^{-1}zy$ is $B(L^0_{E/F})$-generic of level r, from Proposition 4.8.19, we have

$$E\big[y^{-1}zy\big]^{\times}\subset E[b]^{\times}B_{n(E/F)-r}\big(L^0_{E/F}\big).$$

Also, $y^{-1}zy = tk$ for some t in $E[b]^{\times}$ and k in $B_{n(E/F)}(L^0_{E/F})$, we apply Proposition 4.8.19 to get

$$E[b]^{\times}= E[t]^{\times}\subset E\big[y^{-1}zy\big]^{\times}B_{n(E/F)-r}\big(L^0_{E/F}\big),$$

so that

$$E[b]^{\times}B_{n(E/F)-r}\big(L^0_{E/F}\big) = E\big[y^{-1}zy\big]^{\times}B_{n(E/F)-r}\big(L^0_{E/F}\big).$$

Similarly, $E[b]^{\times}B_{n(E/F)-r}(L^0_{E/F}) = E[z]^{\times}B_{n(E/F)-r}(L^0_{E/F})$. It follows that y lies in

$$N_{K([L^0_{E/F}])}\Big(E[z]^{\times}B_{n(E/F)-r}\big(L^0_{E/F}\big)\Big) = N_{K([L^0_{E/F}])}\Big(E[b]^{\times}B_{n(E/F)-r}\big(L^0_{E/F}\big)\Big).$$

Conversely, we assume that y lies in $N_{K([L^0_{E/F}])}(E[b]^{\times}B_{n(E/F)-r}(L^0_{E/F}))$. Write $y^{-1}zy = y^{-1}tyk$ for some t in $E[b]^{\times}$ and k in $B_{n(E/F)}(L^0_{E/F})$. Then k lies in $H(\psi_{b,E/F})$. Thus, we may assume that z lies in $E[b]^{\times}$. Under these circumstances, $z-1$ is an element of $E[b]^{\times}$ that lies in $\mathscr{E}'_r(L^0_{E/F})$. Write $y^{-1}(z-1)y = t_1k_1$ where t_1 lies in $E[b]^{\times}$ and is $\mathscr{E}(L^0_{E/F})$-generic of level r and k_1 is in $B_{n(E/F)-r}(L^0_{E/F})$. We then have

$$y^{-1}zy = 1 + t_1k_1 = (1+t_1)\big(1+(1+t_1)^{-1}t_1(k_1-1)\big).$$

Thus, $y^{-1}zy$ lies in $E[b]^{\times}B_{n(E/F)}(L^0_{E/F}) = H(\psi_{b,E/F})$. This completes the proof.

PROPOSITION 4.8.23. *Let x be a $K([L^0_{E/F}])$ generic, τ-$K([L^0_{E/F}])$ generic element of $H(\psi_{b,E/F})$ that does not lie in $N^{-1}(E^{\times}B_{n(E/F)}(L^0_{E/F}))$ and such that $N(x)$ is F-rational. Then $\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \chi^{(0)}_{\pi}(N(x))$.*

Proof. We recall that

$$\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \text{ in } K([L^0_{E/F}])/H(\psi_{b,E/F})} \dot{\rho}_{E/F}\big(y^{-1}xy^{\tau}\big).$$

Assume then that y lies in $K([L^0_{E/F}])$ and $y^{-1}xy^\tau$ lies in $H(\psi_{b,E/F})$. Then $y^{-1}N(x)y$ lies in $H(\psi_{b,E/F})$. We have that either x and $N(x)$ are $K([L^0_{E/F}])$-generic elements of level $-\infty$, or x and $N(x)$ are $B(L^0_{E/F})$-generic elements of level $r < n$. In the first case, Proposition 4.8.22(a) says that y lies in $N_{K([L^0_{E/F}])}(E[b]^\times B_{n(E/F)}(L^0_{E/F}))$ and then $y^{-1}xy$ lies in $H(\psi_{b,E/F})$ (by Proposition 4.8.22, now applied to the generic element x). By the same argument, Proposition 4.8.22(b) implies that $y^{-1}xy$ lies in $H(\psi_{b,E/F})$ if x and $N(x)$ are $B(L^0_{E/F})$-generic. Thus, $y^{-1}xy$ lies in $H(\psi_{b,E/F})$ if y lies in $K([L^0_{E/F}])$ and $y^{-1}xy^\tau$ lies in $H(\psi_{b,E/F})$. It follows that if y is in Q_E, $y^{-1}y^\tau = (y^{-1}x^{-1}y)(y^{-1}xy^\tau)$ lies in $H(\psi_{b,E/F}) \cap Q_E = Q_{n(E/F),E}$. Now we have by Proposition 4.3.8, since

