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RIGID SETS IN E^n

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We construct rigid embeddings of Cantor sets in E^n ($n \geq 3$) and rigid embeddings of compacta in E^n ($n > 3$). In each case there are uncountably many different rigid embeddings. The results in this paper generalize three-dimensional theorems by Sher, Shilepsky, Bothe, Martin, and Brechner and Mayer.

1. Introduction. Let X be a subset of Euclidean n-space E^n. We say that X is rigid in E^n if the only homeomorphism of X onto itself that is extendable to a homeomorphism of E^n onto itself is the identity homeomorphism. J. Martin has constructed a rigid 2-sphere in E^3 [**M**]. H. G. Bothe has constructed a rigid simple closed curve in E^3 [**Bo**]. Arnold Shilepsky constructed many different rigid Cantor sets in E^3 [**Shil**] by using a result of R. B. Sher on Cantor sets in E^3 [**Sher**]. More recently, Beverly Brechner and John C. Mayer have used Sher's result to construct uncountably many inequivalent embeddings of certain planar continua in E^3 [**B-M**].

In this paper all of these results are generalized to E^n ($n \geq 4$).

The rigid Cantor sets in this paper are in stark contrast to the strongly homogeneous but wildly embedded Cantor sets constructed by R. J. Daverman in E^n ($n \geq 5$) [$\mathbf{D_1}$].

2. Definitions and notations. We use S^n, B^n, and E^n to denote the n-sphere, the n-ball, and Euclidean n-space, respectively. If M is a manifold, we let $\mathrm{Bd}(M)$ and $\mathrm{Int}(M)$ denote the boundary and interior of M, respectively. A *disk with holes* is a compact connected 2-manifold that embeds in E^2. By *map* we will always mean a continuous function. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be a collection of oriented loops in a space X. We shall say Γ *bounds a manifold* if there is a compact oriented 2-manifold M and map $f\colon M \to X$ so that $f|\mathrm{Bd}\, M$ represents the collection Γ. We shall say that Γ is homologically trivial or nontrivial if the element represented by Γ in $H_1(X)$ is, respectively, trivial or nontrivial where the first homology group $H_1(X)$ is computed with integer coefficients. A solid n-torus is a space homeomorphic with $B^2 \times S^1_1 \times S^1_2 \times \cdots \times S^1_{n-2}$ where each S^1_i is a 1-sphere. A solid 3-torus will be called simply a *solid torus*. Let H be a disk with holes and $f\colon H \to M$ a map into a manifold M so that $f(\mathrm{Bd}(H)) \subset \mathrm{Bd}(M)$. Following Daverman [$\mathbf{D}_2$] we call the map f *I-inessential* (interior inessential) if there is a map $\tilde{f}$ from H into $\mathrm{Bd}(M)$ with $f|\mathrm{Bd}(H) = \tilde{f}|\mathrm{Bd}(H)$; otherwise, f is said to be *I-essential.*

Unless otherwise specified *linking* will refer to linking in the sense of homotopy and not homology. If $A \subset E^n$ and γ is a loop or simple closed curve in $E^n - A$, we say that γ links A if γ is not null homotopic in $E^n - A$. If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ is a collection of oriented loops in $E^n - A$ we say that Γ links A in case Γ does not bound a disk with holes; i.e., there is no map f of a disk with holes H into $E^n - A$ so that under some orientation of H, $f|\mathrm{Bd}(H)$ represents the collection Γ.

Let X be a metric space. If $a \in X$ and $\delta > 0$, we let $N(a, \delta)$ denote the set of points in X whose distance from a is less than δ. If $A \subset X$, we let $\mathrm{Cl}(A)$ denote the closure of A in X. Let A be a subset of a metric space X and $a \in \mathrm{Cl}(X - A)$. We say $X - A$ is *locally* 1-*connected* at a, written $X - A$ is 1-LC at a, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each map of S^1 into $(X - A) \cap N(a, \delta)$ extends to a map of B^2 into $(X - A) \cap N(a, \varepsilon)$. We say $X - A$ is *uniformly locally* 1-*connected* and write $X - A$ is 1-ULC if a $\delta > 0$ exists as above independent of the choice of $a \in \mathrm{Cl}(X - A)$. If $A \subset B \subset X$ and $a \in \mathrm{Cl}(X - A)$, we say that $X - B$ is *locally* 1-*connected* in $X - A$ at a and write $X - B$ is 1-LC in $X - A$ at a if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each map of S^1 into $(X - B) \cap N(a, \delta)$ extends to a map of B^2 into $(X - A) \cap N(a, \varepsilon)$. Furthermore, we say $X - B$ is *uniformly locally* 1-*connected* in $X - A$ if the uniform property holds, and we write $X - B$ is 1-ULC in $X - A$.

