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LORENTZIAN ISOPARAMETRIC HYPERSURFACES

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A Lorentzian hypersurface will be called isoparametric if the minimal polynomial of the shape operator is constant. This allows for complex or non-simple principal curvatures (eigenvalues of the shape operator). This paper locally classifies isoparametric hypersurfaces in Lorentz space.

The classification is done by proving Cartan-type identities for the principal curvatures and showing that the hypersurface can have at most one non-zero real principal curvature. Standard examples are given in §3 and the main theorems are in §4.

The hypersurfaces with minimal polynomials $(x-a)^2$ and $(x-a)^3$ are called generalized umbilical hypersurfaces since they have exactly one principal curvature. The classification of these hypersurfaces gives insight into principal curvatures and the effect of the constant principal curvatures on the structure of a hypersurface.

1. Preliminaries. In this paper all manifolds and maps are assumed to be C^∞. $f\colon M \to \tilde{M}$ will always be an immersion but f can be treated locally as an embedding. Thus x will often be identified with $f(x)$ and the mention of f will be supressed.

Lorentz space and its hypersurfaces. Let $\mathbf{L}^{n+1}$ be the $n+1$ dimensional real vector space $\mathbf{R}^{n+1}$ with an inner product of signature $(1, n)$ given by

$$(\vec{x}, \vec{y}) = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$$

for $\vec{x} = (x_0, x_1, \ldots, x_n)$ and $\vec{y} = (y_0, y_1, \ldots, y_n)$. $\mathbf{L}^{n+1}$ is called Lorentz space.

The n-dimensional sphere of radius r in $\mathbf{L}^{n+1}$, $S_1^n(1/r^2)$, is the hypersurface

$$\{\vec{x} \in \mathbf{L}^{n+1} \colon (\vec{x}, \vec{x}) = r^2\}$$

with the induced Lorentzian metric. It has constant sectional curvature $1/r^2$.

Generally a hypersurface M in $\mathbf{L}^{n+1}$ is called a Lorentzian hypersurface if the induced metric has signature $(1, n-1)$. If D is the flat connection on $\mathbf{L}^{n+1}$ the Levi-Civita connection ∇ on M is specified by

$$D_X Y = \nabla_X Y + \alpha(X, Y) \tag{1.1}$$

where X and Y are tangent vector fields on M, $\nabla_X Y$ is the tangential component of $D_X Y$ and $\alpha(X, Y)$ is the normal component. α is called the second fundamental form.

In a neighborhood of each point of M we can find a field ξ of unit normal vectors. Using ξ a field of endomorphisms A on M can be defined by

$$D_X \xi = -AX. \tag{1.2}$$

A is symmetric with respect to the induced Lorentzian metric and is called the shape operator of M.

The curvature tensor R of M is related to A by the Gauss equation

$$R(X, Y) = AX \wedge AY \tag{1.3}$$

where $U \wedge V$ denotes the endomorphism of the tangent space defined by

$$(U \wedge V)W = (V, W)U - (U, W)V.$$

The shape operator satisfies Codazzi's equation

$$\nabla_X(AY) - A(\nabla_X Y) = \nabla_Y(AX) - A(\nabla_Y X). \tag{1.4}$$

Throughout the paper the equation which results from taking the inner product of both sides of (1.4) with a tangent vector Z will be denoted by $\{X, Y\}Z$.

Symmetric Endomorphisms. If V is a vector space with a Lorentzian inner product (,) an *orthonormal* basis $\{E_1, \dots, E_n\}$ is one satisfying

$$(E_1, E_1) = -1, \quad (E_i, E_j) = \delta_{ij}, \quad (E_1, E_i) = 0$$

for $2 \leq i,\ j \leq n$. A pseudo-orthonormal basis is a basis $\{X, Y, E_1, \dots, E_{n-2}\}$ such that

$$(X, X) = 0 = (Y, Y) = (X, E_i) = (Y, E_i), \quad (X, Y) = -1$$

and

$$(E_i, E_j) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq n-2.$$

A symmetric endomorphism A of a vector space V with a Lorentzian inner product can be put into one of four forms [**11**].

$$\text{I.} \quad A \sim \begin{bmatrix} a_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_n \end{bmatrix} \qquad \text{II.} \quad A \sim \begin{bmatrix} a_0 & 0 & & & \mathbf{0} \\ 1 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ \mathbf{0} & & & & a_{n-2} \end{bmatrix}$$

$$\text{III.} \quad A \sim \begin{bmatrix} a_0 & 0 & 0 & & & \\ 0 & a_0 & 1 & & & \\ -1 & 0 & a_0 & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_{n-3} \end{bmatrix}$$

$$\text{IV.} \quad A \sim \begin{bmatrix} a_0 & b_0 & & & \\ -b_0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{bmatrix}$$

Here b_0 is assumed to be non-zero. In cases I and IV A is represented with respect to an orthonormal basis while in cases II and III the basis is pseudo-orthonormal. In cases I, II and III the eigenvalues are real, while $a_0 \pm ib_0$ are eigenvalues in case IV.

2. Cartan's identities. In the case where the shape operator A is diagonalizable a hypersurface is said to be *isoparametric* if A has constant eigenvalues (principal curvatures). If A is not diagonalizable define a hypersurface to be *isoparametric* if the minimal polynomial of the shape operator is constant. Such a hypersurface has constant principal curvatures and A can be put into exactly one of the canonical forms I, II, III, or IV.

Following and simplifying the method in **[10]** we show that in each case the principal curvatures satisfy an identity.

If M^n is a Lorentzian hypersurface in $\mathbf{L}^{n+1}$ with a constant principal curvature a, define a distribution T_a on M by

$$T_a = \{U \in TM \colon AU = aU\}.$$

LEMMA 2.1. *T_a is an integrable distribution.*

Proof. If U and V are in T_a, by 1.4

$$A[U,V] = a[U,V]$$

so that $[U,V]$ is in T_a. □

This doesn't depend on the metric induced on T_a, which may be degenerate.

THEOREM 2.2. *If the shape operator of a Lorentzian hypersurface in $\mathbf{L}^{n+1}$ is diagonalizable and has distinct constant eigenvalues $a_1, \ldots, a_p$ with* *multiplicities* $\nu_1, \ldots, \nu_p$, *then for any* i, $1 \le i \le p$,

$$\sum_{j \neq i} \frac{\nu_j a_i a_j}{a_i - a_j} = 0. \tag{2.1}$$

Proof. For an eigenvalue a of A define the focal map $f_a \colon M \to \mathbf{L}^{n+1}$ by

$$f_a(x) = x + \frac{1}{a}\xi(x),$$

so that

$$(f_a)_* U = U - \frac{1}{a} AU.$$

For any j, $1 \le j \le p$, denote f_{a_j} by f_j and T_{a_j} by T_j. We see that

$$(f_i)_* = 0 \quad \text{on } T_i, \text{ while}$$

$$(f_i)_* U = \frac{a_i - a_j}{a_i} U \quad \text{for } U \text{ in } T_j, j \neq i.$$

Call $f_i(M) = V_i$. This is an $n - \nu_i$ dimensional submanifold of $\mathbf{L}^{n+1}$, at least in a neighborhood of $f_i(x)$. We can identify $T_p(V_i)$ with $[T_i]^\perp$. The line $x(t) = x + t\xi(x)$ is normal to V_i at $f_i(x)$ and $x'(t) = \xi(x)$. For U in $[T_i]^\perp$ we want to calculate the shape operator $B_\xi U$ at $f_i(x)$.

$$D_U \xi = (f_i)_*(-B_\xi U) + \nabla_U^\perp \xi$$

where $\nabla_U^\perp \xi$ is the component of $D_U \xi$ normal to V_i. If U is in T_j

$$D_U \xi = -a_j U$$

so that $B_\xi U = (a_i a_j / a_i - a_j) U$. Therefore

$$\operatorname{tr} B_\xi = \sum_{j \neq i} \frac{\nu_j a_i a_j}{a_i - a_j}. \tag{2.2}$$

Following [**10**] we define a differentiable mapping from M_i, the integral manifold of T_i through x, to $N_p(V_i)$, the normal space to V_i at $f_i(x) = p$. f_i maps M_i to the single point p. Define $g_i \colon M_i \to N_p(V_i)$ by

$$g_i(y) = \xi(y).$$

The differential of g_i at x is

$$(g_i)_* Z = -a_i Z$$

so that g_i is injective. $N_p(V_i)$ is either Euclidean or Lorentzian, depending on M_i. Consider the linear function w on this vector space given by

$$w(V) = \operatorname{tr} B_V.$$

Let $S_p \subset N_p(V_i)$ denote the unit sphere, which is either a Riemannian or Lorentzian manifold. $g_i(M_i)$ is an open subset of S_p containing $\xi(x)$.

By (2.2) w is constant on an open subset of S_p. An easy argument shows that $w \equiv 0$ on $N_p(V_i)$. □

In order to prove the appropriate identity when A falls in case II we need the following lemma.

LEMMA 2.3. *Suppose A is the shape operator of a Lorentzian hypersurface. If A has distinct constant eigenvalues $a_0, a_1, \ldots, a_p$ with multiplicities $\nu_0, \nu_1, \ldots, \nu_p$ and the minimal polynomial of A is $(x - a_0)^2(x - a_1) \cdots (x - a_p)$ then there is a pseudo-orthonormal basis*

$$\{X, Y, Z_1, \ldots, Z_{\nu_0 - 2}, E_{11}, \ldots, E_{1\nu_1}, \ldots, E_{p\nu_p}\}$$

of vector fields in a neighborhood of any point in M with respect to which

$$A = \begin{bmatrix} a_0 & 0 & & & & & \\ 1 & a_0 & & & & & \\ & & \ddots & & & & \\ & & & a_0 & & & \\ & & & & a_1 & & \\ & & & & & \ddots & \\ & & & & & & a_p \end{bmatrix}.$$

Note. The multiplicity of an eigenvalue a_β is the exponent of $(x - a_\beta)$ in the characteristic polynomial. See, for example, [7], p. 236.

Proof. Take such a basis at a point x_0. Extend the basis to vector fields $\{\tilde{X}, \tilde{Y}, \tilde{Z}_1, \ldots, \tilde{E}_{p\nu_p}\}$ in a neighborhood of x_0. Consider

$$(A - a_0)^2(A - a_1) \cdots \widehat{(A - a_j)} \cdots (A - a_p)\tilde{E}_{jk},$$
$$1 \le j \le p, 1 \le k \le \nu_j.$$

For a fixed j these ν_j vector fields span T_j. They can be made orthonormal using the Gram-Schmidt process, yielding $E_{11}, \ldots, E_{p\nu_p}$. Using these we can form $\overline{X}, \overline{Y}, \overline{Z}_1, \ldots, \overline{Z}_{\nu_0 - 2}$ from $\tilde{X}, \tilde{Y}, \ldots, \tilde{Z}_{\nu_0 - 2}$ which are perpendicular to $T_1 \oplus \cdots \oplus T_p$. Now apply Gram-Schmidt to

$$\left\{\frac{\overline{X} + \overline{Y}}{\sqrt{2}}, \frac{\overline{X} - \overline{Y}}{\sqrt{2}}, \overline{Z}_1, \ldots, \overline{Z}_{\nu_0 - 2}\right\}$$

to form $\{W_1, \ldots, W_{\nu_0}\}$, an orthonormal basis of $[T_1 \oplus \cdots \oplus T_p]^\perp$.

