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THE C^*-ALGEBRAS GENERATED BY PAIRS OF SEMIGROUPS OF ISOMETRIES SATISFYING CERTAIN COMMUTATION RELATIONS

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Arising in the computation of the Arveson-Powers index for $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$ is the notion of a pair of one-parameter semigroups of isometries $\mathscr{U} = \{U_t : t \in \Gamma^+\}$ and $\mathscr{S} = \{S_t : t \in \Gamma^+\}$ satisfying the commutation relations $S_t^* U_t = e^{-\lambda t} I$, for Γ the set of real numbers. If Γ is any subgroup of $\mathbb{R}$ we show that the C^*-algebra $\mathfrak{A}_\Gamma$ generated by $\mathscr{U}$ and $\mathscr{S}$ is canonically unique. $\mathfrak{A}_\gamma$ is simple if and only if Γ is dense in $\mathbb{R}$.

I. Introduction. According to the von Neumann-Wold decomposition for an isometry V acting on a Hilbert space $\mathfrak{H}$, $\mathfrak{H}$ may be decomposed into an orthogonal direct sum of reducing Hilbert subspaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ for V, where $V|_{\mathfrak{H}_1}$ is a unitary operator and $V|_{\mathfrak{H}_2}$ is a pure isometry. In [**6**], L. A. Coburn characterized the C^*-algebra $C^*(V)$ generated by an isometry. If V is completely unitary then as is well known, $C^*(V)$ is isometrically $*$-isomorphic to $C(\sigma(V))$, the algebra of complex-valued continuous functions on the spectrum of V. If V has a non-trivial pure isometric part, $C^*(V)$ contains a closed two-sided ideal which is isomorphic to the compact operators $\mathscr{K}$. The quotient algebra $C^*(V)/\mathscr{K}$ is isomorphic to the algebra of continuous functions on the circle, [**6**].

Generalizations of this result (see [**4**], [**7**]–[**10**], [**12**]) made by Coburn and other authors have taken various forms. For example, the study of C^*-algebras generated by a semigroup of isometries has led to interesting developments in the theory of an index for algebras of operators. This theory is modelled on the theory of Fredholm operators in $\mathfrak{B}(\mathfrak{H})$, and has led to some interesting connections between the notions of topological and analytic index, [**8**]–[**10**].

In [**12**], R. G. Douglas analyzed the structure of the C^*-algebras $\mathfrak{A}_\Gamma$ generated by one-parameter semigroups of isometries $\mathscr{V}_\Gamma = \{V_\gamma : \gamma \in \Gamma^+\}$, where Γ is a subgroup of the real numbers. Without making any assumptions about the continuity of the mapping $\gamma \to V_\gamma$, Douglas showed that the C^*-algebra $\mathfrak{A}_\Gamma$ is canonically unique. This analysis was carried out via a characterization of the (commutative) quotient algebras $\mathfrak{A}_\Gamma/C_\Gamma$, where C_Γ is the closed two-sided ideal generated by the commutators in $\mathfrak{A}_\Gamma$. He determined also that $\mathfrak{A}_\Gamma$ and $\mathfrak{A}_{\Gamma'}$ are isomorphic if and only if the corresponding groups Γ, Γ' are order isomorphic. (A similar analysis, using K-theoretic techniques, has recently been carried out on the commutator ideals, **[13]**, see also **[19]**.) This uniqueness result stands in marked contrast to the abundance of isometric representations of the semigroups Γ^+, as shown in **[14]**.

The Cuntz algebras O_n, $n \in (\infty, 2, 3, \ldots)$, are a highly non-commutative generalization of $C^*(V)$. For $n < \infty$, O_n is defined as the C^*-algebra generated by n isometries $S_1, \ldots, S_n$ on a Hilbert space which satisfy the relations $S_i^* S_j = \delta_{ij} I$, and $\sum_{i=1}^n S_i S_i^* = I$. These identities characterize O_n uniquely, up to isomorphism. O_n is a simple C^*-algebra; in fact, it possesses the remarkable property that for any non-zero X in O_n, there are A, $B \in O_n$ satisfying $AXB = I$, **[11**, Theorem 1.13] (see also Theorem 3.9 below).

If one replaces the second equation above with the inequality $\sum_{i=1}^n S_i S_i^* < I$, then the C^*-algebra generated by the polynomials in the S_i's is an extension of O_n by the compact operators (**[11**, Proposition 3.1], see also Theorem 2.4 below). Taking $n = 1$, the C^*-algebra generated by a (non-unitary) isometry fits into this framework.

In this work we study a problem which is a combination, in a sense, of the two generalizations discussed briefly above. For a subgroup Γ of $\mathbb{R}$, let $\mathscr{U}_\Gamma = \{U_\gamma : \gamma \in \Gamma^+\}$ and $\mathscr{S}_\Gamma = \{S_\gamma : \gamma \in \Gamma^+\}$ be a pair of semigroups of isometries on a separable Hilbert space. We assume that $\mathscr{U}_\Gamma$ and $\mathscr{S}_\Gamma$ are related by the Weyl commutation relations

$$S_\gamma^* U_\gamma = e^{-\lambda\gamma} I, \qquad \gamma \in \Gamma^+, \tag{1}$$

for some fixed $\lambda > 0$. Here again we make no assumptions about the continuity of the mappings $\gamma \to S_\gamma$ and $\gamma \to U_\gamma$. We should point out that from (1) it follows that each S_γ must contain a nontrivial pure isometric part, for $\gamma > 0$, since the assertion that S_γ is unitary leads to the equation $1 = \|U_\gamma\| = \|e^{-\lambda\gamma} S_\gamma\| = e^{-\lambda\gamma}$, which is absurd. By symmetry, U_γ also contains a pure isometric part. We show below in Theorem 2.4 that if Γ is a discrete subgroup of $\mathbb{R}$, then the C^*-algebra $\mathfrak{A}_\Gamma$ generated by all operators U_γ, S_γ, for $\gamma \in \Gamma^+$, is an extension, as above, of the algebra O_2 by the ideal of compact operators. If Γ is dense, then $\mathfrak{A}_\Gamma$ is simple: in fact, $\mathfrak{A}_\Gamma$ is strongly simple in the sense shared by the Cuntz algebras that for any $X \neq 0$ there are operators A, B in $\mathfrak{A}_\Gamma$ such that $AXB = I$ (Theorem 3.9). We also show that the C^*-algebras $\mathfrak{A}_\Gamma$ are canonically unique, Theorem 3.12. Our methods of proof of these results rely heavily on some techniques used by J. Cuntz, [11], and R. G. Douglas, [12].