A Cantor set X in E^n (or S^n) is said to be *tame* if there is a homeomorphism of E^n (or S^n) onto itself taking X into a polygonal arc. If X is not tame, we say X is *wild*. We will be most interested in the fact that a tame Cantor set in E^n or S^n ($n \geq 3$) has 1-ULC complement.

Let A, B be homeomorphic subsets of E^n. We say that A and B are *equivalently embedded* if there is a homeomorphism h: $E^n \to E^n$ with $h(A) = B$. If no such homeomorphism exists, we say that A and B are *inequivalently embedded*.

Finally, a subset A of E^n is *rigid* if whenever h: $E^n \to E^n$ is a homeomorphism with $h(A) = A$, then $h(x) = x$ for each $x \in A$.

3. Some 3-dimensional preliminaries. Consider the embedding of k solid tori ($k \geq 4$) $A_1, A_2, \ldots, A_k$ in a solid torus T as shown in Figure 1 (where $k = 6$). We call such an embedding an *Antoine embedding*. We say

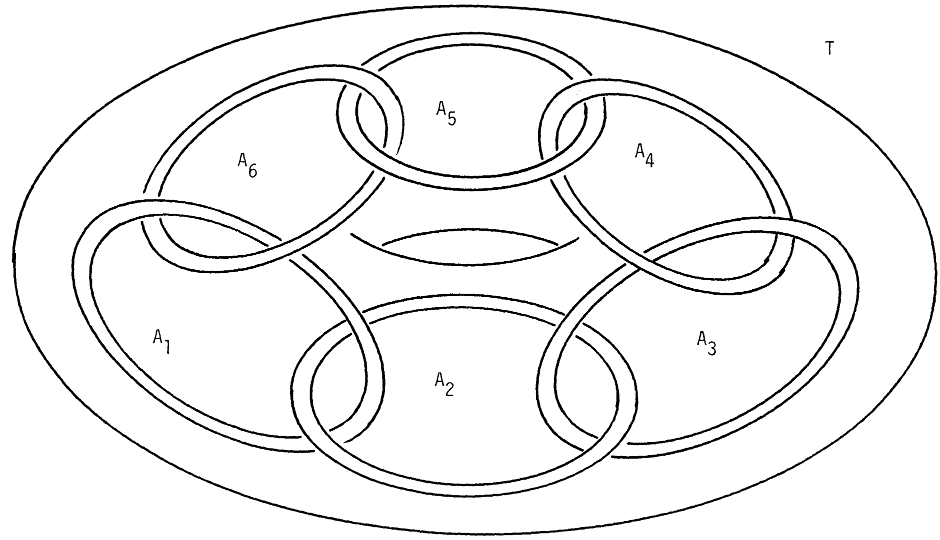

FIGURE 1

that A_i and A_j $(i \neq j)$ are *adjacent* in case A_i is not null homotopic in $T - A_j$. Clearly, each A_i is adjacent to exactly two other solid tori.