$W_1 + W_2$ is lightlike (has length zero), $(A - a_0)(W_1 + W_2) \neq 0$ in a neighborhood of x_0 and $(A - a_0)^2(W_1 + W_2) = 0$. This means that

$$((A - a_0)(W_1 + W_2), (A - a_0)(W_1 + W_2)) = 0 \quad \text{and}$$
$$A((A - a_0)(W_1 + W_2)) = a_0((A - a_0)(W_1 + W_2))$$

so that $(A - a_0)(W_1 + W_2)$ is lightlike and in T_0. Thus there is a multiple of $W_1 + W_2$ such that

$$(c(W_1 + W_2), (A - a_0)(c(W_1 + W_2))) = -1,$$

near x_0. Setting $X = c(W_1 + W_2)$ and $Y = (A - a_0)(c(W_1 + W_2))$ it is easy to complete the desired basis. □

In the statements and proofs of the following theorems the indices i, j and β will have the following ranges: $1 \leq i, j \leq p$ and $0 \leq \beta \leq p$.

THEOREM 2.4. *If the shape operator of a Lorentzian isoparametric hypersurface in $\mathbf{L}^{n+1}$ has minimal polynomial $(x - a_0)^2(x - a_1) \cdots (x - a_p)$ and the eigenvalues have multiplicities $\nu_0, \nu_1, \ldots, \nu_p$ then for any i*

$$\sum_{\beta \neq i} \frac{\nu_\beta a_i a_\beta}{a_i - a_\beta} = 0. \tag{2.3}$$

Proof. Fix i and again consider

$$f_i(x) = x + \frac{1}{a_i}\xi(x).$$

Then

$$(f_i)_* U = \frac{a_i - a_j}{a_i} U \quad \text{for } U \text{ in } T_j, j \neq i,$$

while on $T_* = [T_1 \oplus \cdots \oplus T_p]^\perp$ we have

$$(f_i)_*(X) = \frac{a_i - a_0}{a_i} X - \frac{1}{a_i} Y$$

for X and Y as in Lemma 2.3 and

$$(f_i)_* U = \frac{a_i - a_0}{a_0} U \quad \text{for } U \text{ in } T_0.$$

Therefore $(f_i)_*$ is injective on $[T_i]^\perp$, which can be considered the tangent space to $V_i = f_i(M)$. The line $x(t) = x + t\xi(x)$ with $x'(t) = \xi(x)$ is normal to V_i at $f_i(x)$. We want to calculate the shape operator of V_i in this normal direction. For U in $[T_i]^\perp$

$$D_U \xi = -(f_i)_*(B_\xi U) + \nabla_U^\perp \xi = -AU.$$

With respect to the basis above

$$B_\xi = \begin{bmatrix} \frac{a_0 a_i}{a_i - a_0} & & & \\ \frac{a_i^2}{(a_i - a_0)^2} & \frac{a_0 a_i}{a_i - a_0} & & \\ & & \ddots & \\ & & & \frac{a_p a_i}{a_i - a_0} \end{bmatrix}$$

so that $\operatorname{tr} B_\xi = \sum_{\beta \neq i} \nu_\beta a_i a_\beta / (a_i - a_\beta)$. As in the proof of Theorem 2.2 this constant is zero. □

To tackle case III we need another indefinite Gram-Schmidt lemma.

LEMMA 2.5. *Let A be the shape operator of a Lorentzian hypersurface. If A has distinct constant eigenvalues $a_0, a_1, \ldots, a_p$ with multiplicities $\nu_0, \nu_1, \ldots, \nu_p$ and the minimal polynomial of A is $(x - a_0)^3(x - a_1) \cdots (x - a_p)$ then there is a pseudo-orthonormal basis*

$$\{X, Y, Z, Z_1, \ldots, Z_{\nu_0 - 3}, E_{11}, \ldots, E_{1\nu_1}, \ldots, E_{p\nu_p}\}$$

of vector fields in a neighborhood of any point in M with respect to which

$$A = \begin{bmatrix} a_0 & 0 & 0 & & & & \\ 0 & a_0 & 1 & & & & \\ -1 & 0 & a_0 & & & & \\ & & & \ddots & & & \\ & & & & a_0 & & \\ & & & & & a_1 & \\ & & & & & & \ddots \\ & & & & & & & a_p \end{bmatrix}.$$

Proof. As in Lemma 2.3, find $E_{11}, \ldots, E_{p\nu_p}$ with the desired properties and $\overline{X}, \overline{Y}, \overline{Z}, \overline{Z}_1, \ldots, \overline{Z}_{\nu_0 - 3}$ which are perpendicular to $T_1 \oplus \cdots \oplus T_p$. Apply the Gram-Schmidt process to

$$\left\{ \frac{\overline{X} + \overline{Y}}{\sqrt{2}}, \frac{\overline{X} - \overline{Y}}{\sqrt{2}}, \overline{Z}, \ldots, \overline{Z}_{\nu_0 - 3} \right\}$$

to obtain $\{W_1, \ldots, W_{\nu_0}\}$, an orthonormal basis of $[T_1 \oplus \cdots \oplus T_p]^\perp$. Set $\tilde{X} = (W_1 + W_2)/\sqrt{2}$. $\tilde{X}$ is lightlike, $(A - a_0)^2 \tilde{X} \neq 0$ and $(A - a_0)^3 \tilde{X} = 0$. Now let

$$\hat{X} = \frac{\tilde{X}}{\sqrt{((A - a_0)^2 \tilde{X}, \tilde{X})}}$$

$$\hat{Y} = -(A - a_0)^2 \hat{X} \quad \text{and}$$

$$\hat{Z} = -(A - a_0)\hat{X} - ((A - a_0)\hat{X}, \hat{X})\hat{Y}.$$

These vector fields satisfy: $(\hat{X}, \hat{X}) = 0 = (\hat{Y}, \hat{Y})$, $(\hat{X}, \hat{Y}) = -1$, $(\hat{X}, \hat{Z}) = 0 = (\hat{Y}, \hat{Z})$, $(\hat{Z}, \hat{Z}) = 1$ and

$$\begin{aligned} A\hat{X} &= a_0\hat{X} + C\hat{Y} - \hat{Z} \\ A\hat{Y} &= a_0\hat{Y} \\ A\hat{Z} &= \hat{Y} + a_0\hat{Z}. \end{aligned}$$

C is a possibly non-zero function. Finally set

$$\begin{aligned} X &= \hat{X} + \frac{C^2}{4}\hat{Y} - \frac{C}{2}\hat{Z} \\ Y &= \hat{Y} \\ Z &= \frac{-C}{2}\hat{Y} + \hat{Z}. \end{aligned}$$

As before $Z_1, \ldots, Z_{\nu_0 - 3}$ are simple to find. □

THEOREM 2.6. *If the shape operator of a Lorentzian isoparametric hypersurface in* $\mathbf{L}^{n+1}$ *has minimal polynomial* $(x - a_0)^3(x - a_1) \cdots (x - a_p)$ *and the eigenvalues have multiplicities* $\nu_0, \nu_1, \ldots, \nu_p$, *then for every* i

$$\sum_{\beta \neq i} \frac{\nu_\beta a_i a_\beta}{a_i - a_\beta} = 0. \tag{2.4}$$

Proof. Fix i and look at the focal map f_i.

$$(f_i)_* U = \frac{a_i - a_\beta}{a_i} U \quad \text{for } U \text{ in } T_\beta,$$

while

$$\begin{aligned} (f_i)_* X &= \frac{a_i - a_0}{a_i} X + \frac{1}{a_i} Z \\ (f_i)_* Z &= -\frac{1}{a_i} Y + \frac{a_i - a_0}{a_i} Z. \end{aligned}$$

We calculate as above that the shape operator B_ξ to V_i has the form

$$B_\xi = \begin{bmatrix} \frac{a_i a_0}{a_i - a_0} & 0 & 0 & \cdots & \\ \frac{-a_i a_0}{(a_i - a_0)^3} - \frac{a_i}{(a_i - a_0)^2} & \frac{a_i a_0}{a_i - a_0} & \frac{a_i}{a_i - a_0} + \frac{a_i a_0}{(a_i - a_0)^2} & 0 \cdots 0 & \\ \frac{-a_i a_0}{(a_i - a_0)^2} - \frac{a_i}{a_i - a_0} & 0 & \frac{a_i a_0}{a_i - a_0} & \ddots & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \frac{a_i a_p}{a_i - a_p} \end{bmatrix}$$

As above the trace of $B_\xi = 0$. □

COROLLARY 2.7. *If M^n is a Lorentzian isoparametric hypersurface in $\mathbf{L}^{n+1}$ whose principal curvatures are all real then M has at most one non-zero principal curvature.*

Proof. Suppose that A has more than one non-zero real eigenvalue. By looking at $\pm\xi$ we can assume some are positive. Choose the smallest positive eigenvalue. If it is a_i, $i \neq 0$, then

$$\sum_{\beta \neq i} \frac{\nu_\beta a_i a_\beta}{a_i - a_\beta}$$

has only non-positive summands, so that $a_i a_\beta = 0$ for all $\beta \neq i$. If the shape operator is diagonalizable, this finishes the proof. If not, there is the possibility that a_0 is the smallest positive eigenvalue. In this case we may also assume that all eigenvalues are positive or 0.

If all the non-zero eigenvalues are positive and a_0 is the smallest, let a_p be the largest.

$$\sum_{\beta=0}^{p-1} \frac{\nu_\beta a_p a_\beta}{a_p - a_\beta}$$

has only non-negative summands so $a_p a_\beta = 0$ for all $\beta \neq p$ and only a_0 is non-zero. □

THEOREM 2.8. *If the shape operator of a Lorentzian isoparametric hypersurface in $\mathbf{L}^{n+1}$ has minimal polynomial $[(x - a_0)^2 + b_0^2](x - a_1) \cdots (x - a_p)$, $b_0 \neq 0$, and the real eigenvalues have multiplicities $\nu_1, \ldots, \nu_p$ then for every i*

$$2a_i\left[\frac{(a_i - a_0)a_0 - b_0^2}{(a_i - a_0)^2 + b_0^2}\right] + \sum_{j \neq i} \frac{\nu_j a_i a_j}{a_i - a_j} = 0. \tag{2.5}$$

Proof. Choose an orthonormal basis of vector fields $\{C_1, C_2, E_{11}, \ldots, E_{p\nu_p}\}$ such that

$$AC_1 = a_0 C_1 - b_0 C_2,$$

$$AC_2 = b_0 C_1 + a_0 C_2,$$

$$AE_{jk} = a_j E_{jk}, \qquad 1 \leq j \leq p, 1 \leq k \leq \nu_j.$$

This can be done by complexifying the tangent bundle. Letting f_i be the focal map corresponding to a non-zero real eigenvalue we have

$$(f_i)_* C_1 = \frac{a_i - a_0}{a_i} C_1 + \frac{b_0}{a_i} C_2$$

$$(f_i)_* C_2 = \frac{-b_0}{a_i} C_1 + \frac{a_i - a_0}{a_i} C_2$$

$$(f_i)_* U = \frac{a_i - a_j}{a_i} U \quad \text{for } U \text{ in } T_j.$$

Thus $(f_i)_*$ is injective on $[T_i]^\perp$. It is easy to see that the shape operator B_ξ of V_i in the direction of ξ has

$$\operatorname{tr} B_\xi = 2a_i \left[\frac{(a_i - a_0)a_0 - b_0^2}{(a_i - a_0)^2 + b_0^2} \right] + \sum_{j \neq i} \frac{\nu_j a_i a_j}{a_i - a_j}.$$

As before (2.5) holds. □

Note that (2.5) holds with $a_0 = \frac{1}{2}$, $a_1 = 1$, $b_0 = \frac{1}{2}$ and $i = 1$. Thus A can have more than one non-zero principal curvature and satisfy (2.5). However, A can have at most one non-zero real principal curvature.