The principal motivation for studying this algebra comes from the recent work of R. T. Powers and the author, [17], relating the index theories of Powers and W. B. Arveson on E_0-semigroups of $*$-endomorphism of $\mathfrak{B}(\mathfrak{H})$, [1]–[3], [15]–[17]. Let $\alpha = \{\alpha_t : t \geq 0\}$ be a one-parameter semigroup of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$. Then α is an E_0-semigroup if each α_t is unital, if $\alpha_t(\mathfrak{B}(\mathfrak{H}))$ is properly contained in $\mathfrak{B}(\mathfrak{H})$, and if the mapping $t \to \alpha_t(A)$ is continuous in the weak operator topology for all A in $\mathfrak{B}(\mathfrak{H})$. A strongly continuous one-parameter semigroup $\mathscr{U} = \{U_t : t \geq 0\}$ of operators (not necessarily isometries) in $\mathfrak{B}(\mathfrak{H})$ is said to *intertwine* α, [1], if for all $t \geq 0$ and for all A in $\mathfrak{A}(\mathfrak{H})$, $U_t A = \alpha_t(A) U_t$. It may occur that α has no intertwining semigroups, [16]. However, when intertwining semigroups $\mathscr{U}$ and $\mathscr{S}$ do exist, it follows, [2], that there is a complex number $c(\mathscr{U}, \mathscr{S})$ such that, for all t,

$$S_t^* U_t = \exp(tc(\mathscr{U}, \mathscr{S}))I. \tag{2}$$

Modifying $\mathscr{S}$ and $\mathscr{U}$ through multiplication by scalar-valued semigroups, one may assume that $\mathscr{U}$ and $\mathscr{S}$ are semigroups of isometries satisfying (1), [17].

Let $\mathscr{U}_\alpha$ be the family of all strongly continuous intertwining semigroups of α. Arveson's index for α is obtained by calculating the dimension of the Hilbert space completion of the space of functions $\{f : \mathscr{U}_\alpha \to \mathbb{C} : f$ is finitely non-zero and $\sum_{\mathscr{S} \in \mathscr{U}_\alpha} f(\mathscr{S}) = 0\}$ in the positive semidefinite inner product $(f, g) = \sum_{\mathscr{U}, \mathscr{S} \in U_\alpha} f(\mathscr{U})\overline{g(\mathscr{S})}c(\mathscr{U}, \mathscr{S})$. The Powers' index is obtained by calculating the multiplicity of a certain representation of the dense $*$-subalgebra $\mathfrak{D}(\delta)$ of $\mathfrak{B}(\mathfrak{H})$, where $\mathfrak{D}(\delta)$ is the domain of the infinitesimal generator δ of the one-parameter semigroup α, [15]. The key problem involved in showing that these two versions of index agree is to analyze the structure of a pair of strongly continuous flows of isometries satisfying (1) (see [17] for a proof of the existence of these flows and an analysis of their structure).

We end this section by remarking that W. B. Arveson has defined and analyzed the structure of a separable C^*-algebra, called the *spectral C^*-algebra*, associated with an E_0-semigroup α of endomorphisms. These algebras, which are, along with the index, an outer conjugacy invariant for E_0-semigroups, are constructed from the product systems E corresponding to α, [3]. As noted by Arveson, this family of algebras contains the Wiener-Hopf C^*-algebra as a degenerate case in much the same way that the Toeplitz C^*-algebra studied by Coburn is the degenerate case of the Cuntz algebras.

II. The discrete case. In this section we consider the structure of the C^*-algebra $C^*(U_t, S_t)$ generated by a pair of isometries U_t, S_t acting on a separable Hilbert space and satisfying the relation (1), for fixed t. As we shall see in the next section, the proof of the simplicity of $\mathfrak{A}_\Gamma$, for Γ a dense subgroup of $\mathbb{R}$, depends greatly on the special case considered here.

We begin this section by introducing some notation which shall be used throughout the paper. We denote by $\mathscr{U} = \{U_t : t \geq 0\}$ and by $\mathscr{S} = \{S_t : t \geq 0\}$ a pair of semigroups of isometries on a separable Hilbert space $\mathfrak{H}$ which satisfy, for a fixed positive $\lambda > 0$, the commutation relations (1). An explicit construction in **[17]** shows that such pairs do indeed exist. Let $\mathscr{P}$ be the $*$-algebra of polynomials in the operators U_t, S_t, $t \geq 0$. Using (1) and the fact that U_t, S_t are isometries, one may always write any polynomial $P \in \mathscr{P}$ as a linear combination of terms of the form

$$A = U_{l_1} S_{l_2} \cdots U_{l_{2\mu-1}} S_{l_{2\mu}} S^*_{r_{2\nu}} U^*_{r_{2\nu-1}} \cdots S^*_{r_2} U^*_{r_1} \tag{3}$$

for non-negative real numbers l_1, r_j. We say that a term in this form is a *word in reduced form*. Associated with A are its (left and right) lengths, $l(A)$, $r(A)$, where $l(A) = \sum_{i=1}^{2\mu} l_i$ and $r(A) = \sum_{j=1}^{2\nu} r_j$. As we shall see (Lemma 3.1) a polynomial P has one and only one expression as a linear combination of words in reduced form (where we agree to use the semigroup laws $U_t U_s = U_{t+s}$, $S_t S_s = S_{t+s}$ to combine the terms in A as much as possible), so that the length functions are well-defined on reduced words. We say that a word A is even if $l(A) = r(A)$. By $\Phi_0(P)$ we denote the summand of P consisting of linear combinations of all even words of P. P is said to be even if $\Phi_0(P) = P$. Let $\mathscr{P}_0$ be the subspace of all even polynomials in $\mathscr{P}$. Using the commutation relations (1) one sees that $\mathscr{P}_0$ is actually a $*$-subalgebra of $\mathscr{P}$.

DEFINITION 2.1. For $t > 0$, let F_t be the even polynomial

$$F_t = [U_t U_t^* + S_t S_t^* - e^{-\lambda t}(U_t S_t^* + S_t U_t^*)]/(1 - e^{-2\lambda t}).$$

Let $F_0 = I$. For $t \geq 0$, let $J_t = 1 - F_t$.

