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ON REGULAR SUBDIRECT PRODUCTS OF SIMPLE ARTINIAN RINGS

CHEN-LIAN CHUANG AND PJEK-HWEE LEE

We construct a counterexample to settle simultaneously the following questions all in the negative: (1) Is a regular subdirect product of simple artinian rings unit-regular? (2) If R is a regular ring such that every nonzero ideal of R contains a nonzero ideal of bounded index, is R unit-regular? (3) Is a regular ring with a Hausdorff family of pseudo-rank functions unit-regular? (4) If R is a regular ring which contains no infinite direct sum of nonzero pairwise isomorphic right ideals, is R unit-regular? (5) Is a regular Schur ring unit-regular?

In [**1**] Goodearl proposed a list of open problems on regular rings. Some involve potential sufficient conditions for a regular ring to be unit-regular. The primary aim of this paper is to construct a counterexample for the questions 6, 7, 8, 9 (second part) and 11 in Goodearl's book.

Among others the sixth question asks: *Is a regular subdirect product of simple artinian rings always unit-regular*? In [**4**] Tyukavkin has shown that any regular algebra over an uncountable field, which is a subdirect product of countably many simple artinian rings, is unit-regular. Recently, Goodearl and Menal [**2**] have generalized this result by showing that any regular algebra over an uncountable field, which has no uncountable direct sums of nonzero right or left ideals, must be unit-regular; in particular, any regular algebra over an uncountable field, which has a rank function, is unit-regular. In this paper we shall construct an example of a regular ring which is a subdirect product of countably many simple artinian rings but is *not* unit-regular.

Let F be a countable field, $F[t]$ the ring of polynomials over F in an indeterminate t, and $F(t)$ the quotient field of $F[t]$. Define an exponential valuation ∂ on $F(t)$ by $\partial r(t) = +\infty$ if $r(t) = 0$ and $\partial r(t) = n$ if $r(t) = t^n f(t)/g(t)$ where n is an integer and $f(t), g(t) \in F[t]$ with $t \nmid f(t)g(t)$. Let V be the valuation ring associated with ∂, namely, $V = \{r(t) \in F(t) | \partial r(t) \geq 0\}$. Note that $F[t]$, $F(t)$ and V are all countable. Consequently, V is a countable-dimensional vector space over F.

Let $v_0, v_1, \ldots, v_n, \ldots$ be a basis of V over F. First, we may assume that $\partial v_i \neq \partial v_j$ for $i \neq j$. Suppose that n is the least integer such that $\partial v_n = \partial v_i$ for some $i < n$. Choose $\alpha_i \in F$ so that $v_n/v_i - \alpha_i \in tV$; then $\partial(v_n - \alpha_i v_i) > \partial v_i$. If $\partial(v_n - \alpha_i v_i) = \partial v_j$ for some $j < n$, then $\partial(v_n - \alpha_i v_i - \alpha_j v_j) > \partial v_j$ for some $\alpha_j \in F$. Continuing this process we get a v_n' such that $\partial v_n' \neq \partial v_i$ for all $i < n$ and that $\{v_0, v_1, \ldots, v_{n-1}, v_n'\}$ spans the same subspace as $\{v_0, v_1, \ldots, v_{n-1}, v_n\}$ does. Next, we assume, by reordering, that $\partial v_0 < \partial v_1 < \partial v_2 < \cdots$. For $v = \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots$ with $\alpha_k \neq 0$, we see that $\partial v = \partial v_k$. Since $v_0, v_1, v_2, \ldots$ span the whole space V, we must have $\partial v_0 = 0$, $\partial v_1 = 1$, $\partial v_2 = 2$ and so on.

We begin by constructing a ring which is similar to that in Bergman's example [**1**; Example 4.26]. Let S be the set of those $x \in E = \mathrm{End}_F(V)$ such that $(x - a)t^n V = 0$ for some $a \in F(t)$ and some nonnegative integer n. As in [**1**; p. 47] we observe that a depends only on x, that is, for each $x \in S$ there is a unique element $\varphi x \in F(t)$ such that $(x - \varphi x)t^n V = 0$ for some $n \geq 0$. Also, it can be verified that S is an F-subalgebra of E containing $F[t]$ and that φ is an F-algebra map of S onto $F(t)$. In addition, $\ker\varphi$ is a regular ideal of S and $S/\ker\varphi \simeq F(t)$, and therefore S is a regular ring. However, S is not unit-regular because of the existence of $t \in S$ which is injective but not surjective on V.