$$Q_E \cap K\left(\left[L^0_{E/F}\right]\right)/Q_E \cap H(\psi_{b,E/F})$$

and $K([L^0_{E/F}])/H(\psi_{b,E/F})$ are in natural bijection, that

$$\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \text{ in } Q_F \cap K([L^0_{E/F}])/Q_F \cap H(\psi_{b,E/F})} \dot{\rho}_{E/F}(y^{-1}xy)$$

$$= \sum_{y \text{ in } Q_F \cap K([L^0_F])/Q_{n,F}} \dot{\rho}_{E/F}(y^{-1}xy).$$

Finally, Lemma 4.8.14 implies that

$$\chi^{(0)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \text{ in } Q_F \cap K([L^0_F])/Q_{n,F}} \dot{\rho}(y^{-1}N(x)y) = \chi^{(0)}_{\pi}(N(x)).$$

4.9. For this section only we set

$$\chi^{(m)}_{\tilde{\pi}_{E/F}}(\tau, x) = \sum_{y \text{ in } \mathrm{Gl}_2(O_E) z_m K([L^0_{E/F}])/H(\psi_{b,E/F})} \dot{\rho}_{E/F}(y^{-1}xy^\tau)$$

where z_m lies in $Z_{E/F}$ (see §4.3).

It should be observed that $K_0^{unr'}(L^0_E) = \bigcup_{r=0}^\infty K_r^{unr'}(L^0_E)$, where $K_r^{unr'}(L^0_E)$ consists of the $K_0^{unr}(L_E)$-generic elements of level r. It follows that it is enough to restrict ourselves to τ-unramified generic elements in $N^{-1}(K_r^{unr}(L^0_E) - U_E^r K_{r+1}^{unr}(L^0_E))$.

Proposition 4.9.1. *Let x be a τ-$K^{unr}([L^0_E])$ generic element. Then there is an element h in $\mathrm{Gl}_2(O_E)$ such that $h^{-1}N(x)h$ is F-rational.*

Proof. Similar to the proof in Proposition 4.8.5.

It follows from above that we may assume without loss of generality that if x is τ-$K^{unr}([L^0_E])$ generic then $N(x)$ lies in $\mathrm{Gl}_2(F)$.

PROPOSITION 4.9.2. *Let x be a τ-$K_0^{unr}(L_E^0)$ generic element of level r (i.e., $N(x)$ has level r) such that $N(x)$ is F-rational. Then there is a $K_0^{unr}(L_E^0)$-generic element z of level r such that $N(z) = N(x)$.*

Proof. We recall once more that the restriction of the norm map N to $E[N(x)]$ is the norm of the field extension $E[N(x)]/F[N(x)]$.

We prove the proposition when E/F is ramified. The unramified case is proved similarly.

Suppose first that $r = 0$, i.e., $N(x)$ lies in $U^0_{E[N(x)]}$ but not in $U^0_E U^1_{E[N(x)]}$. It follows that x lies in $U_{E[N(x)]}$ and cannot be an element of $U^0_E U^1_{E[N(x)]}$ (otherwise $N(x)$ becomes an element of $U^0_E U^1_{E[N(x)]}$, which is false). Also, since x commutes with $N(x)$ we have that $E[x] = E[N(x)]$, so that x is unramified generic of level zero.

We now suppose $r \geq 1$. Then $N(x)$ lies in $U^r_{E[N(x)]}$ but not in $U^r_E U^{r+1}_{E[N(x)]}$. It follows that $r = sl$ for some s (because $N(x)$ lies in $U^s_{F[N(x)]}$, some s, so we may take s such that $N(x)$ lies in $U^s_{F[N(x)]} - U^{s+1}_{F[N(x)]}$ from which $r = sl$). Thus, by properties of the norm (see [Se]), there is an element z in $U^{sl}_{E[N(x)]} - U^{sl+1}_{E[N(x)]}$ such that $N(z) = N(x)$. It follows that z lies in fact in $U^r_{E[N(x)]} - U^r_E U^{r+1}_{E[N(x)]}$ (because if z is in $U^r_E U^{r+1}_{E[N(x)]}$ then $N(z)$ is in $U^r_E U^{r+1}_{E[N(x)]}$, which is false). Also, given that x commutes with $N(x)$, we have that x is unramified generic. The level of x is r. This completes the proof.