Using the commutation relations and the isometric properties of U and $\mathscr{S}$, Lemma 2.2.1 below is easily verified. The other assertions follow directly from 2.2.1.

LEMMA 2.2. *The operators F_t, J_t are projections in $\mathscr{P}$ satisfying the following identities, for $s \geq t \geq 0$;*
(1) $F_tU_s = U_s$, *and* $F_tS_s = S_s$,
(2) $J_tU_s = 0 = J_tS_s$,
(3) $F_tF_s = F_sF_t = F_s$, *and*
(4) $J_tJ_s = J_sJ_t = J_s$.

LEMMA 2.3. $J_t \neq 0$, *for* $t > 0$.

Proof. It suffices to show that for some isometry W in $\mathscr{P}$, $W^*F_tW \neq I$, since $I = F_t + J_t$. Let $W = U_{t/2}S_{t/2}$, then $W^*U_t = e^{-\lambda t/2}I = W^*S_t$, so

$$W^*F_tW = [e^{-\lambda t}(2 - 2e^{-\lambda t})/(1 - e^{-2\lambda t})]I \neq I. \qquad \square$$

We may now determine the structure of the algebra $C^*(U_t, S_t) = \mathfrak{A}_t$. We shall show below that this algebra is *not* simple. To see this, define positive numbers $a = a_t = \frac{1}{2}(\sqrt{1 + e^{-\lambda t}} + \sqrt{1 - e^{-\lambda t}})$ and $b = b_t = \frac{1}{2}(\sqrt{1 + e^{-\lambda t}} - \sqrt{1 - e^{-\lambda t}})$, and define operators

(4) $T_{t,1} = (aU_t - bS_t)/(a^2 - b^2)$ and $T_{t,2} = (aS_t - bU_t)/(a^2 - b^2)$.

$\mathfrak{A}_t$ is clearly generated as a C^*-algebra by the operators $T_{t,i}$, $i = 1, 2$, and it is straightforward to show that the $T_{t,i}$ are isometries which satisfy the following identities:

$$T^*_{t,1}T_{t,2} = 0 = T^*_{t,2}T_{t,1}, \tag{5.1}$$

$$T_{t,1}T^*_{t,1} + T_{t,2}T^*_{t,2} = F_t. \tag{5.2}$$

Hence we may apply **[11**, Proposition 3.1] to obtain the following result.

THEOREM 2.4. *For $t > 0$, let $\mathfrak{A}_t$ be the C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$ generated by the isometries U_t and S_t. Then the projection J_t generates a two-sided closed ideal in $\mathfrak{A}_t$ isomorphic to the C^*-algebra of compact operators $\mathscr{K}$, and $\mathfrak{A}_t/\mathscr{K}$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$.*

As one might suspect from this result, the Cuntz algebra $\mathcal{O}_2$ plays a significant role in understanding the structure of the C^*-algebras $\mathfrak{A}_\Gamma$.

III. Simplicity of $\mathfrak{A}_\Gamma$ for semigroups Γ. In this section we show that if Γ is a dense subgroup of the real numbers, then the C^*-algebra $\mathfrak{A}_\Gamma$ generated by the semigroups of isometries $\mathscr{U}_\Gamma$ and $\mathscr{S}_\Gamma$ is simple.

(Unless stated otherwise, we take $\Gamma = \mathbb{R}$ in this section.) Our main tool is to construct a conditional expectation from $\mathfrak{A}_\Gamma$ to the C^*-subalgebra of $\mathfrak{A}_\Gamma$ generated by the even polynomials $\mathscr{P}_0$. In order to show that this construction is well-defined, we need the following lemma.

LEMMA 3.1. *Any polynomial $P \in \mathscr{P}$ has a unique expression as a linear combination of words in reduced form.*

Proof. To prove the lemma it suffices to show that if $P = 0$ is a linear combination $\sum_{i=0}^q c_i A_i$ of words in reduced form, then each coefficient c_i must be 0. If not, let $l = \min_i\{lle(A_i), r(A_i)\}$. Without loss of generality we may assume $l = lle(A_i)$, for some i. Next let r $(\geq l)$ be the minimum length $r(A_j)$, where j ranges over all indices such that $l(A_j) = l$. We may assume $l(A_0) = l$ and $r = r(A_0)$. Using the semigroup properties $U_sU_t = U_{s+t}$, $S_sS_t = S_{s+t}$, we may construct partitions $\{0, l_1, l_1 + l_2, \dots, l_1 + \cdots + l_n\}$ of $[0, l]$ and $\{0, r_1, r_1+r_2, \dots, r_1+\cdots+r_m\}$ of $[0, r]$ such that every term A_i of P having lengths $l(A_i) = l$ and $r(A_i) = r$ may be written as a scalar multiple of a word of the form

$$W_{l_1, a_1} \cdots W_{l_n, a_n} W^*_{r_m, b_m} \cdots W^*_{r_1, b_1} \tag{6}$$

for $a_i, b_j \in \{1, 2\}$ and $W_{t,1} = U_t$, $W_{t,2} = S_t$, for any $t \geq 0$.

Now if A_k is any summand of P such that $l(A_k) > l$ or $r(A_k) > r$, then $C = X^*A_kY$ is a scalar multiple of a word in reduced form with $l(C) > 0$ or $r(C) > 0$, for X any word of the form $W_{l_1, a_1} \cdots W_{l_n, a_n}$ and Y any word of the form $W_{r_1, b_1} \cdots W_{r_m, b_m}$. Using Lemma 2.2.2, there is a positive scalar t_k sufficiently small such that $J_tC = 0$ or $CJ_t = 0$ for $0 < t \leq t_k$. Let t be the minimum of the lengths t_k, where k ranges over the summands of P such that $l(A_k) > l$ or $r(A_k) > r$.