Let us fix a basis $v_0, v_1, v_2, \ldots$ of V over F with $\partial v_n = n$ for all n. Then $v_n, v_{n+1}, \ldots$ form a basis of $t^n V$ over F. Let π_n be the projection of V onto the subspace spanned by $v_0, v_1, \ldots, v_n$ with kernel $t^{n+1}V$. Consider the matrix of $a \in S$ with respect to the basis $v_0, v_1, v_2, \ldots$. Certainly, it is column-finite. That is, for any $m \geq 0$ there exists $n \geq 0$ such that $(1 - \pi_n)a\pi_m = 0$. Also, it is row-finite: for any $m \geq 0$ there exists $n \geq 0$ such that both $(a - \varphi a)t^n V = 0$ and $(\varphi a)t^n V \subseteq t^{m+1}V$, consequently, $a(t^n V) \subseteq t^{m+1}V$ and $\pi_m a(1 - \pi_n) = 0$.

Set $W = S \times \prod_{k=0}^{\infty} \pi_k E \pi_k$ and write elements of W as sequences $w = (w_{-1}, w_0, w_1, \ldots)$ where $w_{-1} \in S$ and $w_k \in \pi_k E \pi_k$ for $k \geq 0$. Let R be the set of elements $w \in W$ satisfying the following two conditions: (i) for any $m \geq 0$ there exists $n \geq 0$ such that $w_k \pi_m = w_{-1}\pi_m$ for all $k \geq n$; (ii) for any $m \geq 0$ there exists $n \geq 0$ such that $\pi_m w_k = \pi_m w_{-1}$ for all $k \geq n$. It is clear that R is an F-subspace of W. To show that R is a ring, we consider any u, $w \in R$ and $m \geq 0$. There exists $n \geq 0$ such that $w_k \pi_m = w_{-1}\pi_m$ for all $k \geq n$. Now because $w_{-1} \in S$ is column-finite, $w_{-1}\pi_m = \pi_j w_{-1}\pi_m$ for some $j \geq 0$. Also, there exists $n' \geq 0$ such that $u_k \pi_j = u_{-1}\pi_j$ for all $k \geq n'$. Then

$u_k w_k \pi_m = u_k w_{-1} \pi_m = u_k \pi_j w_{-1} \pi_m = u_{-1} \pi_j w_{-1} \pi_m = u_{-1} w_{-1} \pi_m$ for all $k \geq \max\{n, n'\}$. Similarly, we can show that there exists $n'' \geq 0$ such that $\pi_m u_k w_k = \pi_m u_{-1} w_{-1}$ for all $k \geq n''$. Thus, $uw \in R$. Therefore R is an F-subalgebra of W.

Let $\alpha\colon R \to \prod_{k=0}^{\infty} \pi_k E \pi_k$ be the projection $(w_{-1}, w_0, w_1, \ldots) \mapsto (w_0, w_1, \ldots)$. Given any $w \in \ker\alpha$, $w_k = 0$ for all $k \geq 0$. For any $m \geq 0$ we have $w_{-1}\pi_m = w_k \pi_m = 0$ for some k. Hence, $w_{-1} = 0$ and so $w = 0$. Thus α is injective. If $w_k \in \pi_k E \pi_k$, $k = 0, 1, \ldots, n$, then $w = (0, w_0, w_1, \ldots, w_n, 0, \ldots) \in R$ and $\alpha w = (w_0, w_1, \ldots, w_n, 0, \ldots)$. In other words, $\bigoplus_{k=0}^{\infty} \pi_k E \pi_k \subseteq \alpha R$.

Let $\beta\colon R \to S$ be the projection $(w_{-1}, w_0, w_1, \ldots) \mapsto w_{-1}$. For $x \in S$, set $w = (x, \pi_0 x \pi_0, \pi_1 x \pi_1, \ldots) \in W$. Let $m \geq 0$. Since x is column-finite, there exists $n \geq 0$ such that $(1 - \pi_k) x \pi_m = 0$ for all $k \geq n$. Then $w_k \pi_m = \pi_k x \pi_k \pi_m = \pi_k x \pi_m = x \pi_m = w_{-1} \pi_m$ for all $k \geq \max\{m, n\}$. Similarly, there exists $n' \geq 0$ such that $\pi_m w_k = \pi_m w_{-1}$ for all $k \geq n'$. Thus $w \in R$ and $\beta w = x$. Hence, β is surjective.