COROLLARY 2.9. *If M^n is a Lorentzian isoparametric hypersurface in $\mathbf{L}^{n+1}$ with complex principal curvatures $a_0 \pm ib_0$, $b_0 \neq 0$ then M^n has at most one non-zero real principal curvature.*

Proof. Assume that M^n has a non-zero real principal curvature and call a_1 the smallest positive one. If $a_1 > 0 \geq a_0$ or $a_0 \geq a_1 > 0$ then by (2.5)

$$2a_1 \left[\frac{(a_1 - a_0)a_0 - b_0^2}{(a_1 - a_0)^2 + b_0^2} \right] + \sum_{j \neq 1} \frac{\nu_j a_1 a_j}{a_1 - a_j} = 0$$

where the distinct real principal curvatures are given by $a_1, a_2, \dots, a_p$. The summands are all non-positive and so $a_1 a_j = 0$ for all $j \neq 1$. In addition

$$(a_1 - a_0)a_0 - b_0^2 = 0. \tag{2.6}$$

Suppose $a_1 > a_0 > 0$. If there were another non-zero real principal curvature there would be either a negative one with smallest absolute value, a_p, or all would be positive and some a_q would be largest. In the first case

$$a_1 > a_0 > 0 > a_p.$$

By considering $A_{-\xi}$ this has been done. In the second case

$$a_q > a_1 > a_0 > 0.$$

Summing over $j \neq 1$ and then over $j \neq q$ in (2.5) gives

$$(a_1 - a_0)a_0 - b_0^2 \geq 0$$

$$(a_q - a_0)a_0 - b_0^2 \leq 0$$

which is impossible. □

3. Examples. The main purpose of this paper is to describe locally the Lorentzian isoparametric hypersurfaces of $\mathbf{L}^{n+1}$. It will turn out that no such hypersurface has complex principal curvatures so only those with one or two real principal curvatures and at most one non-zero principal curvature need be classified.

If the shape operator of M^n is diagonalizable and M^n is complete then M^n is a Lorentzian hyperplane, sphere or one of two types of cylinders:

$$M = \{(x_0, x_1, \ldots, x_n) \colon -x_0^2 + x_1^2 + \cdots + x_k^2 = r^2\} \quad \text{or}$$

$$M = \{(x_0, x_1, \ldots, x_n) \colon x_k^2 + \cdots + x_n^2 = r^2\}, \qquad 1 \leq k < n.$$

If the shape operator is not diagonalizable then M will be called either a *generalized cylinder* or a *generalized umbilical hypersurface*. The following examples give the necessary local models.

A *null curve* $x(s)$ is a curve whose tangent vectors have length zero. A *frame* for a curve $x(s)$ in $\mathbf{L}^{n+1}$ is a set of vector-valued functions $E_1(s), \ldots, E_{n+1}(s)$ such that, for each s, $\{E_1(s), \ldots, E_{n+1}(s)\}$ is a basis of $\mathbf{L}^{n+1}$. If the basis is pseudo-orthonormal it is called a pseudo-orthonormal frame.

We assume $a > 0$ below. If $a < 0$ the examples can be easily modified by requiring that $f(s, \vec{0}) = x(s)$.

EXAMPLE 3.1. Generalized cylinder of type 1.

Take a null curve $x(s)$ in $\mathbf{L}^{r+\rho+3}$ with a pseudo-orthonormal frame $\{X(s), Y(s), C(s), W_1(s), \ldots, W_r(s), Z_1(s), \ldots, Z_\rho(s)\}$ such that

$$\dot{x}(s) = X(s),$$
$$\dot{C}(s) = -B(s)Y(s), \qquad B \neq 0,$$
$$\dot{Z}_\beta(s) \in \operatorname{span}\{Y(s), Z_1(s), \ldots, Z_\rho(s)\}, \qquad 1 \leq \beta \leq \rho.$$

The parametrized hypersurface in $\mathbf{L}^{r+\rho+3}$ defined, in a neighborhood of the origin, by

$$f(s, u, w_1, \ldots, w_r, z_1, \ldots, z_\rho) = x(s) + uY(s) + \sum_j w_j W_j(s) + \sum_\beta z_\beta Z_\beta(s)$$
$$+ \frac{1}{a}C(s) - \sqrt{\frac{1}{a^2} - \sum z_\beta^2}\, C(s)$$

is called a generalized cylinder of type 1.

$$\xi = a\left(\sqrt{\frac{1}{a^2} - \sum z_\beta^2}\right)C(s) - a\sum_\beta z_\beta Z_\beta(s).$$

The minimal polynomial of A_ξ is $x^2(x-a)$ if $\rho \neq 0$ and x^2 if $\rho = 0$.

EXAMPLE 3.2. Generalized cylinder of type 2.

Let $x(s)$ be a null curve in $\mathbf{L}^{r+\rho+3}$ with a pseudo-orthonormal frame $\{X(s), Y(s), C(s), Z_1(s), \dots, Z_r(s), W_1(s), \dots, W_\rho(s)\}$ such that

$$\begin{aligned}
&\dot{x}(s) = X(s),\\
&\dot{C}(s) = -aX(s) - B(s)Y(s), \qquad B \neq 0,\\
&\dot{W}_\beta(s) \in \operatorname{span}\{Y(s), W_1(s), \dots, W_\rho(s)\}.
\end{aligned}$$

The parametrized hypersurface in $\mathbf{L}^{r+\rho+3}$ defined by

$$\begin{aligned}
f(s, u, z_1, \dots, z_r, w_1, \dots, w_\rho) &= x(s) + uY(s) + \sum_j z_j Z_j(s) + \sum_\beta w_\beta W_\beta(s)\\
&\quad + \frac{1}{a}C(s) - \sqrt{\frac{1}{a^2} - \sum z_j^2}\, C(s)
\end{aligned}$$

is called a generalized cylinder of type 2 if $\rho \neq 0$.

$$\xi = -auY(s) + a\sqrt{\frac{1}{a^2} - \sum z_j^2}\, C(s) - a\sum z_j Z_j(s).$$

The minimal polynomial of A_ξ is $(x-a)^2 x$ if $\rho \neq 0$ and $(x-a)^2$ if $\rho = 0$. If $\rho = 0$ the hypersurface is called a generalized umbilical hypersurface.

If $\rho = 0$ then for each s we get a map $f_s(u, z_1, \dots, z_r) = f(s, u, z_1, \dots, z_r)$. The image of f_s is an $r+1$ dimensional submanifold of $\mathbf{R}^{r+3}$ which is contained in the $r+2$ dimensional sphere of radius $1/a$ centered at $x(s) + (1/a)C(s)$. $f_s(u, z_1, \dots, z_r) - x(s) - (1/a)C(s)$ is also perpendicular to $Y(s)$. Therefore $f(s, u, z_1, \dots, z_r)$ consists of those $r+1$ dimensional submanifolds of the appropriate $r+2$ dimensional spheres perpendicular to $Y(s)$ moving along $x(s)$.

EXAMPLE 3.3. Generalized cylinder of type 3.

Start with a null curve $x(s)$ in $\mathbf{L}^{r+\rho+4}$ and a pseudo-orthonormal frame $\{X(s), Y(s), Z(s), C(s), U_1(s), \dots, U_r(s), V_1(s), \dots, V_\rho(s)\}$ satisfying

$$\begin{aligned}
&\dot{x}(s) = X(s),\\
&\dot{C}(s) = B(s)Z(s), \qquad B(s) \neq 0,\\
&\dot{V}_\beta(s) \in \operatorname{span}\{Y(s), V_1(s), \dots, V_\rho(s)\}, \qquad 1 \leq \beta \leq \rho.
\end{aligned}$$

The parametrized hypersurface defined in a neighborhood of the origin by

$$\begin{aligned} f(s, u, z, u_1,\dots,u_r, v_1,\dots,v_\rho) &= x(s) + uY(s) + zZ(s) + \sum u_j U_j(s) \\ &\quad + \sum v_\beta V_\beta(s) + \frac{1}{a}C(s) - \sqrt{\frac{1}{a^2} - \sum v_\beta^2}\, C(s) \end{aligned}$$

is called a generalized cylinder of type 3. The minimal polynomial of the shape operator is $x^3(x-a)$ if $\rho \neq 0$ and x^3 if $\rho = 0$.

EXAMPLE 3.4. Generalized cylinder of type 4.

Take a null curve $x(s)$ in $\mathbf{L}^{r+\rho+4}$ with a pseudo-orthonormal frame $\{X(s), Y(s), Z(s), C(s), V_1(s),\dots, V_r(s), U_1(s),\dots, U_\rho(s)\}$ such that

$$\begin{aligned} &\dot{x}(s) = X(s), \\ &\dot{C}(s) = -aX(s) + B(s)Z(s), \\ &\dot{U}_\beta(s) \in \operatorname{span}\{Y(s), U_1(s),\dots, U_\rho(s)\}, \qquad 1 \le \beta \le \rho. \end{aligned}$$

The parametrized hypersurface given by

$$\begin{aligned} f(s, u, z, v_1,\dots,v_r, u_1,\dots,u_\rho) &= x(s) + uY(s) + zZ(s) + \sum v_j V_j(s) \\ &\quad + \sum u_\beta U_\beta(s) + \frac{1}{a}C(s) - \sqrt{\frac{1}{a^2} - \sum v_j^2 - z^2}\, C(s) \end{aligned}$$

is called a generalized cylinder of type 4, if $\rho \neq 0$. In this case the minimal polynomial of the shape operator is $(x-a)^3 x$. If $\rho = 0$ f is called a generalized umbilical hypersurface and the minimal polynomial is $(x-a)^3$.

In the appendix the existence of framed null curves with the appropriate derivatives is proved.

4. Hypersurfaces in $\mathbf{L}^{n+1}$ with at most one non-zero real principal curvature. In this section isoparametric hypersurfaces of $\mathbf{L}^{n+1}$ with at most one non-zero real principal curvature are shown to be generalized cylinders or generalized umbilical hypersurfaces. A few lemmas are needed to begin.