Consider the operators $Z_{t,1} = U_t - e^{-\lambda t}S_t$ and $Z_{t,2} = S_t - e^{-\lambda t}U_t$. It is straightforward to show that the $Z_{t,i}$ are scalar multiples of isometries and satisfy $Z^*_{t,1}W_{t,2} = 0$, $Z^*_{t,2}W_{t,1} = 0$, and $Z^*_{t,i}W_{t,i} = (1 - e^{-\lambda t})I$. We may suppose that A_0 has the form (6). Let $X = Z_{l_1, a_1} \cdots Z_{l_n, a_n} J_t$, $Y = Z_{r_1, b_1} \cdots Z_{r_m, b_m} J_t$. Then X^*A_0Y is a non-zero scalar multiple of J_t, but $X^*A_jY = 0$ for all other j. But then $0 = X^*PY = X^*A_0Y$, a contradiction, which yields the result. □

Using the uniqueness result above, and following **[11]**, we note that if $\widetilde{\mathscr{U}} = \{\widetilde{U}_t : t \geq 0\}$ and $\widetilde{\mathscr{S}} = \{\widetilde{S}_t : t \geq 0\}$ are a pair of semigroups of isometries on a separable Hilbert space $\tilde{\mathfrak{H}}$ which satisfy (1), then the algebra $\tilde{\mathscr{P}}$ of polynomials in the operators in $\tilde{\mathscr{U}}$ and $\tilde{\mathscr{S}}$ is algebraically isomorphic to $\mathscr{P}$. Hence we may define a norm $\| \ \|_0$ on $\mathscr{P}$ by setting, for $P \in \mathscr{P}$,

$$\|P\|_0 = \sup\{\|\pi(P)\|: \pi \text{ is a separable representation of } \mathscr{P}\}.$$

We shall denote by $\mathscr{L}$ the C^*-algebra obtained by completing $\mathscr{P}$ in the $\| \ \|_0$-norm, and by $\mathscr{L}_0$ we shall denote the completion of the subalgebra $\mathscr{P}_0$ of even polynomials in $\mathscr{P}$, see **[11**, 1.9].

The result above also shows that there is a unique way of extending the mappings $U_t \to e^{i\gamma t}U_t$ and $S_t \to e^{i\gamma t}S_t$ to $*$-homomorphisms α_γ of $\mathscr{P}$, for all $\gamma \in \mathbb{R}$. We observe that $\alpha_\gamma(P) = P$ for all $\gamma \in \mathbb{R}$ if, and only if, $P \in \mathscr{P}_0$. We also note that the mappings α_γ are in fact $*$-automorphisms of $\mathscr{P}$, since clearly $\alpha_{-\gamma} \circ \alpha_\gamma = \iota = \alpha_\gamma \circ \alpha_{-\gamma}$. Moreover, if π is a separable $*$-representation of $\mathscr{P}$ then so is $\pi \circ \alpha_\gamma$, whence $\|P\|_0 = \|\alpha_\gamma(P)\|_0$ for all $p \in \mathscr{P}$. Hence there is a unique extension of α_γ (which we also denote by α_γ) to a $*$-automorphism of $\mathscr{L}$, and from the obvious group law $\alpha_\gamma \circ \alpha_{\gamma_0} = \alpha_{\gamma+\gamma_0}$ on $\mathscr{P}$, the family $\alpha = \{\alpha_\gamma: \gamma \in \mathbb{R}\}$ is a one-parameter group of automorphisms of $\mathscr{L}$. α is in fact a strongly continuous family; clearly $\|\alpha_\gamma(P) - P\|_0 \to 0$ as $\gamma \to 0$ for $P \in \mathscr{P}$ (note that $\alpha_\gamma(A) = \exp(i\gamma[l(A) - r(a)])A$ for reduced words A). For general X in $\mathscr{L}$, the convergence $\|\alpha_\gamma(X) - X\|_0 \to 0$ as $\gamma \to 0$ follows from the uniform density of $\mathscr{P}$ in $\mathscr{L}$. Summing up, we have:

LEMMA 3.2. *Let $\mathscr{L}$ be the C^*-algebra obtained as the completion of $\mathscr{P}$ in the norm $\| \ \|_0$. Then there exists a unique strongly continuous one-parameter group $\alpha = \{\alpha_\gamma: \gamma \in \mathbb{R}\}$ of $*$-automorphisms on $\mathscr{L}$ defined by $\alpha_\gamma(U_t) = e^{it\gamma}U_t$ and $\alpha_\gamma(S_t) = e^{it\gamma}S_t$.*

THEOREM 3.3. *For any $X \in \mathscr{L}$, $\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\alpha_\gamma(X)\,d\gamma$ converges uniformly to an element $\Phi_0(X) \in \mathscr{L}_0$. The linear mapping $\Phi_0: \mathscr{L} \to \mathscr{L}_0$ is a conditional expectation from $\mathscr{L}$ to $\mathscr{L}_0$.*

Proof. If A is an even reduced word then $\alpha_\gamma(A) = A$, so $\Phi_0(A) = A$. If A is uneven, $\alpha_\gamma(A) = \exp(i\gamma[l(A) - r(A)])A$, so $\Phi_0(A) = 0$. Hence $\Phi_0(P)$ exists for $P \in \mathscr{P}$, $\Phi_0(P)$ is the sum of the even terms comprising P, so $\Phi_0(P) \in \mathscr{P}_0$. Since $\mathscr{P}$ is uniformly dense in $\mathscr{L}$ it is clear that $\Phi_0(X)$ exists for all $X \in \mathscr{L}$, and moreover,

$$\begin{aligned}\|\Phi_0(X)\|_0 &= \lim_{T\to\infty}(2T)^{-1}\left\|\int_{-T}^{T}\alpha_\gamma(X)\,d\gamma\right\|_0\\ &\le \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\|\alpha_\gamma(X)\|\,d\gamma = \|X\|_0.\end{aligned}$$

Clearly Φ_0 preserves positivity.