LEMMA 3.1. *Let P be a polyhedron in a solid torus T such that any loop in P is null homotopic in T. Then there is a collection Γ of loops in* $\mathrm{Bd}\, T - P$ *so that Γ is homologically nontrivial in* $\mathrm{Bd}(T)$ *but homologically trivial in $T - P$. Furthermore, each loop of Γ is nullhomotopic in T.*

Proof. Let $\tilde{T}$ be the universal covering space for T. The group of covering transformations of $\tilde{T}$ is isomorphic with the integers. Let ϕ_i be the covering transformation that corresponds with the integer i under some isomorphism. Consider $\tilde{T}$ as a subspace of B^3 so that $B^3 - \tilde{T}$ consists of exactly two points. Let p be any point in $\tilde{T}$. Let ∞ denote the point in $B^3 - \tilde{T}$ so that the sequence $\phi_1(p), \phi_2(p), \phi_3(p), \ldots$ converges to ∞. The other point in $B^3 - \tilde{T}$ will be denoted by $-\infty$. Since each loop in P is null homotopic in T, there is a lift $f\colon P \to \tilde{T}$. Let $A = f(P)$. The set A is compact, and $\phi_i(A) \cap \phi_j(A) = \varnothing$ for $i \neq j$. Setting $X = \{\infty\} \cup \phi_0(A) \cup \phi_1(A) \cup \phi_2(A) \cup \cdots$ and $Y = \{-\infty\} \cup \phi_{-1}(A) \cup \phi_{-2}(A) \cup \phi_{-3}(A) \cup \cdots$, we obtain two disjoint closed sets in B^3. Hence, there is a properly embedded compact connected piecewise-linear 2-manifold M in B^3 that separates X from Y. Since M is two sided any arc from $-\infty$ to ∞

in general position with M must pierce M algebraically once. Therefore, homological linking arguments **[Do]** show that $\mathrm{Bd}(M)$ (thought of as a 1-cycle induced from some fixed orientation of M) must be nontrivial in $\mathrm{Bd}(\tilde{T})$. The projection of $\mathrm{Bd}(M)$ into T gives the desired collection Γ.

4. Blankinship Cantor sets in E^n. We describe Cantor sets in E^n similar to the generalizations of Antoine's Necklace **[An]** given by W. A. Blankinship **[Bl]**. Our description will be brief. An excellent description of such Cantor sets has been given by W. T. Eaton **[E]**.

Let $M_0 \supseteq \mathrm{Int}\, M_0 \supseteq M_1 \supseteq \mathrm{Int}\, M_1 \supseteq M_2 \supseteq \cdots$ be a nest of compact n-manifolds in E^n such that

1. M_0 is any solid n-torus in E^n.
2. Each component of each M_i is a solid n-torus.
3. For each component N of each M_i there is a projection P of N onto a $B^2 \times S^1$ factor so that $M_{i+1} \cap N$ is $P^{-1}(A)$ for some Antoine embedding A in $B^2 \times S^1$. (See §3.)
4. The diameters of the components of M_i tend to zero as i aproaches infinity.

The intersection X of such a nest of manifolds is called a *Blankinship Cantor set*. The M_i form a *canonical defining sequence* for X. We state the following facts about such Cantor sets.

(I) Such Cantor sets may be formed so that the number of components of M_{i+1} in a given component of M_i is any desired integer $k \geq 4$. (In fact $k = 2, 3$ is also possible. We do not allow $k = 2$ so that our theory on linking Cantor sets will work. For $k = 3$ see (II).)

(II) If $f\colon H \to M_i$ is a map of a disk with holes such that $f(\mathrm{Bd}(H)) \subset \mathrm{Bd}(M_i)$ and $f(H) \cap X = \varnothing$, then f is I-inessential **[$\mathbf{D_2}$]**. If the component of M_i that contains $f(H)$ contains only three components of M_{i+1}, we require the additional hypothesis that f restricted to each boundary curve of H is trivial in M_i. This additional condition may not be necessary. We choose to avoid this case by considering only values of $k \geq 4$ in (I).

(III) If Y is a proper subset of X, then the inclusion induced homomorphism

$$\pi_1(E^n - \mathrm{Int}\, M_0) \to \pi_1(E^n - Y)$$

is trivial.

(IV) If Z is a closed nowhere dense subset of X, then $E^n - Z$ is 1-ULC.

DEFINITION 4.1. Let X_1 and X_2 be disjoint Cantor sets in E^n. We say X_1 *links* X_2 if for each connected compact manifold $W \subset E^n$ such that

(i) $X_2 \subset \operatorname{Int} W$,

(ii) $X_1 \cap W = \varnothing$, and

(iii) the fundamental group of W is abelian, then the inclusion induced homomorphism

$$\pi_1(E^n - \operatorname{Int} W) \to \pi_1(E^n - X_2)$$

is nontrivial.