LEMMA 4.1. *If M^n is a Lorentzian isoparametric hypersurface in $\mathbf{L}^{n+1}$ then the kernel of the shape operator A is a totally geodesic distribution on M.*

Proof. Let W_1, W_2 be vector fields in $\ker A$, and V be any vector field on M.

$$(AV, W_1) = 0$$

so by (1.4)

$$\begin{aligned} 0 &= W_2(AV, W_1) = \big(\nabla_{W_2}(AV), W_1\big) + \big(AV, \nabla_{W_2}W_1\big) \\ &= \big(A(\nabla_{W_2}V), W_1\big) + (\nabla_V(AW_2), W_1) \\ &\quad - (A(\nabla_V W_2), W_1) + \big(AV, \nabla_{W_2}W_1\big) \\ &= \big(AV, \nabla_{W_2}W_1\big) \end{aligned}$$

so $A(\nabla_{W_2}W_1) \perp TM$. □

LEMMA 4.2. *If M^n is a manifold with a vector field X and two integrable distributions T_1 and T_2 satisfying*

(1) $X \oplus T_1 \oplus T_2 = TM$

(2) $\nabla_{T_i}T_j \subseteq T_j$, $i, j = 1, 2$, *then for every point $\gamma(0)$ in M there is a coordinate system $(s, v_1, \ldots, v_p, w_1, \ldots, w_q)$ with origin $\gamma(0)$ such that*

(i) $\{\partial/\partial v_1, \ldots, \partial/\partial v_p\}$ *forms a local basis of T_1*

(ii) $\{\partial/\partial w_1, \ldots, \partial/\partial w_q\}$ *forms a local basis of T_2*

(iii) $(s, 0, \ldots, 0)$ *is an integral curve of X.*

Proof. By hypothesis $T_1 \oplus T_2$ is integrable, so by the lemma in [5] vol. I, p. 182 there is a coordinate system $(s, y_1, \ldots, y_{n-1})$ with origin $\gamma(0)$ such that $s = c$ defines an integral manifold of $T_1 \oplus T_2$ while $y_j = c_j$, $1 \le j \le n-1$, defines an integral curve of X. Thus the curve $\gamma(s) = (s, 0, 0, \ldots, 0)$ is an integral curve for X. For each s let $N(s)$ denote the integral manifold of $T_1 \oplus T_2$ passing through $\gamma(s)$. $N(s)$ has two complementary integrable totally geodesic distributions T_1 and T_2. Again by the lemma in [5] there is a coordinate system $\{t_1, \ldots, t_p, u_1, \ldots, u_q\}$ on $N(s)$ with origin $\gamma(s)$ such that $t_j = c_j$ is an integral manifold of T_2 and $u_k = d_k$ is an integral manifold of T_1. As in [5] vol. I, p. 183 there is an open neighborhood $\mathcal{O}(s)$ of $\gamma(s)$ in $N(s)$ such that $\mathcal{O}(s) = \mathcal{O}_1(s) \times \mathcal{O}_2(s)$ where $\mathcal{O}_j(s)$ is open in $M_j(s)$ and $M_j(s)$ is integral to T_j. Now let $V_1(s), \ldots, V_p(s)$, $W_1(s), \ldots, W_q(s)$ be smooth vector fields along $\gamma(s)$ with $V_j(s)$ in T_1 and $W_j(s)$ in T_2. We have, possibly by making $\mathcal{O}_j(s)$ smaller,

$$\mathcal{O}_1(s) = \exp_{\gamma(s)}\left(\sum v_j V_j(s)\right)$$

$$\mathcal{O}_2(s) = \exp_{\gamma(s)}\left(\sum w_k W_k(s)\right)$$

so

$$\mathcal{O}(s) = \left(\exp_{\gamma(s)} \sum v_j V_j(s), \exp_{\gamma(s)} \sum w_k W_k(s)\right)$$

and $(s, v_1, \ldots, v_p, w_1, \ldots, w_q)$ is the desired coordinate system. □

If M^n is a Lorentzian manifold satisfying the hypotheses of (4.2) and $f\colon M^n \to \mathbf{L}^{n+1}$ is an isometric immersion then we can make the following definitions. For the coordinate system above set $x(s) = f(\gamma(s))$ and let

$$f_{1,s}\colon \mathcal{O}_1(s) \to \mathbf{L}^{n+1} \quad \text{be } \vec{v} \mapsto f(s, \vec{v}, 0) \quad \text{and}$$
$$f_{2,s}\colon \mathcal{O}_2(s) \to \mathbf{L}^{n+1} \quad \text{be } \vec{w} \mapsto f(s, 0, \vec{w}).$$

LEMMA 4.3. *Let M^n be a Lorentzian manifold satisfying the hypotheses of* (4.2). *If $f\colon M^n \to \mathbf{L}^{n+1}$ is an isometric immersion and $\alpha(T_1, T_2) = 0$ then f can be written locally as*

$$f(s, \vec{v}, \vec{w}) = -x(s) + f_{1,s}(\vec{v}) + f_{2,s}(\vec{w}).$$

Proof. Let $(s, v_1, \ldots, v_p, w_1, \ldots, w_q)$ be the coordinate system obtained above. For a fixed s we have $\mathcal{O}(s) = \mathcal{O}_1(s) \times \mathcal{O}_2(s)$. To employ the proof of "Moore's Lemma" in **[2]**, p. 386 we must show that

$$\nabla_{\partial/\partial v_j} \frac{\partial}{\partial w_k} = 0 = \nabla_{\partial/\partial w_k} \frac{\partial}{\partial v_j} \quad \text{for all } j, k.$$

Note that

$$0 = \left[\frac{\partial}{\partial v_j}, \frac{\partial}{\partial w_k}\right] = \nabla_{\partial/\partial v_j} \frac{\partial}{\partial w_k} - \nabla_{\partial/\partial w_k} \frac{\partial}{\partial v_j}$$

so

$$\nabla_{\partial/\partial v_j} \frac{\partial}{\partial w_k} = \nabla_{\partial/\partial w_k} \frac{\partial}{\partial v_j}.$$

But the left-hand side of this equation is in T_2 while the right-hand side is in T_1, so both are zero.

Therefore

$$f(s, \vec{v}, \vec{w}) - x(s) = f_{1,s}(\vec{v}) - x(s) + f_{2,s}(\vec{w}) - x(s)$$

and

$$f(s, \vec{v}, \vec{w}) = -x(s) + f_{1,s}(\vec{v}) + f_{2,s}(\vec{w}).$$ □

THEOREM 4.4. *If M^n is a Lorentzian hypersurface isometrically immersed in $\mathbf{L}^{n+1}$ whose shape operator has minimal polynomial $(x - a)x^2$,*

$a \neq 0$, then, in a neighborhood of any point, M^n is a generalized cylinder of type 1.

Proof. In a neighborhood of $\gamma(0)$ in M take a pseudo-orthonormal basis of vector fields $\{X, Y, W_1, \ldots, W_r, Z_1, \ldots, Z_\rho\}$ such that $AX = BY$, $B \neq 0$, $AY = 0 = AW_j$ and $AZ_\beta = aZ_\beta$ for $1 \leq j \leq r$ and $1 \leq \beta \leq \rho$. First note that Y can be assumed to be a geodesic vector field. To see this denote $\ker A$ by T_0 and $\ker(A - a)$ by T_a. If $U \in T_0$ then $[U, Y] \in T_0$ and so $(Y, [U, Y]) = 0 = (Y, \nabla_U Y) - (Y, \nabla_Y U) = (\nabla_Y Y, U)$ so that $\nabla_Y Y \perp T_0$. Therefore $\nabla_Y Y$ is in span$\{Y\}$ and is pregeodesic. Multiplication by a function makes it geodesic.

Next we show that the hypotheses of (4.2) and (4.3) hold. First define a new distribution $T_* = \operatorname{span}\{X\} \oplus T_0$. We recall the notation $\{X, Y\}Z$ which is explained after equation (1.4). From

$$\begin{array}{lllll} (1) & \{X, Y\}Z_\beta & & & ([X, Y], Z_\beta) = 0 \\ & & \text{we have} & & \\ (2) & \{X, W_j\}Z_\beta & & & ([X, W_j], Z_\beta) = 0 \end{array}$$

so that T_* is integrable.

Three instances of Codazzi's equation show that T_a is totally geodesic.

$$(3) \qquad \{Y, Z_\beta\}Z_\gamma, \quad (\nabla_{Z_\beta} Z_\gamma, Y) = 0,$$

$$(4) \qquad \{X, Z_\beta\}Z_\gamma, \quad a(\nabla_{Z_\beta} X, Z_\gamma) = B(\nabla_{Z_\beta} Z_\gamma, Y) = 0,$$

$$(5) \qquad \{W_i, Z_\beta\}Z_\gamma, \quad (\nabla_{Z_\beta} Z_\gamma, W_i) = 0.$$

If $(\nabla_{Z_\beta} W_i, Y) = 0$ then by

$$(6) \qquad \{X, Z_\beta\}W_i, \quad a(\nabla_X Z_\beta, W_i) = B(\nabla_{Z_\beta} Y, W_i),$$

and (2) $(\nabla_{W_i} X, Z_\beta) = 0$. Then (1), (3), (5) and

$$(7) \qquad \{X, Z_\beta\}Y, \quad (\nabla_X Y, Z_\beta) = 0,$$

would show that $\nabla_{T_a} T_0 \subseteq T_0$ and $\nabla_{T_0} T_a \subseteq T_a$.

To show that $(\nabla_{Z_\beta} W_i, Y) = 0$ set

$$\nabla_{Z_\beta} Y = \phi_\beta Y + \sum_{j=1}^{r} \phi_{\beta j} W_j$$

$$\nabla_{W_j} Z_\beta = \theta_{j\beta} Y + \sum_{\gamma=1}^{\rho} \theta_{j\beta}^{\gamma} Z_\gamma .$$

From $\{W_i, Z_\beta\} X$, $a\theta_{j\beta} = -B\phi_{\beta j}$. Because $AY = 0$, T_a is totally geodesic and $\nabla_Y T_a \subseteq T_a$

$$0 = \left(R(Y, Z_\beta)Z_\beta, X\right) = \left(\nabla_Y \nabla_{Z_\beta} Z_\beta - \nabla_{Z_\beta}\nabla_Y Z_\beta - \nabla_{\nabla_Y Z_\beta - \nabla_{Z_\beta} Y} Z_\beta, X\right)$$

$$= \left(\nabla_{\nabla_{Z_\beta} Y} Z_\beta, X\right) = \sum_{j=1}^{r} \phi_{\beta j}\left(\nabla_{W_j} Z_\beta, X\right) = \sum_{j=1}^{r} \phi_{\beta j}\left(\frac{B}{a}\right)\phi_{\beta j}.$$

Therefore $\phi_{\beta j} = 0 = \theta_{\beta j}$. With (2) we see that $(\nabla_X Z_\beta, W_j) = 0$ and by (7) $(\nabla_X Z_\beta, Y) = 0$.

We know then that the immersion f splits as

$$f(s, u, w_1, \ldots, w_r, z_1, \ldots, z_\rho) = -x(s) + f_{0,s}(u, \vec{w}) + f_{a,s}(\vec{z})$$

with $f_{0,s}$: $M_0(s) \to \mathbf{L}^{n+1}$ and $f_{0,a}$: $M_a(s) \to \mathbf{L}^{n+1}$. Here, of course, $M_0(s)$ is the leaf of T_0 through $\gamma(s)$ and $M_a(s)$ is the leaf of T_a through $\gamma(s)$.

Now restrict $f_*(X), f_*(Y), f_*(W_1), \ldots, f_*(Z_\rho)$ to $x(s) = f(\gamma(s))$ and denote the restrictions by $X(s), Y(s), W_1(s), \ldots, Z_\rho(s)$. Denote $\xi(x(s))$ by $C(s)$.

We'll see that $f_{0,s}(M_0(s))$ maps onto an open subset of the $r + 1$ dimensional plane spanned by $Y(s), W_1(s), \ldots, W_r(s)$. For each fixed s $M_0(s)$ is a totally geodesic submanifold of M, so that each geodesic in $M_0(s)$ is a geodesic in M. Furthermore $f(M_0(s))$ is a totally geodesic submanifold in $\mathbf{L}^{n+1}$. In fact if $w(t)$ is a geodesic in $M_0(s)$ then

$$D_t f_*(\dot{w}(t)) = f_*(\nabla_t \dot{w}(t)) + \alpha(\dot{w}(t), \dot{w}(t)) = 0,$$

and $f(w(t))$ is a geodesic in $\mathbf{L}^{n+1}$. Therefore $f(M_0(s))$ is an open subset of an $r + 1$ dimensional plane in $\mathbf{L}^{n+1}$ passing through $x(s)$ and can be written

$$f_{0,s}(u, w_1, \ldots, w_r) = x(s) + uY(s) + \sum w_j W_j(s).$$

$f_{a,s}(M_a(s))$ is an open subset of the ρ-dimensional sphere passing through $x(s)$ contained in the subspace perpendicular to $f_*(T_*(s))$ with center $x(s) + (1/a)C(s)$ and radius $1/a$. By equation (4) and $\nabla_{T_a} T_0 \subseteq T_0$ we see that $\nabla_{T_a} T_* \subseteq T_*$.