Now suppose (P_n) is a sequence of polynomials converging uniformly to X. Then $\|\Phi_0(X)-\Phi_0(P_n)\|_0 \le \|X-P_n\|_0$, so $\Phi_0(X)$ is the uniform limit of even polynomials of $\mathscr{P}$. Hence $\Phi_0(X) \in \mathscr{L}_0$. Conversely, if $X \in \mathscr{L}_0$, then since $X = \lim_{n\to\infty} P_n$ for a sequence of even polynomials, $\Phi_0(X) = \lim_{n\to\infty}\Phi_0(P_n) = \lim_{n\to\infty} P_n = X$, so that Φ_0 is surjective and $\Phi_0 \circ \Phi_0 = \Phi_0$. Hence Φ_0 is a conditional expectation on γ. □

Using some elementary results on almost periodic functions we show (see also **[12]**) that the mapping Φ_0 is one-to-one on the positive elements. We shall assume $\mathscr{L}$ to be unitally embedded in $\mathfrak{B}(\mathfrak{H}')$ for some Hilbert space $\mathfrak{H}'$. If $P \in \mathscr{P}$ is written as a linear combination of reduced words, $P = \sum_{j=1}^{q} c_j A_j$, then from the expression $\alpha_\gamma(P) = \sum_{j=1}^{q} c_j e^{i\gamma\xi_j} A_j$, where $\xi_j = l(A_j) - r(A_j)$, it is clear that the mapping $\gamma \to (\alpha_\gamma(P)f, g)$ is an almost periodic function of γ, for any $f, g \in \mathfrak{H}'$. For $X \in \mathscr{L}$, consider the function $\varphi(\gamma) = (\alpha_\gamma(X)f, g)$; and define $\varphi_m(\gamma) = (\alpha_\gamma(P_m)f, g)$ for some sequence of polynomials $\{P_m\}$ converging uniformly to X. Then for $\gamma \in \mathbb{R}$,

$$\begin{aligned}|\varphi(\gamma) - \varphi_m(\gamma)| &= |(\alpha_\gamma(X)f, g) - (\alpha_\gamma(P_m)f, g)|\\ &\le \|\alpha_\gamma(X - P_m)\|_0\|f\|\,\|g\|,\end{aligned}$$

so that φ is the uniform limit of a sequence of almost periodic functions. Hence φ is itself almost periodic, [**5**, Theorem 49.V]. Now if X is a non-zero positive element of $\mathscr{L}$ we may choose a vector $f = g$ in $\mathfrak{H}'$ such that $\varphi(0) = (Xf, f) > 0$. But then $\varphi(\gamma)$ is a non-negative, almost periodic function which is not identically equal to 0, so that its mean, $\mathfrak{M}(\varphi)$, is strictly positive, [**5**, Theorem 72]. But

$$\begin{aligned}\mathfrak{M}(\varphi) &= \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\varphi(\gamma)\,d\gamma\\ &= \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}(\alpha_\gamma(X)f, f)\,d\gamma = (\Phi_0(X)f, f),\end{aligned}$$

so that $\Phi_0(X)$ is a non-zero positive element of $\mathscr{L}_0$. Hence we have established the following (cf. [**12**, Proposition 2]).

PROPOSITION 3.4. *The condition expectation* $\Phi_0\colon \mathscr{L} \to \mathscr{L}_0$ *is one-to-one on the positive elements of* $\mathscr{L}$.

As in the previous section let $\mathscr{U}$ and $\mathscr{S}$ be a pair of semigroups of isometries acting on the Hilbert space $\mathfrak{H}$, and let $\mathfrak{A}$ be the C^*-algebraic completion of $\mathscr{P}$ in $\mathfrak{B}(\mathfrak{H})$. We shall show that the completion of $\mathscr{P}_0$ in $\mathfrak{B}(\mathfrak{H})$ is isometrically $*$-isomorphic to the completion $\mathscr{L}_0$ of $\mathscr{P}_0$ in $\mathscr{L}$. To begin this, suppose $P = \sum_{j=1}^{q} d_j A_j$ is the unique decomposition of an even polynomial P in $\mathfrak{A}$ into a sum of (even) terms in reduced form. Let $L = \max\{l(A_j)\colon 1 \le j \le q\}$ $(= \max\{r(A_j)\colon 1 \le j \le q\})$. For each j, if A_j has the form (3), then let R_j be the partition of $[0, L]$ formed as the union of the partitions

$$\{0, L-(l_1+\cdots+l_{2\mu-1}), L-(l_1+\cdots+l_{2\mu-2}), \dots, L-l_1, L\} \quad \text{and}$$
$$\{0, L-(r_1+\cdots+r_{2\nu-1}), L-(r_1+\cdots+r_{2\nu-2}), \dots, L-r_1, L\}.$$

Let R be the union of all of the partitions R_j, $1 \le j \le q$. Then there are positive real numbers $c_1, c_2, \dots, c_n$, for some n, such that

$$R = \{0, L-(c_1+\cdots+c_{n-1}), L-(c_1+\cdots+c_{n-2}), \dots, L-c_1, L)$$

(and $0 = L-(c_1+\cdots+c_n)$). Then clearly any A_j may be written in the form

$$A_j = W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W^*_{c_{k_j}, b_{k_j}} \cdots W^*_{c_1, b_1}$$

where $a_i, b_i \in \{1, 2\}$ depend on A_j, for $1 \le i \le k_j$, where $k_j \le n$ satisfies $\sum_{i=1}^{k_j} c_i = l(A_j)$ $(= r(A_j))$, and as above, $W_{t,1} = U_t$, $W_{t,2} = S_t$. If $k_j < n$, then we may rewrite A_j as

$$\begin{aligned} A_j = {} & W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} J_{c_{k_j+1}} W^*_{c_{k_j}, b_{k_j}} \cdots W^*_{c_1, b_1} \\ & + W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} F_{c_{k_j+1}} W^*_{c_{k_j}, b_{k_j}} \cdots W^*_{c_1, b_1} \end{aligned}$$

From Definition 2.1, the second term above may be rewritten as a linear combination of four terms, each of the form

$$W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j+1}, a_{k_j+1}} W^*_{c_{k_j+1}, b_{k_j+1}} W^*_{c_{k_j}, b_{k_j}} \cdots W^*_{c_1, b_1}.$$

If $k_j + 1 = n$ we do nothing; otherwise, we rewrite each of the four terms as the sum of two terms

$$\begin{aligned} & W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j+1}, a_{k_j+1}} J_{c_{k_j+2}} W^*_{c_{k_j+1}, b_{k_j+1}} W^*_{c_{k_j}, b_{k_j}} \cdots W^*_{c_1, b_1} \\ & \quad + W_{c_1, a_1} \cdots W_{c_{k_j}, a_{k_j}} W_{c_{k_j+1}, a_{k_j+1}} F_{c_{k_j+2}} W^*_{c_{k_j+1}, b_{k_j+1}} W^*_{c_{k_j}, b_{k_j}} \cdots W^*_{c_1, b_1}. \end{aligned}$$