LEMMA 4.9.3. *Let k be a quadratic extension of the finite field $\mathbf{F}_q$ of q elements. Let l be an odd prime number and suppose that for $c \neq 0$, lying in $\mathbf{F}_q$, there is β in k such that $\beta^l = c$. Then there is an element γ in $\mathbf{F}_q$ such that $\gamma^l = c$.*

Proof. Suppose first that l does not divide $q - 1$. Then the function $\gamma \mapsto \gamma^l$ is an injective endomorphism of $\mathbf{F}_q^\times$, so that is an automorphism. It follows that given c in $\mathbf{F}_q^\times$, there is γ in $\mathbf{F}_q^\times$ such that $\gamma^l = c$.

Consider now when l divides $q - 1$. Since $\beta^l = c$ lies in $\mathbf{F}_q$ we have that $\beta^{l(q-1)} = 1$. Also, $\beta^{q^2-1} = 1$. Thus, the order of β in $k^\times$ divides g.c.d.$(q^2 - 1, l(q - 1)) = q - 1$ (because l does not divide $q + 1$, since l is odd). We then have $\beta^{q-1} = 1$. The uniqueness of the subgroup of a given order in a cyclic group implies then that β lies in $\mathbf{F}_q$. This completes the proof.

PROPOSITION 4.9.4. *Let x be a τ-$K^{unr}([L_E^0])$-generic element such that has F-rational norm. Then there is an element α in $E^\times$ and a τ-$K_0^{unr}([L_E^0])$ generic element g such that x is τ-conjugate with αg.*

Proof. $N(x)$ has the form βh where β lies in $E^\times$ and h is $K_0^{unr}(L_E^0)$-generic of level r for some $r \geq 0$. So

$$N(x) = \begin{bmatrix} \beta u & \beta a \\ \beta c & \beta v \end{bmatrix}$$

where one of u, v, a or c is a unit. Suppose u is a unit. Set $r = \beta u$. Then r lies in F and $r^{-1}N(x) = r^{-1}\beta h$ is F-rational. We have $r^{-1}\beta h = u^{-1}h$ and if we set $h_1 = u^{-1}h$, it follows that h_1 is $K_0^{unr}(L_E^0)$-generic of level r. We observe that $N(x) = \beta h = rh_1$ so we may assume without loss of generality that $N(x) = \beta h$ where β lies in F and h is $K_0^{unr}(L_E^0)$-generic.

Suppose E/F is unramified. By properties of the norm there is a $K_0^{unr}(L_E^0)$-generic element g such that $N(g) = h$. It follows that β is a $E[N(x)]/F[N(x)]$-norm, but we need to prove that β is an E/F-norm. Since

$$N_{E/F}E^\times \subset N_{E[N(x)]/F[N(x)]}E[N(x)] \cap F^\times \subset F^\times$$

and $[F^\times : N_{E/F}E^\times] = l$ (see proof of Proposition 4.8.7), it remains to prove that this last inclusion is proper. The element π_F^2 has valuation two in $F[N(x)]$, so it is not in $N_{E[N(x)]/F[N(x)]}E[N(x)]$, because in th is last set the elements have valuations which are multiples of l. From this the proposition follows for E/F unramified.

Now suppose E/F is ramified. Set $A = F[N(x)]$. Then $\nu_F(N_{E/F}\pi_E^{\nu_F(\beta)}) = \nu_F(\beta)$ so we may scale x by $\pi_E^{\nu_F(\beta)}$ and we may then assume $N(x)$ is in U_A. Suppose first that $N(x)$ lies in U_A but does not lie in $U_F U_A^1$. We have then that x is a $K_0^{unr}(L_E^0)$-generic element lying in $U_A - U_F U_A^1$. Next, if $N(x)$ lies in U_A^1 and β lies in U_F^1, then β is an E/F-norm (see [Se]). Thus, h is a AE/A-norm and by Proposition 4.9.2, we are done in this case. Finally, suppose that $N(x)$ lies in $U_F U_A^1$ but not in U_A^1. Then $N(x) = \beta h$ where h lies in U_A^1 and β lies in U_F. It follows that h is an AE/A-norm, i.e., there is an element h_1 in U_{AE}^1 such that $N(h_1) = h$. So β is an AE/A-norm and the proposition will be proved once we prove that β is, in fact, an E/F-norm. Denote by $\bar{\beta}$ the image of β in k_A (the residue field of A). Consider the diagram

$$\begin{tikzcd} & k_{AE} \arrow[dr, "N"] & \\ k_E \arrow[ur, dash] & & k_A \\ & k_F \arrow[ul, dash] \arrow[ur, dash, "2"'] & \end{tikzcd}$$

$\bar{\beta}$ is an lth-power in k_A and it lies in k_F. Thus, $\bar{\beta}$ is an lth-power in k_F (by Lemma 4.9.3). It follows that β is an E/F-norm.