If $V(0)$ is in $T_*(s)$ and $z(t)$ is a curve in $M_a(s)$ passing through $\gamma(s)$ let $V(t)$ be the parallel translation of $V(0)$ along $z(t)$.

$$D_t f_*(V(t)) = \alpha(V(t), \dot{z}(t)) = 0$$

which shows that $f_*(V(t))$ is a constant vector in $\mathbf{L}^{n+1}$. Now

$$\frac{d}{dt}(f(z(t)) - f(\gamma(s)), f_*(V(0))) = (f_*(\dot{z}(t)), f_*(V(0)))$$
$$= (f_*(\dot{z}(t)), f_*(V(t))) = 0.$$

Therefore $f(z(t))$ is contained in $[f_*(T_*(s))]^\perp$. $f_{a,s}(M_a(s))$ is an umbilical immersion in this $\rho + 1$ dimensional space and so is an open subset of the sphere of radius $1/a$, with center $x(s) + (1/a)C(s)$. Therefore locally M is a generalized cylinder of type 1. □

If the minimal polynomial of A were x^2 in the hypothesis of Theorem 3.4 it is easy to see that M is a generalized cylinder of type 1 with $\rho = 0$. In **[2]** complete isometric immersions with this hypothesis are classified.

THEOREM 4.5. *If the shape operator of a Lorentzian hypersurface M^n in $\mathbf{L}^{n+1}$ has $(x - a)^2$, $a \neq 0$, as its minimal polynomial then, in a neighborhood of any point, M^n is a generalized umbilical hypersurface as in Example 3.2.*

Proof. We take a pseudo-orthonormal basis $\{X, Y, Z_1, \dots, Z_r\}$, $r = n - 2$, for TM in a neighborhood of $x(0)$ such that $AX = aX + BY$, $AY = aY$ and $AZ_j = aZ_j$, with $B \neq 0$.

Let T_a denote the integrable, degenerate distribution $\ker(A - a)$ on M. Treating M^n as an embedded hypersurface, let $x(s)$ be an integral curve of X and indicate $X(x(s))$ by $X(s)$, etc. If $C(s) = \xi(x(s))$ where ξ is the unit normal then

$$D_s C(s) = -aX(s) - B(s)Y(s).$$

We show that $M_a(s)$, the leaf of T_a through $x(s)$, is an $n - 1$ dimensional submanifold of the n-dimensional indefinite sphere centered at $x(s) + (1/a)C(s)$ with radius $1/a$. Fix s and let $x(s) + \beta(t)$ with $\beta(0) = \vec{0}$ be a curve in $M_a(s)$ so that $\beta'(t) \in T_a(x(s) + \beta(t))$.

$$D_t\left(x(s) + \beta(t) + \frac{1}{a}\xi(x(s) + \beta(t))\right) = 0$$

so $x(s) + \beta(t) + (1/a)\xi(x(s) + \beta(t))$ is a constant vector equal to its value at $t = 0$, $x(s) + (1/a)C(s)$. Therefore for each t

$$x(s) + \beta(t) + \frac{1}{a}\xi(x(s) + \beta(t)) = x(s) + \frac{1}{a}C(s)$$

giving

$$\beta(t) - \frac{1}{a}C(s) = -\frac{1}{a}\xi(x(s) + \beta(t)) \quad \text{and} \tag{1}$$

$$\left(\beta(t) - \frac{1}{a}C(s),\ \beta(t) - \frac{1}{a}C(s)\right) = \frac{1}{a^2}. \tag{2}$$

From (2) $M_a(s)$ is contained in the appropriate sphere.

Consider now Codazzi's equation with U in T_a and X.
$\nabla_U(AX) - A(\nabla_U X) = \nabla_X(AU) - A(\nabla_X U)$ gives

$$B(\nabla_U Y) = (A - a)\nabla_U X - (A - a)\nabla_X U - (UB)Y.$$

The image of $A - a$ is contained in span Y so $\nabla_U Y$ is in span Y for all U in T_a, i.e., Y is parallel along T_a. In addition

$$D_U Y = \nabla_U Y + (AY, U)\xi = \nabla_U Y$$

so Y is parallel along T_a in $\mathbf{L}^{n+1}$. From (1)

$$\left(Y(x(s) + \beta(t)), \beta(t) - \frac{1}{a}C(s)\right) = 0.$$

Because Y is parallel along T_a

$$\left(Y(s), \beta(t) - \frac{1}{a}C(s)\right) = 0$$

and M^n is a generalized umbilical hypersurface. □

THEOREM 4.6. *If M^n is a Lorentzian hypersurface isometrically immersed in $\mathbf{L}^{n+1}$ whose shape operator has minimal polynomial $(x - a)^2 x$, $a \neq 0$, then, in a neighborhood of any point, M^n is a generalized cylinder of type* 2.

Proof. Denote $\ker(A - a)$ by T_a, $\ker A$ by T_0 and $[T_0]^\perp$ by T_*. For any point $\gamma(0)$ in M choose a pseudo-orthonormal basis $\{X, Y, Z_1, \dots, Z_r, W_1, \dots, W_\rho\}$ of vector fields near $\gamma(0)$ such that $AX = aX + BY$, $B \neq 0$, $AY = aY$, $AZ_j = aZ_j$ and $AW_\beta = 0$. As in 4.4 $(\nabla_Y Y, U) = 0$ for U in T_a. Examining $\{Y, W_\beta\}Y$ gives $(\nabla_Y W_\beta, Y) = 0$, so we may assume that Y is a geodesic vector field.

Next we show that M^n satisfies the hypotheses of Lemmas 4.2 and 4.3.

T_0 and T_a are integrable. Using

(1) $\{X, Y\}W_\beta$, $\quad ([X, Y], W_\beta) = 0$,

(2) $\{Z_i, W_\beta\}Y$, $\quad (\nabla_{Z_i} W_\beta, Y) = 0$,

(3) $\{X, Z_i\}W_\beta$, $\quad ([X, Z_i], W_\beta) = B(\nabla_{Z_i} Y, W_\beta) = 0$,

we have the integrability of T_*.

To see that $\nabla_{T_a} T_0 \subseteq T_0$ note that $(\nabla_Y W_\beta, X) = 0$ by (1) and

(4) $\{X, W_\beta\}Y$, $\quad (\nabla_X W_\beta, Y) = 0$.

$(\nabla_Y W_\beta, Y) = 0$ because Y is geodesic.

$$(5) \qquad \{Y, W_\beta\} Z_i, \quad (\nabla_Y W_\beta, Z_i) = 0,$$

$$(6) \qquad \{Y, Z_i\} W_\beta, \quad (\nabla_{Z_i} W_\beta, Y) = 0,$$

$$(7) \qquad \{Z_i, W_\beta\} Z_j, \quad (\nabla_{Z_i} W_\beta, Z_j) = 0$$

would finish $\nabla_{T_a} T_0 \subseteq T_0$ if we knew $(\nabla_{Z_i} W_\beta, X) = 0$, which will be shown later.

To see that T_a is parallel along T_0 note that $(\nabla_{W_\beta} Y, W_\gamma) = 0 = (\nabla_{W_\beta} Z_i, W_\gamma)$ because T_0 is totally geodesic. We also have $(\nabla_{W_\beta} Y, Y) = 0$ so we need only $(\nabla_{W_\beta} Z_i, Y) = 0$. This can be done by expanding

$$0 = \big(R(Y, W_\beta) W_\beta, X\big) = \big(\nabla_Y \nabla_{W_\beta} W_\beta - \nabla_{W_\beta} \nabla_Y W_\beta - \nabla_{[Y, W_\beta]} W_\beta, X\big).$$

T_0 is totally geodesic and $\nabla_Y T_0 \subseteq T_0$ so this reduces to

$$0 = \big(\nabla_{\nabla_{W_\beta} Y} W_\beta, X\big).$$

Set $\nabla_{W_\beta} Y = bY + \sum_{j=1}^r b_j Z_j$. The equation becomes

$$0 = \sum b_j \big(\nabla_{Z_j} W_\beta, X\big).$$

By

$$(8) \qquad \{Z_j, W_\beta\} X, \quad a\big(\nabla_{Z_j} W_\beta, X\big) = B\big(\nabla_{W_\beta} Z_j, Y\big),$$

we have $(\nabla_{Z_j} W_\beta, X) = -\frac{B}{a}(\nabla_{W_\beta} Y, Z_j)$ so

$$0 = -\frac{B}{a} \sum_{j=1}^r b_j^2.$$

Therefore $b_j = 0 = (\nabla_{W_\beta} Z_j, Y)$ and $(\nabla_{Z_j} W_\beta, X) = 0$ which completes the proofs that $\nabla_{T_a} T_0 \subseteq T_0$ and $\nabla_{T_0} T_a \subseteq T_a$.

In order to prove that T_a is a totally geodesic foliation note that

$$\begin{aligned} &(\nabla_Y Z_i, Y) = 0, \quad (\nabla_Y Z_i, W_\beta) = 0 \quad \text{by (5)}, \\ &(\nabla_{Z_i} Y, Y) = 0, \quad (\nabla_{Z_i} Y, W_\beta) = 0 \quad \text{by (6)}, \\ &(\nabla_{Z_i} Z_j, W_\beta) = 0 \quad \text{by (7)} \end{aligned}$$

and we have

$$(9) \qquad \{X, Z_i\} Z_j, \quad (\nabla_{Z_i} Z_j, Y) = 0.$$

By Lemmas 4.2 and 4.3 we know that, locally, the immersion f splits,

$$f(s, u, z_1, \ldots, z_r, w_1, \ldots, w_\rho) = -x(s) + f_{0,s}(u, z_1, \ldots, z_r) + f_{a,s}(w_1, \ldots, w_\rho).$$

Restrict the vector fields $f_*(X), f_*(Y), \ldots, f_*(W_\rho)$ to $x(s)$ and denote the results by $X(s)$, $Y(s), \ldots, W_\rho(s)$. As before let $\xi(x(s)) = C(s)$. We can see that $D_s W_\beta(s) \in \operatorname{span}\{Y(s)\} \oplus T_0$ using (4), (3) and $(\nabla_{Z_i} W_\beta, X) = 0$.

As in the proof of (4.4) $f_{0,s}(M_0(s))$ is an open subset of the ρ-dimensional plane passing through $x(s)$ spanned by $W_1(s), \ldots, W_\rho(s)$.

Also as in (4.4) $f_{a,s}(M_a(s))$ is contained in the subspace of $\mathbf{L}^{n+1}$ through $x(s)$ perpendicular to $f_*(T_0(s))$. Following the proof of (4.5) we see that M^n is a generalized cylinder of type 2. □

The following theorems involve shape operators with minimal polynomials that have x^3 or $(x-a)^3$ as factors. As the polynomials become more complicated so do the proofs.

THEOREM 4.7. *If M^n is a Lorentzian hypersurface isometrically immersed in $\mathbf{L}^{n+1}$ whose shape operator has minimal polynomial $x^3(x-a)$, $a \neq 0$, then locally M^n is a generalized cylinder of type* 3.

Proof. Let $\{X, Y, Z, U_1, \ldots, U_r, V_1, \ldots, V_\rho\}$ be a pseudo-orthonormal basis of vector fields near $\gamma(0)$ satisfying $AX = -BZ$, $AY = 0 = AU_j$, $AZ = BY$ and $AV_\beta = aV_\beta$ where $B \neq 0$. We can assume Y is geodesic.