Continuing this process, we may rewrite P as a linear combination of terms each of which takes one of the following three forms:

$$J_{c_1} \tag{7.1}$$

$$W_{c_1,a_1}\cdots W_{c_r,a_r}J_{c_{r+1}}W^*_{c_r,b_r}\cdots W^*_{c_1,b_1}, \qquad 0<r<n, \tag{7.2}$$

$$W_{c_1,a_1}\cdots W_{c_n,a_n}W^*_{c_n,b_n}\cdots W^*_{c_1,b_1}. \tag{7.3}$$

Using the identities (4), we may further decompose (7.2) and (7.3) so that P may be rewritten as a linear combination of terms, each of which takes one of the following three forms:

$$J_{c_1}, \tag{8.1}$$

$$T_{c_1,a_1}\cdots T_{c_r,a_r}J_{c_{r+1}}T^*_{c_r,b_r}\cdots T^*_{c_1,b_1}, \qquad 0<r<n, \tag{8.2}$$

$$T_{c_1,a_1}\cdots T_{c_n,a_n}T^*_{c_n,b_n}\cdots T^*_{c_1,b_1}. \tag{8.3}$$

Note that any two distinct terms above (with either the same or different forms) have product 0; this follows from Lemma 2.2.2. Using the commutation relations and (5) shows that for fixed r, $0\le r\le n$, the 4^r terms in $\mathscr{P}$ having the form (8.1) if $r=0$, (8.2) if $0<r<n$, and (8.3) if $r=n$, are matrix units for a $2^r\times 2^r$ matrix subalgebra $\mathfrak{M}_r$ of $\mathscr{P}$. Since $BC=0$ for any elements $B\in\mathfrak{M}_r$ and $C\in\mathfrak{M}_{r_0}$, for $r\ne r_0$, the totality of terms of the form in (8) are matrix units for a finite-dimensional C^*-subalgebra of $\mathscr{P}$. Since P lies in this algebra, we may reassemble P as a sum of polynomials $\sum_{r=0}^n P_r$, where $P_r\in\mathfrak{M}_r$. Since the subalgebras $\mathfrak{M}_r$ are mutually orthogonal, it is now clear that $\|P\|=\max\{\|P_r\|: 0\le r\le n\}$. Hence we have:

Proposition 3.5. *Let $\mathscr{U}$ and $\mathscr{S}$ be a pair of one-parameter semigroups of isometries on a Hilbert space $\mathfrak{H}$, satisfying* (1). *Let $\mathscr{P}$ be the algebra of polynomials in these isometries. If $P\in\mathscr{P}_0$ there is a finite-dimensional C^*-subalgebra of $\mathscr{P}$ containing P.*

Using the decomposition of P above we see that for any even polynomial P, $\|P\|$ is the same in any representation of the semigroups $\mathscr{U}$ and $\mathscr{S}$, by the uniqueness of the C^*-algebraic norm on finite-dimensional matrix algebras. In particular, if $\mathfrak{A}$ is the C^*-algebraic completion of $\mathscr{U}$ and $\mathscr{S}$ in $\mathfrak{B}(\mathfrak{H})$, as above, with norm $\|\ \ \|$, then for all $P\in\mathscr{P}_0$, $\|P\|=\|P\|_0$ (cf. **[11**, 1.9]). This yields the following result.

Theorem 3.6. *Let $\mathscr{U}$ and $\mathscr{S}$ be a pair of one-parameter semigroups of isometries on $\mathfrak{B}(\mathfrak{H})$ satisfying the commutation relations* (1), *and let*

$\mathfrak{A}$ be the C^*-algebra obtained as the uniform closure of the polynomial algebra $\mathscr{P}$ in the isometries U_t, S_t, $t \geq 0$. Let $\mathfrak{A}_0$ be the C^*-subalgebra of $\mathfrak{A}$ obtained as the completion of the even polynomials $\mathscr{P}_0$ in the norm. Then there exists a $*$-isometric isomorphism from $\mathfrak{A}_0$ to $\mathscr{L}_0$.

THEOREM 3.7. *Let $\mathscr{U}$, $\mathscr{S}$, and $\mathfrak{A}$ be as above. For any element $P \in \mathscr{P}$ $(\subset \mathfrak{A})$ there exists a projection $Q \in \mathscr{P}_0$, depending on P, such that $QPQ \in \mathscr{P}_0$ and $\|QPQ\| = \|\Phi_0(P)\|$.*

Proof. Let $P = \sum_{j=1}^{q} d_j A_j$ be a decomposition of P into a linear combination of words in reduced form. If $\Phi_0(P) = 0$, then we may choose $Q = 0$. Hence, we may assume $P \neq 0$ and that there are even reduced words A_j in the decomposition of P. Let $L \geq 0$ be the maximum length $(L = l(A) = r(A))$ among all of the even words. Note that if $\Phi_0(P)$ is just a scalar multiple of I, then $L = 0$. First suppose $L > 0$. For each reduced word (even or uneven) A_j, form a partition R_j of $[0, L]$ as follows: if A_j has the form (3), let $n_j + 1$ be the first index such that $\sum_{i=1}^{n_j+1} l_i \geq L$, let $m_j + 1$ be the first index such that $\sum_{i=1}^{m_j+1} r_i \geq L$, and set R_j to be the union of the partitions $\{0, L - (l_1 + \cdots + l_{n_j}), \ldots, L - l_1, L\}$ and $\{0, L - (r_1 + \cdots + r_{m_j}), \ldots, L - r_1, L\}$. Let $R = \{0, L - (c_1 + \cdots + c_{n-1}), \ldots, L - c_1, L\}$ be the union of these partitions, and let $c_n = L - (c_1 + \cdots + c_{n-1})$. As in the proof of Proposition 3.5, each of the even terms may be decomposed into a linear combination of terms of the form (7), which in turn may be rewritten as a linear combination of the terms appearing in (8).