It remains to show that R is regular. But since $R/\ker\beta \simeq S$ is regular, it suffices to show the regularity of $\ker\beta$. Let $w \in \ker\beta$. For each $m \geq 0$ there exist $n_m \geq 0$ such that $w_k \pi_m = \pi_m w_k = 0$ for all $k \geq n_m$. Without loss of generality, we may assume that $0 < n_0 < n_1 < \cdots$. For $0 \leq k < n_0$, choose $u_k \in \pi_k E \pi_k$ such that $w_k u_k w_k = w_k$. For $n_m \leq k < n_{m+1}$, we have $w_k \in (1 - \pi_m) \pi_k E \pi_k (1 - \pi_m)$, and so choose $u_k \in (1 - \pi_m) \pi_k E \pi_k (1 - \pi_m)$ such that $w_k u_k w_k = w_k$. Thus $u = (0, u_0, u_1, \ldots) \in W$ and $wuw = w$. Moreover, $u_k \pi_m = \pi_m u_k = 0$ for all $k \geq n_m$ by construction. Hence, $u \in R$ and so $u \in \ker\beta$. Therefore, $\ker\beta$ is regular, and so R is regular. On the other hand, S, which is not unit-regular, is a homomorphic image of R. Consequently, R cannot be unit-regular.

Thus, we have constructed a regular ring R which is not unit-regular. Since $\bigoplus_{k=0}^{\infty} \pi_k E \pi_k \subseteq \alpha R \subseteq \prod_{k=0}^{\infty} \pi_k E \pi_k$, where α is a monomorphism and $\pi_k E \pi_k \simeq M_k(F)$, R is a subdirect product of simple artinian rings. This settles Question 6 in the negative.

A ring R is said to be of bounded index if there exists a positive integer n such that $x^n = 0$ for all nilpotent elements x in R. The seventh question is: *If R is a regular ring such that every nonzero two-sided ideal of R contains a nonzero two-sided ideal of bounded index, is R unit-regular?* This question is in fact equivalent to Question 6. Instead of showing this, one can verify easily that the example constructed above satisfies the condition of this question. Let I be a nonzero two-sided ideal of αR. Let $w = (w_0, w_1, w_2, \ldots) \in I$ with

$w_n \neq 0$ for some $n \geq 0$. Since $\bigoplus_{k=0}^{\infty} \pi_k E \pi_k \subseteq \alpha R$ and $\pi_n E \pi_n$ is simple, it follows that I contains a nonzero two-sided ideal isomorphic to $\pi_n E \pi_n$ which is clearly of bounded index. This gives a negative answer to Question 7.

A *pseudo-rank function* on a regular ring R is a map $N : R \to [0, 1]$ such that (a) $N(1) = 1$, (b) $N(xy) \leq \min\{N(x), N(y)\}$ for all $x, y \in R$, (c) $N(e+f) = N(e) + N(f)$ for all orthogonal idempotents $e, f \in R$. If, in addition, $N(x) = 0$ only if $x = 0$, N is called a *rank function* on R. The set of all pseudo-rank functions on R is denoted by $\mathbf{P}(R)$. Given a family $X \subseteq \mathbf{P}(R)$, we use $\ker(X)$ to denote the kernel of X, namely, $\ker(X) = \{x \in R | N(x) = 0 \text{ for all } N \in X\}$. Since all simple artinian rings have rank functions [**1**; Corollary 16.6], then $\sum_{k=0}^{\infty}(1/2^{k+1})N_k$ defines a rank function on $\prod_{k=0}^{\infty} M_k(F)$, where N_k is a rank function on $M_k(F)$. Thus any regular subdirect product R of $\prod_{k=0}^{\infty} M_k(F)$ has a rank function and hence $\ker(\mathbf{P}(R)) = 0$. Therefore we have obtained a counterexample to the eighth question: *If R is a regular ring such that* $\ker(\mathbf{P}(R)) = 0$, *is R unit-regular*? Since a regular ring with a rank function contains no infinite direct sums of nonzero pairwise isomorphic right or left ideals [**1**; Proposition 16.11], the second part of Question 9 is also settled: *If R is a regular ring which contains no infinite direct sums of nonzero pairwise isomorphic right ideals, is R unit-regular*? Finally, a regular ring with a rank function satisfies the hypothesis of Question 11 [**3**; Theorem 5]: *Let R be a regular ring, and assume that whenever $x, y \in R$ such that $xy = yx$ and $xR + yR = R$, then $Rx + Ry = R$. Is R unit-regular*? Thus our example also provides a negative answer to this question.

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NATIONAL TAIWAN UNIVERSITY
TAIPEI, TAIWAN 10674