We use the following notation:

$$T_0 = \ker A$$

$$T_0^2 = \ker A^2$$

$$T_a = \ker(A - a) \quad \text{and}$$

$$T_* = [T_a]^\perp.$$

Next we show that the hypotheses of 4.2 and 4.3 hold. From

(1) $\{Y, V_\beta\}V_\gamma, \quad (\nabla_{V_\beta} V_\gamma, Y) = 0,$

(2) $\{Z, V_\beta\}V_\gamma, \quad B(\nabla_{V_\beta} Y, V_\gamma) - a(\nabla_{V_\beta} Z, V_\gamma) = 0,$

(3) $\{X, V_\beta\}V_\gamma, \quad B(\nabla_{V_\beta} Z, V_\gamma) + a(\nabla_{V_\beta} X, V_\gamma) = 0,$

(4) $\{U_j, V_\beta\}V_\gamma, \quad (\nabla_{V_\beta} V_\gamma, U_j) = 0,$

we see that T_a is totally geodesic.

In order to prove that T_0^2 is totally geodesic the covariant derivatives of any two vector fields in T_0^2 must be perpendicular to Y and V_β, $\beta = 1, 2, \ldots, \rho$. Some of the necessary equations come from the following instances of Codazzi's equation.

(5) $\{X, Z\}Y, \quad (\nabla_Z Z, Y) = 0,$

(6) $\{Y, Z\}V_\beta, \quad (\nabla_Y Z, V_\beta) = (\nabla_Z Y, V_\beta),$

(7) $\{Y, V_\beta\}Z, \quad (\nabla_Y V_\beta, Z) = 0,$

(8) $\{Z, U_j\}Z, \quad (\nabla_Z U_j, Y) = 0.$

Because T_0 is a totally geodesic we must only prove

$$(\nabla_Z Z, V_\beta) = 0, \quad (\nabla_Z U_j, V_\beta) = 0 \quad \text{and} \quad (\nabla_{U_j} Z, V_\beta) = 0.$$

This can be accomplished as follows. Consider M^n as the tube of radius $1/a$ over $f_a(M^n)$. If B_{V_β} is the shape operator in the direction of V_β of $f_a(M^n)$ and W is a tangent vector to $f_a(M^n)$ then

$$AW = (I - aB_{V_\beta})^{-1} B_{V_\beta} W$$

where A is the shape operator of M. (See [**1**] for this computation.) We know that $A^3 W = 0$ for all W and so $B^3_{V_\beta} W = 0$. For a fixed β we write $\nabla_- V_\beta$ restricted to T_* as a matrix with respect to the basis $\{X, Y, Z, U_1, \ldots, U_r\}$.

$$\nabla_- V_\beta = \left[\begin{array}{ccc|cccc} a^\beta_{11} & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a^\beta_{21} & a^\beta_{22} & a^\beta_{23} & c^\beta_{21} & \cdot & \cdot & \cdot & c^\beta_{2r} \\ a^\beta_{31} & 0 & a^\beta_{33} & c^\beta_{31} & \cdot & \cdot & \cdot & c^\beta_{3r} \\ \hline b^\beta_{11} & 0 & b^\beta_{13} & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & \mathbf{0} & & \\ \cdot & \cdot & \cdot & & & & & \\ b^\beta_{r1} & 0 & b^\beta_{r3} & & & & & \end{array}\right]$$

From

(8) $\{X, Y\}V_\beta, \quad a(\nabla_X Y, V_\beta) = B(\nabla_Y Z, V_\beta) + a(\nabla_Y X, V_\beta),$

(9) $\{X, Z\}V_\beta, \quad a(\nabla_X Z, V_\beta) - B(\nabla_X Y, V_\beta)$
$\qquad = a(\nabla_Z X, V_\beta) + B(\nabla_Z Z, V_\beta),$

(10) $\{X, U_j\}V_\beta, \quad a(\nabla_X U_j, V_\beta) = a(\nabla_{U_j} X, V_\beta) + B(\nabla_{U_j} Z, V_\beta),$

(11) $\{Z, U_j\}V_\beta, \quad (\nabla_Z U_j, V_\beta) = (\nabla_{U_j} Z, V_\beta),$

we get the following relations:

$$a^\beta_{11} = a^\beta_{22},$$
$$-aa^\beta_{31} - Ba^\beta_{11} = aa^\beta_{23} - Ba^\beta_{33},$$
$$-ab^\beta_{j1} = ac^\beta_{2j} - Bc^\beta_{3j},$$
$$b^\beta_{j3} = c^\beta_{j3}.$$

By substituting in and cubing the matrix, which must equal zero since $B^3_{V_\beta} = 0$, we see it has the form

$$\nabla_{-}V_\beta = \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a^\beta_{21} & 0 & a^\beta_{23} & -b^\beta_{11} & \cdot & \cdot & \cdot & -b^\beta_{r1} \\ -a^\beta_{23} & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \hline b^\beta_{11} & 0 & 0 & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & \mathbf{0} & & \\ \cdot & \cdot & \cdot & & & & & \\ b^\beta_{r1} & 0 & 0 & & & & & \end{array}\right]$$

Therefore T_0^2 is totally geodesic.

Combining what was just proved with the following two equations we see also that $\nabla_{T_a} T_0^2 \subseteq T_0^2$.

$$\{Z, V_\beta\}Z, \quad a(\nabla_Z V_\beta, Z) = -2B(\nabla_{V_\beta} Z, Y), \tag{12}$$

$$\{U_j, V_\beta\}Z, \quad a(\nabla_{U_j} V_\beta, Z) = -B(\nabla_{V_\beta} U_j, Y). \tag{13}$$

The integrability of T_* follows from (9), (10) and

$$\{X, Y\}V_\beta, \qquad a([X, Y], V_\beta) = B(\nabla_Y Z, V_\beta). \tag{14}$$

Using the notation above, i.e.,

$$a^\beta_{23} = -(\nabla_Z V_\beta, X) = -(\nabla_X V_\beta, Z)$$
$$b^\beta_{j1} = (\nabla_X V_\beta, U_j) = (\nabla_{U_j} V_\beta, X)$$

we must prove $a^\beta_{23} = 0 = b^\beta_{j1}$ in order to have $\nabla_{T_0^2} T_a \subseteq T_a$. This will be done by showing that a^β_{23} and $b^\beta_{j1}, j = 1, \ldots, r$, $\beta = 1, \ldots, \rho$ satisfy a certain over-determined system of partial differential equations.

$$\begin{aligned} Ya^\beta_{23} &= -Y(\nabla_X V_\beta, Z) = -(\nabla_Y \nabla_X V_\beta, Z) - (\nabla_X V_\beta, \nabla_Y Z) \\ &= -(\nabla_Y \nabla_X V_\beta, Z) = -(\nabla_X \nabla_Y V_\beta, Z) - (\nabla_{[Y, X]} V_\beta, Z) \end{aligned}$$

since

$$\begin{aligned} R(Y, X)V_\beta = 0 = &-\left(\nabla_X\left(\sum_\gamma (\nabla_Y V_\beta, V_\gamma)V_\gamma\right), Z\right) \\ &+([Y, X], Y)(\nabla_X V_\beta, Z) \\ = &-\sum_\gamma (\nabla_Y V_\beta, V_\gamma)(\nabla_X V_\gamma, Z) + ([Y, X], Y)(\nabla_X V_\beta, Z) \end{aligned}$$

because $[Y, X] \perp T_a$ and $(\nabla_W V_\beta, Z) = 0$ if W is in T_0^2. So

$$Ya_{23}^{\beta} = \sum_{\gamma} (\nabla_Y V_\beta, V_\gamma) a_{23}^{\gamma} - ([Y, X], Y) a_{23}^{\beta}.$$

Similarly

$$Za_{23}^{\beta} = -(\nabla_X Y, Z) a_{23}^{\beta} + \sum_{\gamma} (\nabla_Z V_\beta, V_\gamma) a_{23}^{\gamma} - ([Z, X], Y) a_{23}^{\beta} - \sum_{j} (\nabla_Z Z, U_j) b_{j1}^{\beta}$$

$$U_j a_{23}^{\beta} = (\nabla_X Y, Z) b_{j1}^{\beta} + \sum_{\gamma} (\nabla_{U_j} V_\beta, V_\gamma) a_{23}^{\gamma} - ([U_j, X], Y) a_{23}^{\beta}$$

$$Yb_{j1}^{\beta} = \sum_{\gamma} (\nabla_Y V_\beta, V_\gamma) b_{j1}^{\gamma} - ([Y, X], Y) b_{j1}^{\beta} + \sum_{k} (\nabla_Y U_j, U_k) b_{k1}^{\beta}$$

$$Zb_{j1}^{\beta} = (\nabla_X Y, U_j) a_{23}^{\beta} + \sum_{\gamma} (\nabla_Z V_\beta, V_\gamma) b_{j1}^{\gamma} - (\nabla_Z U_j, Z) a_{23}^{\beta} - ([Z, X], Y) b_{j1}^{\beta} + \sum_{k} (\nabla_Z U_j, U_k) b_{k1}^{\beta}$$

$$U_j b_{k1}^{\beta} = -(\nabla_X Y, U_k) b_{j1}^{\beta} + \sum_{\gamma} (\nabla_{U_j} V_\beta, V_\gamma) b_{k1}^{\gamma} - ([U_j, X], Y) b_{k1}^{\beta} + \sum_{i} (\nabla_{U_j} U_k, U_i) b_{i1}^{\beta}.$$

A leaf of T_0^2 is an $r+2$ dimensional euclidean space. These six sets of partial differential equations give an over-determined system in $\mathbf{R}^{r+2}$. A unique solution is determined once the values of a_{23}^{β}, b_{j1}^{β} are determined at one point. One possible solution is clearly $a_{23}^{\beta} \equiv 0 \equiv b_{j1}^{\beta}$. From the definitions of the functions we see that their value at $\gamma(0)$ is determined only by $X(\gamma(0)), Y(\gamma(0)), \ldots, V_\rho(\gamma(0))$ and not by any extensions of these vector fields. Consider then a normal coordinate system with $X(\gamma(0)), \ldots, V_\rho(\gamma(0))$ as initial conditions. For this coordinate system the Christoffel symbols are zero and so $a_{23}^{\beta}(\gamma(0)) = 0 = b_{j1}^{\beta}(\gamma(0))$, showing that the functions are identically zero. Therefore $\nabla_{T_0^2} T_a \subseteq T_a$. The theorem follows as in the proof of (4.4). □

THEOREM 4.8. *If the shape operator of a Lorentzian hypersurface M^n in $\mathbf{L}^{n+1}$ has $(x-a)^3$, $a \neq 0$, as its minimal polynomial, then, in a neighborhood of any point, M^n is a generalized umbilical hypersurface as in Example 3.4.*

Proof. Choose a pseudo-orthonormal basis $\{X, Y, Z, V_1, \ldots, V_r\}$, $r = n-3$, in a neighborhood of $x(0)$ such that $AX = aX - BZ$, $B \neq 0$,

$AY = aY$, $AZ = BY + aZ$ and $AV_j = aV_j$. Let $T_a = \ker(A - a)$ and $T_a^2 = \ker(A - a)^2$.

If U is in T_a^2, $AU = aU + B(U, Z)Y$ so from

$$\text{(1)} \qquad \{X, U\}Y, \quad (\nabla_U Z, Y) = 0 \quad \text{and}$$

$$\text{(2)} \qquad \{Z, U_1\}U_2, \quad (\nabla_{U_1} Y, U_2) = (U_1, Z)(\nabla_Z Y, U_2) + (U_2, Z)(\nabla_Z Y, U_1),$$

where U_1, U_2 are in T_a^2 we get $\nabla_Y Y$ is in span Y and T_a^2 is a totally geodesic distribution. We assume that Y is a geodesic vector field.