Suppose $A = A_j$ is an uneven term in the decomposition of P. If $l(A) \geq L$ and $r(A) \geq L$, A may be rewritten in the form

$$W_{c_1, a_1} \cdots W_{c_n, a_n} W V^* W^*_{c_n, b_n} \cdots W^*_{c_1, b_1} \tag{9.1}$$

where W and V are words in reduced form such that $l(W) > 0$ or $l(V) > 0$, and $r(W) = r(V) = 0$. If $l(A) < L$ (respectively, $r(A) < L$), $l(A) = \sum_{i=1}^{k_j} c_i$ (resp., $r(A) = \sum_{i=1}^{k_j} c_i$) for some $k_j < N$, then by using a procedure similar to that used in the proof of Theorem 3.6, we may decompose A into a linear combination of terms taking one of the forms below (where W is a reduced word with $l(W) > 0$

and $r(W) = 0$)

(9.2) $\qquad J_{c_1} W^*, \quad \text{if } l(A) = 0,$

(9.2′) $\qquad W J_{c_1}, \quad \text{if } r(A) = 0,$

(9.3) $\qquad W_{c_1,a_1} \cdots W_{c_r,a_r} J_{c_{r+1}} W^* W^*_{c_r,b_r} \cdots W^*_{c_1,b_1}, \qquad 0 < r < n,$

(9.3′) $\qquad W_{c_1,a_1} \cdots W_{c_r,a_r} W J_{c_{r+1}} W^*_{c_r,b_r} \cdots W^*_{c_1,b_1}, \qquad 0 < r < n.$

From the proof of Proposition 3.5, $\Phi_0(P)$ decomposes into a sum $\sum_{r=0}^{n} P_r$ of even polynomials, where each P_r is in turn a linear combination of terms each of which has the form of one of the elements in (8). Also we have shown that $\|\Phi_0(P)\| = \max \|P_r\|$. Choose r such that $\|\Phi_0(P)\| = \|P_r\|$. If $r = 0$, set $Q = Q_0 = J_{c_1}$. If $0 < r < n$, set

$$Q = Q_r = \sum_{a_1,\dots,a_1=1} T_{c_1,a_1} \cdots T_{c_r,a_r} J_{c_{r+1}} T^*_{c_r,a_r} \cdots T^*_{c_1,a_1}.$$

Then it is clear, using the relations (5), that Q_r is a projection. It is also straightforward to show, appealing to Lemma 2.2.2 (and recalling that $T_{t,1}$ is a linear combination of U_t and S_t) that if B is any term in (9) arising from the decomposition of an uneven reduced term in the expression for P, that $QBQ = 0$. Hence $QA_jQ = 0$ for all uneven terms A_j. Using the argument establishing that $P_r P_{r_0}$ for $r \neq r_0$ in the proof of the proposition above, we also conclude that $Q_r P_{r_0} Q_r = 0$ for $r \neq r_0$. Finally, if B is any term in the decomposition of P_r, then it is easy to see, using (5), that $Q_r B Q_r = B$, whence $Q_r P_r Q_r = P_r$. Assembling these equations we obtain $Q_r P Q_r = P_r$.

Now suppose $r = n$. Then we modify an argument in **[11]** to show that there is a projection $Q_n \in \mathscr{P}_0$ such that $\|Q_n P Q_n\| = \|P_n\|$. Consider the matrix units (8.3) constructed in the proof of the proposition for the $2^n \times 2^n$ matrix algebra $\mathfrak{M}_n$ For any $\varepsilon > 0$ it is straightforward to verify that if

$$Q = \sum_{e_1,\dots,e_n=1}^{2} T_{c_1,e_1} \cdots T_{c_n,e_n} J_\varepsilon T^*_{c_n,e_n} \cdots T^*_{c_1,e_1},$$

then Q is a projection in $\mathscr{P}_0$, and the mapping $D \to QDQ$ on $\mathfrak{M}_n$ is an isomorphism from $\mathfrak{M}_n$ to another matrix subalgebra, $Q\mathfrak{M}_nQ$, of $\mathscr{P}$. It is also easy to verify that if B is any even term of the form in (8.1) or (8.2), then $QBQ = 0$ by using Lemma 2.2.2. Now suppose B is one of the terms of the form in (9) arising from the decomposition of an uneven term A_j of P. It is clear, again from Lemma 2.2.2, that for any term B of the form in (9.2), (9.3), (9.2′), or (9.3′), $QBQ = 0$. Suppose B is a term of the form (9.1). We have

$$
\begin{aligned}
J_\varepsilon T^*_{c_n,e_n} \cdots T^*_{c_1,e_1} B T_{c_1,e_1}, \cdots T_{c_n e_n}, J_\varepsilon
&= J_\varepsilon T^*_{c_n,e_n} \cdots T^*_{c_1,e_1} W_{c_1,a_1} \cdots W_{c_n,a_n} W V^* W^*_{c_n,b_n} \cdots W^*_{c_1,b_1} T_{c_1,e_1}, \\
&\qquad \cdots T_{c_n,e_n}, J_\varepsilon \\
&= \gamma J_\varepsilon W V^* J_\varepsilon,
\end{aligned}
$$

where γ is some scalar whose value is determined by (4) and the commutation relations (1). Since $l(W) > 0$ or $l(V) > 0$, we may use Lemma 2.2 to prescribe a value of ε sufficiently small such that $J_\varepsilon W V^* J_\varepsilon = 0$. But this shows that there is an $\varepsilon > 0$ small enough so that, choosing $Q = Q_n$ of the form indicated above, $Q_n B Q_n = 0$. Combining all of these results shows that $Q_n A_j Q_n = 0$ for all uneven terms in the decomposition of P; that $Q_n P_r Q_n = 0$ for $0 \le r < n$; and, since $D \to Q_n D Q_n$ is an isomorphism on $\mathfrak{M}_n$, $\|Q_n P_n Q_n\| = \|P_n\|$. □

Corollary 3.8. *If $P \in \mathscr{P}$, $\|\Phi_0(P)\| \le \|P\|$.*

Proof. This is clear since $\|\Phi_0(P)\| = \|QPQ\|$ for some projection Q. □

Using the results above allows us to prove that $\mathscr{L}$ is simple. We show in fact that $\mathscr{L}$ is simple in the very strong sense that the Cuntz algebras $\mathcal{Q}_0$ are simple. The proof of the following theorem uses some techniques in [**11**, Theorem 1.13].

Theorem 3.9. *For any non-zero element X of $\mathscr{L}$ there exist A, $B \in \mathscr{L}$ such that $AXB = I$.*

Proof. We may assume without loss of generality that $X > 0$; for if there are A', $B' \in \mathscr{L}$ such that $A'X^*XB' = I$ we simply take $A = A'X^*$, $B = B'$. Hence $\Phi_0(X)$ is a positive (non-zero, by Proposition 3.4) element of $\mathscr{L}_0$. We may assume without loss of generality that $\|\Phi_0(X)\| = 1$.