DEFINITION 4.2. Let X and Y be disjoint Cantor sets in E^n. We say X and Y are *linked* if X links Y and Y links X.

DEFINITION 4.3. Let M_j be a canonical defining sequence for a Blankinship Cantor set in E^n. We say that components R and S of M_{i+1} are adjacent if

(i) R, S lie in some component N of M_i

(ii) under the projection P of N onto a $B^2 \times S^1$ factor so that $M_{i+1} \cap N$ is $P^{-1}(A)$ for an Antoine embedding A in $B^2 \times S^1$, the sets $P(R)$ and $P(S)$ are linked solid tori in $B^2 \times S^1$.

In the theorem that follows, let M_j be a canonical defining sequence for a Blankinship Cantor set X in E^n. Let R, S be different components of M_{i+1}. We consider Cantor sets $X_1 \subset R \cap X$ and $X_2 \subset S \cap X$.

THEOREM 4.4. *The Cantor sets X_1 and X_2 are linked if and only if R, S are adjacent, $X_1 = R \cap X$, and $X_2 = S \cap X$.*

Proof. If R and S are not adjacent, it is possible to find a compact n-manifold W in E^n homeomorphic with $B^3 \times S_1^1 \times S_2^1 \times \cdots \times S_{n-3}^1$ (S_i^1 is a 1-sphere) so that

(i) $S \subset \operatorname{Int} W$ and, hence, $X_2 \subset \operatorname{Int} W$

(ii) $R \cap W = \varnothing$ and, hence, $X_1 \cap W = \varnothing$.

Since the fundamental group of W is abelian and $\pi_1(E^n - \operatorname{Int} W)$ is trivial we see that X_1 and X_2 are not linked.

If $X_1 \neq R \cap X$, then $X_1 \subset \operatorname{Int} R$, $R \cap X_2 = \varnothing$, and $\pi_1(R)$ is abelian. However, by property (III) of Blankinship Cantor sets, the inclusion induced homomorphism

$$\pi_1(E^n - \operatorname{Int} R) \to \pi_1(E^n - X_1)$$

is trivial, and we see that X_1 and X_2 are not linked. A similar argument holds if $X_2 \neq S \cap X$.

We now assume $X_1 = R \cap X$, $X_2 = S \cap X$ and show that X_1 and X_2 are linked. By the definition of adjacent, R and S lie in a component N of M_i so that under a projection P of N onto a $B^2 \times S^1$ factor $P(R)$ and $P(S)$ are linked solid tori in $B^2 \times S^1$. By abuse of notation we consider $B^2 \times S^1$ a subset of N. The 3-torus $T = P(R)$ is, therefore, also a subset of E^n. We observe that any nontrivial loop in T links X_2. Now let W be a compact connected n-manifold in E^n so that $X_2 \subset \operatorname{Int} W$, $X_1 \cap W = \varnothing$, and $\pi_1(W)$ is abelian. In W choose a small open collar C of $\operatorname{Bd} W$ (a set homeomorphic to $\operatorname{Bd} W \times [0,1)$) that misses X_2. Set $W^- = W - C$. Let K be a polyhedron in T that contains $T - W$ and misses W^-.