Assuming M^n is embedded in $\mathbf{L}^{n+1}$ let $x(s)$ be an integral curve of X through $x(0)$. For a fixed s let $M_a(s)$ be the leaf of T_a^2 through $x(s)$. We will show that $M_a(s)$ is contained in the sphere of radius $1/a$ centered at $x(s) + (1/a)\xi(x(s)) =: x(s) + (1/a)C(s)$. To do this a function $k(x)$ near $x(s)$ is constructed which satisfies

$$\text{(3)} \qquad \begin{aligned} &(Yk) = 0 \\ &(Zk)Y - \frac{B}{a}Y + k\nabla_Z Y = 0 \\ &(V_j k)Y + k(\nabla_{V_j} Y) = 0 \\ &k(x(s)) = 0. \end{aligned}$$

It is possible to find such a function because $\nabla_Z Y$ and $\nabla_{V_j} Y$ are in span Y.

Given such a $k(x)$

$$D_{T_a^2}\left(x + \frac{1}{a}\xi(x) + k(x)Y(x)\right) = 0.$$

Therefore, if $x(s) + \beta(t)$ is a curve in $M_a(s)$ with $\beta(0) = \vec{0}$

$$x(s) + \beta(t) + \frac{1}{a}\xi(x(s) + \beta(t)) + k(x(s) + \beta(t))Y(x(s) + \beta(t))$$

is a constant vector equal to $x(s) + (1/a)C(s)$. This yields

$$\text{(4)} \qquad \beta(t) - \frac{1}{a}C(s) = -\frac{1}{a}\xi(x(s) + \beta(t)) - k(x(s) + \beta(t))Y(x(s) + \beta(t))$$

$$\text{(5)} \qquad \left(\beta(t) - \frac{1}{a}C(s),\ \beta(t) - \frac{1}{a}C(s)\right) = \frac{1}{a^2}.$$

As in the proof of (4.5)

$$\text{(6)} \qquad \left(Y(s),\ \beta(t) - \frac{1}{a}C(s)\right) = 0$$

and M^n is a generalized umbilical hypersurface.

To construct $k(x)$ let

$$L(x) = \left(X(s), \frac{1}{a}\xi(x) + x - \frac{1}{a}\xi(x(s)) - x(s)\right).$$

$L(x(s)) = 0$ and $WL = (X(s), W - (1/a)AW)$, so that

$$YL = 0 \tag{7}$$

$$V_jL = 0 \tag{8}$$

$$ZL = -\frac{B}{a}(X(s), Y). \tag{9}$$

We also define $g(x) = -(X(s), Y(x))$, so $g(x(s)) = 1$.

$$Yg = 0 \tag{10}$$

$$Zg = -(\nabla_Z Y, X)g \tag{11}$$

$$V_j g = -\left(\nabla_{V_j} Y, X\right)g. \tag{12}$$

For example,

$$Zg = -(X(s), \nabla_Z Y) = +(X(s), (\nabla_Z Y, X)Y) = (\nabla_Z Y, X)(X(s), Y).$$

Finally, set $k = L/g$. □

THEOREM 4.9. *If M^n is a Lorentzian hypersurface isometrically immersed in $\mathbf{L}^{n+1}$ whose shape operator has $(x - a)^3x$, $a \neq 0$, as its minimal polynomial then, locally, M^n is a generalized cylinder of type* 4.

Proof. Choose a pseudo-orthonormal basis $\{X, Y, Z, V_1, \ldots, V_r, U_1, \ldots, U_\rho\}$ such that $AX = aX - BZ$, $B \neq 0$, $AY = aY$, $AZ = BY + aZ$, $AV_i = aV_i$ and $AU_\beta = 0$. Let $T_0 = \ker A$, $\ker(A - a) = T_a$, $\ker(A - a)^2 = T_a^2$ and $T_* = T_0^\perp$.

Using

$$\{X, Y\}Y, \quad (\nabla_Y Z, Y) = 0 \tag{1}$$

$$\{Y, Z\}V_j, \quad \left(\nabla_Y Y, V_j\right) = 0 \tag{2}$$

$$\{Y, U_\beta\}Y, \quad \left(\nabla_Y U_\beta, Y\right) = 0 \tag{3}$$

we can assume that Y is a geodesic vector field. Next we show that M^n decomposes. We first show that certain covariant derivatives are zero using (1.3) and (1.4).

By (3)

$$\{Y, U_\beta\}Z, \qquad \left(\nabla_Y U_\beta, Z\right) = 0 \tag{4}$$

and

$$\{Y, U_\beta\}V_j, \qquad \left(\nabla_Y U_\beta, V_j\right) = 0. \tag{5}$$

$\nabla_Y U_\beta$ is in span$\{Y\} \oplus T_0$. T_0 is of course totally geodesic and so $\nabla_{U_\beta} Y$ is in T_a^2. Therefore we can write

$$(6) \qquad \nabla_Y U_\beta = -(\nabla_Y U_\beta, X)Y + \sum_\gamma (\nabla_Y U_\beta, U_\gamma)U_\gamma$$

$$(7) \qquad \nabla_{U_\beta} Y = -(\nabla_{U_\beta} Y, X)Y + (\nabla_{U_\beta} Y, Z)Z + \sum_j (\nabla_{U_\beta} Y, V_j)V_j.$$

Expand

$$0 = (R(Y, U_\beta)U_\beta, Z)$$

$$0 = (\nabla_{U_\beta} Y, Z)[(\nabla_Z U_\beta, Z) + (\nabla_Y U_\beta, X)] + \sum_j (\nabla_{U_\beta} Y, V_j)(\nabla_{V_j} U_\beta, Z).$$

Given

$$(8) \qquad \{Y, U_\beta\}X, \qquad a(\nabla_Y U_\beta, X) = -B(\nabla_{U_\beta} Y, Z)$$

and

$$(9) \qquad \{Z, U_\beta\}Z, \qquad a(\nabla_Z U_\beta, Z) = 2B(\nabla_{U_\beta} Z, Y)$$

the equation becomes

$$0 = \frac{-3B}{a}(\nabla_{U_\beta} Y, Z)^2 + \sum_j (\nabla_{U_\beta} Y, V_j)(\nabla_{V_j} U_\beta, Z).$$

Using (5) and

$$(10) \qquad \{Y, V_j\}U_\beta, \qquad (\nabla_Y V_j, U_\beta) = (\nabla_{V_j} Y, U_\beta)$$

we have $(\nabla_{V_j} Y, U_\beta) = 0$. Combining this with

$$(11) \qquad \{Z, V_j\}U_\beta, \qquad a(\nabla_Z V_j, U_\beta) = B(\nabla_{V_j} Y, U_\beta) + a(\nabla_{V_j} Z, U_\beta)$$

and

$$(12) \qquad \{Z, U_\beta\}V_j, \qquad a(\nabla_Z U_\beta, V_j) = -B(\nabla_{U_\beta} Y, V_j)$$

the equation is

$$0 = \frac{-3B}{a}(\nabla_{U_\beta} Y, Z)^2 - \frac{B}{a}\sum_j (\nabla_{U_\beta} Y, V_j)^2$$

so

$$(13) \qquad (\nabla_{U_\beta} Y, Z) = 0 = (\nabla_{U_\beta} Y, V_j) = (\nabla_Z V_j, U_\beta)$$
$$= (\nabla_{V_j} Z, U_\beta) = (\nabla_Z U_\beta, Z) = (\nabla_Y U_\beta, X).$$

In order to see that T_a^2 is totally geodesic we need several more instances of Codazzi's equation, as well as (4), (5), and (13).

(14) $\{Y, Z\}U_\beta, \quad ([Y, Z], U_\beta) = 0.$

(15) $\{X, Z\}Y, \quad (\nabla_Z Z, Y) = 0.$

(16) $\{X, V_j\}Y, \quad (\nabla_{V_j} Z, Y) = 0.$

(17) $\{Z, V_j\}Z, \quad (\nabla_Z V_j, Y) = 2(\nabla_{V_j} Z, Y).$

(18) $\{Z, U_\beta\}V_j, \quad a(\nabla_Z U_\beta, V_j) = -B(\nabla_{U_\beta} Y, V_j).$

(19) $\{V_j, U_\beta\}Y, \quad (\nabla_{V_j} U_\beta, Y) = 0.$

(20) $\{Z, V_j\}V_k, \quad (\nabla_{V_j} Y, V_k) = 0.$

(21) $\{V_j, U_\beta\}V_k, \quad (\nabla_{V_j} U_\beta, V_k) = 0.$

Using (13) we have $\nabla_{T_0} T_a^2 \subseteq T_a^2$ and $\nabla_{T_0} T_* \subseteq T_*$ because T_0 is totally geodesic.

From equations (4), (5), (13), (14), (19) and (21) we can almost conclude that $\nabla_{T_a^2} T_0 \subseteq T_0$; the only additional information needed is that

$$(\nabla_Z U_\beta, X) = 0 = (\nabla_{V_j} U_\beta, X).$$

From (13) we have $(\nabla_Y X, U_\beta) = 0$. In conjunction with

(22) $\{X, Y\}U_\beta, \quad (\nabla_X Y, U_\beta) = (\nabla_Y X, U_\beta)$

and

(23) $\{X, Z\}U_\beta, \quad B(\nabla_X Y, U_\beta) + a(\nabla_X Z, U_\beta) = a(\nabla_Z X, U_\beta)$

this gives

$$(\nabla_Z U_\beta, X) = (\nabla_X U_\beta, Z).$$

We also have from $\{X, V_j\}U_\beta$ that

$$(\nabla_{V_j} U_\beta, X) = (\nabla_X U_\beta, V_j).$$

Set

$$(\nabla_X U_\beta, Z) = a_\beta \quad \text{and} \quad (\nabla_X U_\beta, V_j) = b_{\beta j}.$$

We will show that a_β and $b_{\beta j}$ are solutions to an over-determined system of partial differential equations on T_0, and are identically zero as in Theorem 4.7.

(24) $U_\gamma a_\beta = \sum_\delta (\nabla_{U_\gamma} U_\beta, U_\delta) a_\delta - ([U_\gamma, X], Y) a_\beta + \sum_j (\nabla_{U_\gamma} Z, V_j) b_{\beta j}.$

(25)
$$U_\gamma b_{\beta j} = \sum_\delta \left(\nabla_{U_\gamma} U_\beta, U_\delta\right) b_{\delta j} - \left(\left[U_\gamma, X\right], Y\right) b_{\beta j} + \left(\nabla_{U_\gamma} V_j, Z\right) a_\beta + \sum_k \left(\nabla_{U_\gamma} V_j, V_k\right) b_{\beta k}.$$

Using the techniques of the previous theorems M^n is a generalized cylinder of type 4. □

THEOREM 4.10. *If M^n is a Lorentzian isoparametric hypersurface isometrically immersed in $\mathbf{L}^{n+1}$ then its shape operator cannot have a complex eigenvalue.*

Proof. If such a hypersurface existed its shape operator would have one of four possible minimal polynomials:

$$\left((x-a)^2+b^2\right), \left((x-a)^2+b^2\right)x, \left((x-a)^2+b^2\right)(x-c)$$

or

$$\left((x-a)^2+b^2\right)(x)(x-c), bc \neq 0.$$

The first would be attached to a surface in $\mathbf{L}^3$. Using the techniques of **[6]** it is easy to see that such a surface cannot exist.