From the above we see that we may assume that x is τ-$K_0^{unr}(L_E^0)$ generic, $K_0^{unr}(L_E^0)$-generic, $N(x)$ is F-rational and level x = level Nx.

PROPOSITION 4.9.5. *Let x be K_0^{unr}-generic of level r and let y be in $\mathrm{Gl}_2(O_E)z_mK([L_{E/F}^0])$. Then $y^{-1}xy$ lies in $B_n(L_{E/F}^0)$ if and only if $-m + r \geq [n/2] + (1 + e(E/F))/2$.*

Proof. Write

$$x = 1 + \pi_E^r \begin{bmatrix} a & u \\ v & d \end{bmatrix}$$

with u, v in U_E and a, d in O_E. Also, let $y = hz_mk$ where h lies in $\mathrm{Gl}_2(O_E)$ and k is in $K([L_{E/F}^0])$.

We have $y^{-1}xy = k^{-1}z_m^{-1}h^{-1}xhz_mk$ lies in $B_n(L_{E/F}^0)$ if and only if $z_m^{-1}h^{-1}xhz_m$ lies in $B_n(L_{E/F}^0)$. But since x and $h^{-1}xh$ are unramified generic of the same level, it is enough to prove that $z_m^{-1}xz_m$ lies in $B_n(L_{E/F}^0)$ if and only if $-m + r \geq [n/2] + (1 + e(E/F))/2$.

It follows that

$$z_m^{-1}xz_m = 1 + \begin{bmatrix} a\pi_E^r & u\pi_E^{m+r} \\ v\pi_E^{-m+r} & d\pi_E^r \end{bmatrix}$$

lies in $B_n(L_{E/F}^0)$ if and only if $-m + r \geq [n/2] + (1 + e(E/F))/2$.

COROLLARY 4.9.6. *Let x be $K_0^{unr}(L_E^0)$-generic and τ-$K_0^{unr}(L_E^0)$ generic, level x = level $N(x)$ and let $N(x)$ be F-rational. Then $y^{-1}N(x)y$ lies in $B_n(L_{E/F}^0)$ if and only if $y^{-1}xy$ lies in $B_n(L_{E/F}^0)$.*

PROPOSITION 4.9.7. *Let x be $K_0^{unr}(L_E^0)$-generic and y be an element of $\mathrm{Gl}_2(O_E)z_mK([L_{E/F}^0])$. If $y^{-1}xy$ lies in $E^\times B_n(L_{E/F}^0)$ then $y^{-1}xy$ lies in $B_n(L_{E/F}^0)$.*

Proof. Write $y^{-1}xy = \alpha c$ where α is in $E^\times$ and $c = [c_{ij}]$ lies in $B_n(L_{E/F}^0)$.

Since for k in $K([L_{E/F}^0])$ we have that $k^{-1}zk$ lies in $E^\times B_n(L_{E/F}^0)$ if and only if z lies in $E^\times B_n(L_{E/F}^0)$, we may assume that y has the form $y = hz_m$ with h in $\mathrm{Gl}_2(O_E)$.

Given that $h^{-1}xh$ is unramified generic, we may write

$$h^{-1}xh = 1 + \pi_E^r \begin{bmatrix} a & u \\ v & d \end{bmatrix}$$

where u, v are units. It follows that

$$y^{-1}xy = z_m^{-1}h^{-1}xhz_m = 1 + \pi_E^r \begin{bmatrix} a & \pi_E^m u \\ \pi_E^{-m}v & d \end{bmatrix}.$$

We get $1 + \pi_E^r a = \alpha c_{11}$. Since c_{11} is a unit, it follows that $\nu_E(\alpha) \geq 0$. Also,

$$-m + r = \nu_E(\alpha) + \nu_E(c_{21}) \geq \left[\frac{n}{2}\right] + \frac{1 + e(E/F)}{2}.$$

The result now follows from Proposition 4.9.5.