Suppose K contains only loops that are null homotopic in T. Then Lemma 3.1 promises the existence of a collection of loops Γ in $\operatorname{Bd} T - K$ so that Γ is nontrivial in the meridional direction of $\operatorname{Bd} T$ but homologically trivial in $T - K$. But $T - K \subset W$; so Γ is homologically trivial in W. Since $\pi_1(W)$ is abelian, Γ bounds a disk with holes in W and hence Γ bounds a disk with holes in the complement of X_1. Since Γ is nontrivial in the meridional direction of $\operatorname{Bd} T$, we find from linking theory that Γ links $\operatorname{Int} R$, homologically. Let H be a disk with holes and $f\colon H \to E^n - X_1$ a map so that $f|\operatorname{Bd} H$ represents Γ. We may assume, by general position, that each component of $f^{-1}(R)$ is a disk with holes. By property II we find that the restriction of f to each component of $f^{-1}(R)$ is I-inessential. We may therefore redefine f on $\operatorname{Int}(H)$ so that $f(H) \cap \operatorname{Int}(R) = \varnothing$. But this contradicts the fact that Γ links $\operatorname{Int}(R)$. This contradiction arose from supposing K contains only loops that are null homotopic in T. Hence, we can find a loop γ in K that is not null homotopic in T. The loop γ misses W^-. Using the collar C, γ is homotopic to a loop in $E^n - \operatorname{Int}(W)$ with a homotopy that misses X_2. Since γ links X_2, we see that the inclusion induced homomorphism

$$\pi_1(E^n - \operatorname{Int} W) \to \pi_1(E^n - X_2)$$

is nontrivial. Therefore, we have shown that X_1 links X_2. Similarly, X_2 links X_1, and we see that X_1 and X_2 are linked.

Let X, Y be Blankinship Cantor sets in E^n with canonical defining sequences M_i and N_i, respectively. We let $R_1, R_2, \ldots, R_p$ denote the components of M_1 and $S_1, S_2, \ldots, S_q$ denote the components of N_1. The following lemma and theorem will use the above notation.

LEMMA 4.5. *Let $h\colon E^n \to E^n$ be a homeomorphism such that for each i, $1 \le i \le p$, there is a j, $1 \le j \le q$, such that $h(R_i \cap X) \subset S_j \cap Y$. Then either $h(X) \subset S_j$ for some fixed j or (after possible resubscripting) $h(R_i \cap X) = S_i \cap Y$ for each i, $p = q$, and R_i, R_j are adjacent if and only if S_i and S_j are adjacent.*

Proof. If it is not the case that $h(X) \subset S_j$ for some fixed j, then there are adjacent components that we will call (after possible resubscripting) R_1 and R_2 so that $h(R_1 \cap X) \subset S_1 \cap Y$ and $h(R_2 \cap X) \subset S_2 \cap Y$. Since $R_1 \cap X$ and $R_2 \cap X$ are linked, $h(R_1 \cap X)$ and $h(R_2 \cap X)$ are also linked. But Theorem 4.4 shows that $h(R_1 \cap X) = S_1 \cap Y$, $h(R_2 \cap X) = S_2 \cap Y$, and that S_1 and S_2 are adjacent. By using induction and Theorem 4.4 it is now an easy matter to complete the proof.

THEOREM 4.6. *Let h: $E^n \to E^n$ be a homeomorphism such that $h(X) \subset Y$. Then either $h(X) \subset S_j$ for some fixed j or (after possible resubscripting) $h(R_i \cap X) = S_i \cap Y$ for each i, $p = q$, and R_i, R_j are adjacent if and only if S_i and S_j are adjacent.*

Proof. Let m be the smallest integer so that for each component R of M_m, $h(R \cap X) \subset S_j$ for some j. If $m = 0$ or 1 we are done by Lemma 4.6.

We suppose $m \geq 2$. This implies that there is a component N of M_{m-1} so that for each component R of $M_m \cap N$ $h(R \cap X) \subset S_j$ for some j, but that $h(N \cap X) \not\subset S_j$ for some j. By Lemma 4.6 $h(N \cap X) = Y$. However, since $m > 2$, we see $X - N \neq \varnothing$, and $h(X - N) \subset Y$. This contradicts the fact that h is a homeomorphism, and we see that $m \geq 2$ is impossible.

5. Distinguishing Blankinship Cantor sets. An *Antoine graph* G is a graph G so that G is the countable union of nested subgraphs $\varnothing = G_{-1} \subset G_0 \subset G_1 \subset \cdots$. The graph G_0 is a single vertex. For each vertex v of $G_i - G_{i-1}$ there is a polygonal simple closed curve with at least four vertices $P(v)$ in $G_{i+1} - G_i$ so that if v and w are distinct vertices of $G_i - G_{i-1}$, then $P(v) \cap P(w) = \varnothing$. The graph G_{i+1} consists of G_i plus the union of $P(v)$, v a vertex of $G_i - G_{i-1}$, plus edges running between v and the vertices of $P(v)$. See Figure 2. For an Antoine graph G, the subgraphs $G_0, G_1, G_2, \ldots$ are uniquely determined since the vertex in G_0 is the only vertex that does not separate G. For a fixed vertex v of G we associate the unique Antoine subgraph $G(v)$ of G so that $G(v)_0 = \{v\}$.