For the remaining three cases choose an orthonormal basis $\{C_1, C_2, Z_1, \ldots, Z_r, W_1, \ldots, W_\rho\}$, where ρ or r may be zero, satisfying $AC_1 = aC_1 - bC_2$, $AC_2 = bC_1 + aC_2$, $AZ_j = cZ_j$, and $AW_\beta = 0$. The different minimal polynomials correspond, in order, to $r = 0$, $\rho = 0$, and $r\rho \neq 0$. In addition, if $r \neq 0$ then $c = (a^2+b^2)/a$.

To simplify the calculations which follow, note that $T_0 = \ker A$ is totally geodesic and that $T_c = \ker(A - c)$ is integrable. In addition we have:

(1) $$\{C_1, C_2\} C_1, \quad \left(\nabla_{C_2} C_2, C_1\right) = 0$$

(2) $$\{C_1, C_2\} C_2, \quad \left(\nabla_{C_1} C_1, C_2\right) = 0.$$

Using

(3) $$\{C_1, Z_j\} Z_k, \qquad (c-a)\left(\nabla_{Z_j} C_1, Z_k\right) + b\left(\nabla_{Z_j} C_2, Z_k\right) = 0$$

and

(4) $$\{C_2, Z_j\} Z_k, \qquad -b\left(\nabla_{Z_j} C_1, Z_k\right) + (c-a)\left(\nabla_{Z_j} C_2, Z_k\right) = 0$$

we see that, because $(c-a)^2 + b^2 \neq 0$, $\left(\nabla_{Z_j} Z_k, C_1\right) = 0 = \left(\nabla_{Z_j} Z_k, C_2\right)$. With

(5) $$\{Z_j, W_\beta\} Z_k, \qquad \left(\nabla_{Z_j} Z_k, W_\beta\right) = 0$$

this shows that T_c is totally geodesic.

Let us introduce the following notation

$$\begin{aligned} \left(\nabla_{C_1}C_1, W_\beta\right) &= a_\beta, & \left(\nabla_{C_1}C_1, Z_j\right) &= \alpha_j, \\ \left(\nabla_{C_1}C_2, W_\beta\right) &= b_\beta, & \left(\nabla_{C_1}C_2, Z_j\right) &= \beta_j, \\ \left(\nabla_{C_2}C_1, W_\beta\right) &= c_\beta, & \left(\nabla_{C_2}C_1, Z_j\right) &= \gamma_j, \\ \left(\nabla_{C_2}C_2, W_\beta\right) &= d_\beta, & \left(\nabla_{C_2}C_2, Z_j\right) &= \delta_j, \\ \left(\nabla_{W_\beta}C_1, C_2\right) &= e_\beta, & \left(\nabla_{Z_j}C_1, C_2\right) &= \varepsilon_j. \end{aligned}$$

Using $\{C_1, W_\beta\}C_1$, $\{C_1, W_\beta\}C_2$, $\{C_2, W_\beta\}C_1$ and $\{C_2, W_\beta\}C_2$ we have

$$\text{(6)} \qquad \begin{aligned} a_\beta = d_\beta &= \frac{2ab}{a^2+b^2}e_\beta \\ -b_\beta = c_\beta &= \frac{2b^2}{a^2+b^2}e_\beta. \end{aligned}$$

From $\{C_1, Z_j\}C_1$, $\{C_1, Z_j\}C_2$, $\{C_2, Z_j\}C_1$ and $\{C_2, Z_j\}C_2$ follows

$$\text{(7)} \qquad \begin{aligned} \alpha_j = \delta_j &= \frac{-2ab}{a^2+b^2}\varepsilon_j \\ -\beta_j = \gamma_j &= \frac{2a^2}{a^2+b^2}\varepsilon_j. \end{aligned}$$

Defining

$$\begin{aligned} \left(\nabla_{W_\beta}Z_j, C_1\right) &= a_{\beta j}, & \left(\nabla_{C_2}Z_j, W_\beta\right) &= d_{j\beta}, \\ \left(\nabla_{W_\beta}Z_j, C_2\right) &= b_{\beta j}, & \left(\nabla_{Z_j}W_\beta, C_1\right) &= e_{j\beta}, \\ \left(\nabla_{C_1}Z_j, W_\beta\right) &= c_{j\beta}, & \left(\nabla_{Z_j}W_\beta, C_2\right) &= f_{j\beta}, \end{aligned}$$

and using $\{Z_j, W_\beta\}C_1$, $\{Z_j, W_\beta\}C_2$, $\{C_1, Z_j\}W_\beta$, $\{C_1, W_\beta\}Z_j$, $\{C_2, W_\beta\}Z_j$ and $\{C_2, Z_j\}W_\beta$ we find

$$\text{(8)} \qquad \begin{aligned} e_{j\beta} &= \frac{-b}{a}b_{\beta j}, \qquad f_{j\beta} = \frac{b}{a}a_{\beta j}, \\ c_{j\beta} &= \frac{c-a}{c}a_{\beta j} + \frac{b}{c}b_{\beta j}, \\ d_{j\beta} &= \frac{c-a}{c}b_{\beta j} - \frac{b}{c}a_{\beta j}. \end{aligned}$$

First assume that $r = 0$. We then have $0 = (R(C_1, W_\beta)C_1 - R(C_2, W_\beta)C_2, W_\beta) = -2a_\beta^2 - 2b_\beta^2$, so that $a_\beta = 0 = b_\beta$. In this case the immersion would split into $f_1 \times f_2 = M^2 \times \mathbf{R}^{n-2} \to \mathbf{L}^3 \times \mathbf{R}^{n-2}$ and the principal curvatures of f_1 would be complex, a contradiction.

Next let $\rho = 0$. Expanding

$$2ac = 2(a^2 + b^2) = \left(R(C_1, Z_j)C_1 - R(C_2, Z_j)C_2, Z_j\right)$$

we obtain $2(a^2 + b^2) = -2\alpha_j^2 - 2\beta_j^2$, which is impossible since $b \neq 0$.

Finally assume $\rho r \neq 0$.

$$\left(R(C_1, Z_j)C_1 - R(C_2, Z_j)C_2, Z_j\right) = 2(a^2 + b^2)$$

gives

$$(9) \quad 2(a^2 + b^2) = -2\alpha_j^2 - 2\beta_j^2 - 2\left[\frac{b(c-a)}{ac} - \frac{b}{c} - \frac{b}{a}\right]\sum_\beta a_{\beta j} b_{\beta j} + \left[\frac{c-a}{c} + \frac{b^2}{ac}\right]\left[\sum_\beta \left(a_{\beta j}^2 - b_{\beta j}^2\right)\right].$$

$(R(Z_j, W_\beta)Z_j, W_\beta) = 0$ yields

$$(10) \qquad \left(\frac{2b}{a} + \frac{2b^3}{a(a^2 + b^2)} + \frac{2ab}{(a^2 + b^2)}\right) a_{\beta j} b_{\beta j} = 0.$$

This means that $a_{\beta j} b_{\beta j} = 0$. If $a_{\beta j} = 0$ then (8) shows that $b = 0$, which is impossible. We assume then that $b_{\beta j} = 0$. Under this assumption $(R(Z_j, W_\beta)Z_j, C_2) = 0$ is

$$(11) \qquad -a_{\beta j}\varepsilon_j + f_{j\beta}\alpha_j + a_{\beta j}\beta_j = 0.$$

This implies $a_{\beta j}\varepsilon_j = 0$, so assume $\varepsilon_j = 0$. $(R(W_\beta, Z_j)C_1, W_\beta) = 0$ implies that $e_\beta a_{\beta j} = 0$. This means $e_\beta = 0$. Recalculating

$$\left(R(C_1, W_\beta)C_1, W_\beta\right) = 0$$

under the assumptions $\varepsilon_j = 0 = e_\beta = b_{\beta j}$ we see that $a_{\beta j} = 0$ and no such hypersurfaces exist. □

5. Appendix. In order to guarantee the existence of the examples in this paper, we need to find null curves with pseudo-orthonormal frames having prescribed derivatives. This can be done as soon as certain necessary conditions hold.

If $x(s)$ is a null curve in $\mathbf{L}^{k+2}$ with a pseudo-orthonormal frame $\{A(s), B(s), C_1(s), \ldots, C_k(s)\}$ and $\dot{x}(s) = A(s)$ then the fixed inner products give

$$(1) \qquad \begin{aligned} &(A(s), \dot{A}(s)) = 0 \\ &(B(s), \dot{B}(s)) = 0 \\ &(A(s), \dot{B}(s)) + (\dot{A}(s), B(s)) = 0 \\ &(A(s), \dot{C}_i(s)) + (\dot{A}(s), C_i(s)) = 0 \\ &(B(s), \dot{C}_i(s)) + (\dot{B}(s), C_i(s)) = 0 \\ &(C_i(s), \dot{C}_j(s)) + (\dot{C}_i(s), C_j(s)) = 0 \end{aligned}$$

for $1 \leq i, j \leq k$.

Therefore

$$(2)\quad \begin{aligned} \dot{A}(s) &= a(s)A(s) + 0 + b_{10}(s)C_1(s) + \cdots + b_{k0}(s)C_k(s) \\ \dot{B}(s) &= -a(s)B(s) + b_{11}(s)C_1(s) + \cdots + b_{k1}(s)C_k(s) \\ \dot{C}_j(s) &= b_{j1}(s)A(s) + b_{j0}(s)B(s) + \sum_i d_{ij}(s)C_i(s) \end{aligned}$$

where $[d_{ij}(s)]$ is a skew-symmetric $k \times k$ matrix.

Now let

$$M(s) = \left[\begin{array}{cc|ccc} a(s) & 0 & b_{11}(s) & \cdots & b_{k1}(s) \\ 0 & -a(s) & b_{10}(s) & \cdots & b_{k0}(s) \\ \hline b_{10}(s) & b_{11}(s) & & & \\ \vdots & \vdots & & d_{ij}(s) & \\ b_{k0}(s) & b_{k1}(s) & & & \end{array}\right]$$

where the entries are smooth functions of (s) and $[d_{ij}]$ is skew.

THEOREM 5.1. *Let $M(s)$ be the matrix above. There is a null curve $x(s)$ in $\mathbf{L}^{k+2}$ with a pseudo-orthonormal frame field $\{A(s), B(s), C_1(s), \ldots, C_k(s)\}$ such that*

$$\dot{x}(s) = A(s) \quad \textit{and} \quad (2)\ \textit{holds}.$$

Proof. Following [4] p. 14–15 we see that there is a $k+2 \times k+2$ matrix $X(s)$ which solves

$$(3)\qquad \dot{X}(s) = X(s)M(s), \quad X(0) = \mathrm{Id}.$$

Set

$$T = \begin{bmatrix} 0 & -1 & \mathbf{0} \\ -1 & 0 & \\ \mathbf{0} & & I_k \end{bmatrix}.$$

$M(s)$ satisfies $M(s)T + T({}^tM(s)) = 0$, where ${}^tM(s)$ is the transpose of $M(s)$. The solution $X(s)$ satisfies $X(s)T\,{}^tX(s) = T$. In fact

$$(X(s)\dot{T}\,{}^tX(s)) = \dot{X}(s)T\,{}^tX(s) + X(s)T({}^t\dot{X}(s))$$

using (3)

$$\begin{aligned} &= X(s)M(s)T\,{}^tX(s) + X(s)T\,{}^tM(s)\,{}^tX(s) \\ &= X(s)[M(s)T + T\,{}^tM(s)]\,{}^tX(s) = 0 \end{aligned}$$

and at $s = 0$, $X(0)T^tX(0) = T$. Therefore the columns of $X(s) =:$ $[A(s), B(s), C_1(s), \ldots, C_k(s)]$ form a pseudo-orthonormal basis of $\mathbf{L}^{k+2}$ with metric given by T. Let $x(s) = \int_0^s A(t)\,dt$. □

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