For positive $\varepsilon \le 1/4$, let $P \in \mathscr{P}$ be a self-adjoint polynomial such that $\|X - P\|_0 < \varepsilon$. By Theorem 3.3, $\|\Phi_0(X - P)\|_0 < \varepsilon$, so $1 + \varepsilon > \|\Phi_0(P)\|_0 > 1 - \varepsilon$. Let Q be a projection in $\mathscr{P}_0$ such that $QPQ \in \mathscr{P}_0$ and $\|QPQ\|_0 = \|\Phi_0(P)\|_0$. From the proof of the preceding theorem, either $QPQ = \gamma J_c$, for some $c > 0$; or there are positive real numbers $c_1, c_2, \ldots, c_r, c$, such that QPQ is a self-adjoint operator in the $2^r \times 2^r$ matrix algebra $\mathfrak{M}$ generated by matrix units of the form $T_{c_1,a_1}, \cdots T_{c_r,a_r} J_c T^*_{c_r,b_r} \cdots T^*_{c_1,b_1}$. Let $\sum_{k=1}^q \gamma_k E_k$ be the spectral decomposition of QPQ in $\mathfrak{M}$, where the E_k are rank one orthogonal projections in $\mathfrak{M}$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_q$. From the inequalities above, $\gamma_1 > 1-\varepsilon$, and $\|QPQ\| = \gamma_1$. Let $V \in \mathfrak{M}$ be a partial isometry such that

$$VV^* = E_1 \quad \text{and} \quad V^*V = E_1' = T_{c_1,1} \cdots T_{c_r,1} J_c T^*_{c_r,1} \cdots T^*_{c_1,1}.$$

Setting $W = T_{c_1,1} \cdots T_{c_r,1}$, we have $W^*V^*QPQVW = \gamma_1 W^* E_1' W = \gamma_1 J_c$. Finally define $Y_i = Z_{c/2,i}/\sqrt{1-e^{-\lambda c}}$, $i = 1, 2$, where $Z_{t,i}$ is defined as in Lemma 3.1. Then Y_1 and Y_2 are isometries satisfying $Y_2^* Y_1^* F_c Y_1 Y_2 = 0$, so setting $Y = Y_1 Y_2$, $Y^* J_c Y = I$. Hence $Y^*W^*QPQVW = \gamma_1 I$. Let $D = QVW$. Then $\|D\|_0 \leq 1$, so

$$\begin{aligned}\|D^*XD - I\|_0 &\leq \|D^*XD - D^*PD\|_0 + \|D^*PD - I\|_0 \\ &\leq \|X - P\|_0 + \|\gamma_1 I - I\|_0 < 2\varepsilon,\end{aligned}$$

so D^*XD is invertible, and we are done. □

COROLLARY 3.10. *$\mathscr{L}$ is a simple C^*-algebra.*

We may now prove the following uniqueness result.

COROLLARY 3.11. *Let $\mathscr{U}$ and $\mathscr{S}$ be a pair of one-parameter semigroups of isometries acting on a separable Hilbert space $\mathfrak{H}$ and satisfying the commutation relations* (1). *Let $\mathfrak{A} \subset \mathfrak{B}(\mathfrak{H})$ be the C^*-algebraic completion of the polynomial $*$-algebra $\mathscr{P}$ in the operators U_t, S_t, $t \geq 0$. Then $\mathscr{L}$ and $\mathfrak{A}$ are isomorphic.*

Proof. From the definition of $\mathscr{L}$ it follows that $\mathfrak{A}$ must be a quotient of $\mathscr{L}$, i.e., $\mathfrak{A} = \pi(\mathscr{L}) \cong \mathscr{L}/\ker(\pi)$, for some representation π. But $\ker(\pi) = 0$. □

Suppose Γ is a subgroup of $\mathbb{R}$, and $\mathscr{U}_\Gamma = \{U_t : t \in \Gamma^+\}$, $\mathscr{S}_\Gamma = \{S_t : t \in \Gamma^+\}$ are semigroups of isometries on a Hilbert spaces $\mathfrak{H}$ which satisfy the commutation relations

$$S_t^* U_t = e^{-\lambda t} I, \qquad t \in \Gamma^+.$$

Then we may consider the polynomial $*$-algebra $\mathscr{P}_\Gamma$ generated by the operators U_t, S_t, $t \in \Gamma^+$, and we define $\mathfrak{A}_\Gamma$ to be the C^*-algebraic completion of $\mathscr{P}_\Gamma$ in the norm on $\mathfrak{B}(\mathfrak{H})$. It is easy to see that the techniques used to prove the results above for the case $\Gamma = \mathbb{R}$ may be applied virtually without change to show that $\mathfrak{A}_\Gamma$ is a simple C^*-algebra, if Γ is dense in $\mathbb{R}$. Combining Theorem 2.4 with these observations, we arrive at the following extension of the results above.

THEOREM 3.12. *Let Γ be a subgroup of $\mathbb{R}$ with corresponding C^*-algebra $\mathfrak{A}_\Gamma$. If Γ is discrete, $\mathfrak{A}_\Gamma$ contains a maximal closed two-sided ideal isomorphic to the C^*-algebra of compact operators $\mathscr{K}$, and $\mathfrak{A}_\Gamma/\mathscr{K}$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$. If Γ is dense in $\mathbb{R}$, then $\mathfrak{A}_\Gamma$ is a simple C^*-algebra, and the C^*-algebra generated by pairs of semigroups of isometries $\mathscr{U}_\Gamma$, $\mathscr{S}_\Gamma$ acting on a Hilbert space is canonically unique.*

It would be interesting to obtain necessary and sufficient conditions on a pair of dense semigroups Γ^+, Γ_0^+ of $\mathbb{R}^+$ for the corresponding C^*-algebras $\mathfrak{A}_\Gamma$, $\mathfrak{A}_{\Gamma_0}$ to be isomorphic. In the situation where $\mathfrak{B}_\Gamma$, $\mathfrak{B}_{\Gamma_0}$ are the C^*-algebras generated by single one-parameter semigroups $\mathscr{U}_\Gamma$, $\mathscr{U}_{\Gamma_0}$ of isometries, R. G. Douglas has shown in **[12]** that $\mathfrak{B}_\Gamma$ and $\mathfrak{B}_{\Gamma_0}$ are isomorphic if and only if Γ and Γ_0 are order isomorphic. We suspect that the isomorphism classes of algebras $\mathfrak{A}_\Gamma$ are also determined by order isomorphism classes of semigroups.

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