Given a solid n-torus M in E^n and an Antoine graph G, we think of the graph as a set of instructions for constructing a canonical defining sequence M_i for a Blankinship Cantor set in E^n. The vertex in G_0 corresponds to $M_0 = M$. The vertices of $G_i - G_{i-1}$ correspond to components of M_i. If v is a vertex of $G_i - G_{i-1}$ that corresponds to the component N of M_i, then $M_{i+1} \cap N$ contains components corresponding to the vertices of $P(v)$. Furthermore, the components of $M_{i+1} \cap N$ are adjacent if and only if the corresponding vertices bound an edge.

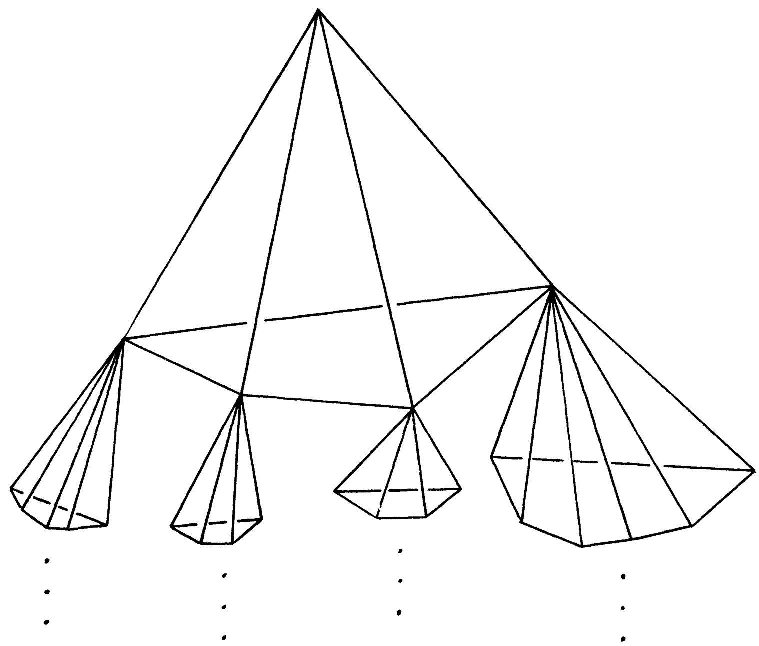

FIGURE 2

On the other hand, given a canonical defining sequence M_i for a Blankinship Cantor set X, we may associate an Antoine graph $G(X)$ so that the defining sequence follows the instructions of the graph as in the previous paragraph. Theorem 4.6 with $X = Y$ and $h =$ identity yields the fact that any two canonical defining sequences for X yield isomorphic Antoine graphs. Hence, $G(X)$ is well defined. We immediately obtain the following theorem and corollary.

THEOREM 5.1. *If X and Y are Blankinship Cantor sets so that $G(X)$ is not isomorphic to $G(Y)$, then X and Y are not equivalently embedded in E^n.*

COROLLARY 5.2. *There are uncountably many inequivalently embedded Blankinship Cantor sets in E^n.*

THEOREM 5.3. *Suppose X and Y are Blankinship Cantor sets in E^n and $h\colon E^n \to E^n$ is a homeomorphism. If $h(X) \cap Y$ contains an open subset of either $h(X)$ or Y, then $G(X)$ and $G(Y)$ have isomorphic Antoine subgraphs.*

Proof. Without loss of generality we assume $h(X) \cap Y$ contains an open subset of $h(X)$. Let M_i be a canonical defining sequence for X. Hence there is an integer r and a component T of M_r so that $h(T \cap X) \subset Y$. Theorem 4.6 shows that $G(T \cap X)$ is isomorphic to an Antoine subgraph of $G(Y)$. Clearly $G(T \cap X)$ is isomorphic to an Antoine subgraph of $G(X)$, and the theorem is proved.