PROPOSITION 4.9.8. *Let x be as in 4.9.6. If $y^{-1}xy^\tau$ lies in $H(\psi_{b,E/F})$ then $y^{-1}xy$ lies in $B_{n(E/F)}(L_{E/F}^0)$.*

Proof. Since $y^{-1}xy^\tau$ lies in $H(\psi_{b,E/F})$, we have that $y^{-1}N(x)y$ lies in $H(\psi_{b,E/F})$. Also, $y^{-1}N(x)y$ is unramified and

$$H(\psi_{b,E/F}) - E^\times B_{n(E/F)}\left(L_{E/F}^0\right)$$

consists only of ramified generic elements, so that $y^{-1}N(x)y$ lies in $E^\times B_{n(E/F)}(L_{E/F}^0)$. By Proposition 4.9.7, $y^{-1}N(x)y$ lies in $B_{n(E/F)}(L_{E/F}^0)$. Finally, by Corollary 4.9.6, $y^{-1}xy$ lies in $B_{n(E/F)}(L_{E/F}^0)$. This completes the proof.

It follows from the above that then $y^{-1}y^\tau$ lies in $H(\psi_{b,E/F})$ and if y is in Q_E, then, in fact, $y^{-1}y^\tau$ lies in $H(\psi_{b,E/F}) \cap Q_E = Q_{n(E/F),E/F}$. Now, the analogs of Propositions 4.8.11 and 4.8.12 follow. Finally, with arguments similar to ones made in §4.8 we get

PROPOSITION 4.9.9. *Let x be as in 4.9.6. Then*

$$\chi_{\tilde{\pi}_{E/F}}^{(m)}(\tau, x) = \chi_\pi^{(m)}(N(x)).$$

4.10. We have at this point all the elements we need in order to present our main result.

THEOREM 4.10.1. *Let F be a p-field. Let E/F be a prime cyclic extension of odd degree l such that $p \neq l$. Let π be an irreducible admissible ramified supercuspidal representation of $\mathrm{Gl}_2(F)$. Let $\pi_{E/F}$ and Π be defined, respectively, as in 2.1.9 and 3.3.1. Then the representations $\pi_{E/F}$ and Π of $\mathrm{Gl}_2(E)$ are equivalent.*

REFERENCES

[B] A. Borel, *Automorphic L-functions, automorphic forms, representations and L-functions*, Amer. Math. Soc., (1979).

[Cl] A. B. Clifford, *Representations induced in an invariant subgroup*, Annals of Math., **38** (1937), 533–550.

[Go] R. Godement, *Notes on Jacquet-Langlands' Theory*, The Institute for Advanced Study, 1970.

[Ge] P. Gérardin, *Construction de Séries Disgrètes p-adiques*, Lecture Notes in Mathematics, Vol. 462.

[G-L] P. Gérardin and J.-P. Labesse, *The solution of a base change problem for* Gl(2), *Automorphic forms, Representations and L-functions*, Amer. Math. Soc., (1979).

[H-C] Harish-Chandra, *Harmonic Analysis on Reductive p-adic Groups*, Lecture Notes in Mathematics, Vol. 162.

[J-L] H. Jacquet and R. P. Langlands, *Automorphic Forms on* Gl(2), Lecture Notes in Mathematics, Vol. 114.

[K1] P. Kutzko, *Mackey's theorem for non-unitary representations*, Proc. Amer. Math. Soc., No. **64** (1977), 173–175.

[K2] _____, *On the supercuspidal representations of* Gl_2, Amer. J. Math., **100**, No. 1, pp. 43–60.

[K3] _____, *On the supercuspidal representations of* Gl_2, Amer. J. Math., **100**, No. 4, pp. 705–716.

[K4] _____, *The Langlands conjecture for* Gl_2 *of a local field*, Annals of Math., **112** (1980), 381–412.

[K5] _____, Private Communication.

[L] R. P. Langlands, *Base Change for* Gl(2), Annals of Math. Studies, Princeton University Press, New Jersey, 1980.

[S] P. J. Sally, Jr., *Character formulas for* Sl_2, Amer. Math. Soc. Proc. Symp. Pure Math., **XXVI** (1972), 395–400.

[Sa] H. Saito, *Automorphic Forms and Algebraic Extensions of Number Fields*, Lectures in Math., No. 8, Kyoto Univ., Ky oto, Japan, 1975.

[Se] J.-P. Serre, *Corps Locaux*, Publications de Mathematique de l'Université de Nancago, Hermann, 1962.

[Sh] J. A. Shalika, *Some conjectures in class field theory*, Proc. of Symposia in Pure Math., Amer. Math. Soc., **20**, pp. 115–122.

[Sp] T. A. Springer, *Reductive groups, automorphic forms, representations, and L-functions*, Amer. Math. Soc., (1979).

Received August 10, 1983.

UNIVERSIDAD DE VALPARAÍSO
GRAN BRETAÑA 1041 P. ANCHA
VALPARAÍSO, CHILE