6. Rigid Cantor sets in E^n. An Antoine graph G is said to be *rigid* if for each pair of distinct vertices v, w in G, the Antoine subgraphs $G(v)$ and $G(w)$ are not isomorphic.

THEOREM 6.1. *Suppose X is a Blankinship Cantor set in E^n so that $G(X)$ is rigid. Then X is rigid.*

Proof. Suppose h: $E^n \to E^n$ is a homeomorphism so that $h(X) = X$ and $h|X \neq$ identity. We can find disjoint manifolds M, N that are components of a defining sequence for X such that $h(M \cap X) \subset N \cap X$. By Theorem 5.3 $G(M \cap X)$ and $G(N \cap X)$ have isomorphic Antoine subgraphs. But these correspond to distinct Antoine subgraphs of $G(X)$ which is a contradiction.

THEOREM 6.2. *There are uncountably many rigid Blankinship Cantor sets X_α in E^n $(n \geq 3)$ such that for $\alpha \neq \beta$ and any homeomorphism h: $E^n \to E^n$, $h(X_\alpha) \cap X_\beta$ is a nowhere dense subset of each.*

Proof. We leave it as a manageable exercise for the reader to construct uncountably many rigid Antoine subgraphs G_α so that for $\alpha \neq \beta$, G_α and G_β do not have isomorphic Antoine subgraphs. The theorem then follows from Theorems 6.1 and 5.3.

7. Rigid Sets in E^n $(n \geq 4)$. We state our main theorem of this section.

THEOREM 7.1. *Let W be a compactum in S^{n-1} $(n \geq 4)$ with no isolated points. There are uncountably many inequivalent embeddings of W in E^n each of which is rigid.*

We will use as a tool the following theorem of J. W. Cannon [**C**].

THEOREM 7.2. *Suppose X is a compact subset of an $(n-1)$-sphere Σ in E^n and* $\dim X \leq n-3$. *Then $E^n - \Sigma$ is 1-ULC in $E^n - X$ if and only if $E^n - X$ is 1-ULC.*

The following theorem will provide us with the means to construct our embeddings.

THEOREM 7.3. *Let W be a compactum in $S^{n-1} = \operatorname{Bd} B^n$ with no isolated points. Given a sequence $X_1, X_2, X_3, \ldots$ of Cantor sets in E^n, there is an embedding $e\colon B^n \to E^n$ and disjoint Cantor sets $Y_1, Y_2, Y_3, \ldots$ in E^n such that setting $\Sigma = e(S^{n-1})$*

(i) *$\bigcup Y_i$ is a dense subset of $e(W)$,*

(ii) *Y_i is equivalent embedded as X_i,*

(iii) *for each integer $k > 0$ there is an embedding $f_k\colon \Sigma \to E^n$ such that*

(a) *f_k moves points less than $1/k$,*

(b) *$f_k | \bigcup_{i=1}^k Y_i$ is the identity,*

(c) *$f_k(\Sigma - \bigcup_{i=1}^k Y_i)$ is locally flat,*

(d) *$\bigcup_{i=1}^k Y_i$ is a tame subset of $f_k(\Sigma)$*

(e) *$f_k(\Sigma) \cap e(B^n) = \bigcup_{i=1}^k Y_i$*

Proof. By standard techniques **[Al]**, **[Bl]**, **[O]** it is well known that given a tame Cantor set Z in S^{n-1} and any Cantor set Y in $E^n - B^n$ there exists an embedding h of B^n in E^n so that $h(Z) = Y$ and $h(S^{n-1}) - Y$ is locally flat. Furthermore, if $Z \cup Y$ is contained in an open n-ball U whose intersection with S^{n-1} is an open $(n-1)$-ball, then h can be chosen so $h(x) = x$ if $x \notin U$ and $h(U \cap S^{n-1}) \subset U$.

Given any Cantor set Y in E^n and any open set U, it is an easy matter to find a Cantor set in U that is equivalently embedded as Y. The above theorem follows from this fact by taking a limit of embeddings e_k using the standard techniques. Care must be taken so that the limit e of the e_k is an embedding. The functions f_k are constructed along with the e_k.

Addendum to Theorem 7.3. Let $W = \Sigma - \bigcup_{i=1}^{\infty} Y_i$. Then $E^n - \Sigma$ is 1-ULC in $E^n - W$. If $n \geq 4$ and $Z_i \subset Y_i$ is a compact subset so that $E^n - Z_i$ is 1-ULC, then, setting $W' = \Sigma - \bigcup (Y_i - Z_i)$, $E^n - \Sigma$ is 1-ULC in $E^n - W'$.

Proof. The Addendum is proved by using the f_k and the fact that $f_k(\Sigma) - \bigcup_{i=1}^k Y_i$ is 1-ULC for $n \geq 4$.

Let W be a compactum in S^{n-1} with no isolated points; $X_1, X_2, \ldots$ be a sequence of Cantor sets in E^n; and Σ an $(n-1)$-sphere constructed as in Theorem 7.3. The following lemma will make the proof of Theorem 7.1 transparent.

LEMMA 7.4. *Let Y be a Cantor set in E^n $(n \geq 4)$ such that for each homeomorphism h: $E^n \to E^n$, $E^n - (h(Y) \cap X_i)$ is* 1-ULC *for each i. If g: $E^n \to E^n$ is a homeomorphism such that $g(Y) \subset \Sigma$, then $E^n - Y$ is* 1-ULC.

Proof. We use Y_i to denote the Cantor sets as in Theorem 7.3 and set $Z_i = g(Y) \cap Y_i$. Then $E^n - Z_i$ is 1-ULC for each i. Since $g(Y) \subset W' = \Sigma - \bigcup(Y_i - Z_i)$, the Addendum to Theorem 7.3 shows that $E^n - \Sigma$ is 1-ULC in $E^n - g(Y)$. Hence Cannon's theorem, Theorem 7.2, shows that $E^n - g(Y)$ is 1-ULC. Clearly $E^n - Y$ is 1-ULC.

Proof of Theorem 7.1. Let W be a compactum in S^{n-1} $(n \geq 4)$ with no isolated points. Using Theorem 6.2, choose rigid Blankinship Cantor sets $X_1, X_2, X_3, \ldots$ and $X_1', X_2', X_3', \ldots$ in E^n such that for each pair of Cantor sets the associated Antoine graphs do not have isomorphic Antoine subgraphs. Let Σ and Σ' be the $(n-1)$-spheres promised by Theorem 7.3 corresponding to the Cantor sets X_i and X_i', respectively. Let $X \subset \Sigma$ and $X' \subset \Sigma'$ be the resulting embeddings of W. We need to show that X and X' are inequivalently embedded and that each is rigid. To this end we let Y_i and Y_i' be the Cantor sets promised by 7.3 such that $\bigcup Y_i$ is dense in X, $\bigcup Y_i'$ is dense in X', and Y_i (respectively Y_i') is equivalently embedded as X_i (respectively X_i').

From Theorem 6.2 and fact (IV) about Blankinship Cantor sets we see that for each homeomorphism h: $E^n \to E^n$, $E^n - (h(X_i) \cap X_j')$ is 1-ULC for all i, j. If g: $E^n \to E^n$ is a homeomorphism such that $g(X) = X'$, then $g(Y_i) \subset \Sigma'$ and, by Lemma 7.4, $E^n - Y_i$ is 1-ULC, a contradiction. Hence, we see that X and X' are not equivalently embedded.

The proof that X and X' are rigid is similar to the above proof with minor modifications.

Since there are uncountably many rigid Blankinship Cantor sets as described in Theorem 6.2, it is now a simple matter to find uncountably many inequivalent embeddings of W in E^n each of which is rigid.

Notice that Theorem 7.3 applies in E^3, but the Addendum does not. Thus we are led to the following question.

Question 7.5. Let W be a compactum in S^2 with no isolated points. Are there uncountably many inequivalent embeddings of W in E^3 each of which is rigid?

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