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L-HARMONIC FUNCTIONS AND THE EXPONENTIAL SQUARE CLASS

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It is proved for a restricted class of second order linear differential operators L if $Lu = 0$ in $\mathbf{R}_+^{d+1}$, $u|_{\mathbf{R}^d} = f$ then if the Lusin area integral of u, $Su \in L^\infty$, f is in the exponential square class. This extends the work of Chang, Wilson and Wolff who proved the same result for harmonic u [3].

1. Introduction. Let

$$L = \sum_{i,j=1}^{d+1} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

be a second order differential operator in divergence form whose coefficients a_{ij} are bounded and measurable functions on $\mathbf{R}_+^{d+1}$, $a_{ij} = a_{ji}$. L is strictly elliptic, i.e., $\exists\lambda > 0$ such that

$$\frac{1}{\lambda}|\xi|^2 \le \sum_{i,j=1}^{d+1} \xi_i a_{ij}\xi_j \le \lambda|\xi|^2.$$

Then if u is a function where $Lu = 0$ in $\mathbf{R}_+^{d+1}$, $u|_{\mathbf{R}^d} = f$, u is said to be the L-harmonic extension of f. (Note: In what follows the summation convention will be used. Sums are $i, j = 1, 2, \dots, d+1$ unless otherwise indicated.)

As in the case $L = \Delta =$ the Laplacian there is a measure associated with L, called L-harmonic measure, written $d\omega$.

There has been a considerable body of work in the last 30 years on the extension of results for harmonic functions to L-harmonic functions. The purpose of this paper is to extend a recent result of Chang, Wilson, Wolff, to the L-harmonic case.

Let u be a harmonic (or L-harmonic) function; let

$$\Gamma_\alpha(x) = \{(y, t) \in \mathbf{R}_+^{d+1} \mid |x - y| < \alpha t\}$$

be the cone in $\mathbf{R}_+^{d+1}$ over $x \in \mathbf{R}^d$ of aperture α;

$$u^*(x) = \sup_{(y,t)\in\Gamma_\alpha(x)} |u(y, t)|$$

be the non-tangential maximal function of f;

$$S_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(y, t)|^2 t^{1-d}\, dy\, dt\right)^{1/2}$$

be the Lusin area integral of f.

In 1971 Burkholder and Gundy proved for harmonic u and $0 < p < \infty$

$$\|u^*\|_p \sim \|Su\|_p. \qquad [1]$$

If $p = \infty$, the correspondence is false so the question arose if $Su \in L^\infty$ was there some class that f was in? Recently Chang, Wilson and Wolff proved the following result. Let $f_Q = \frac{1}{|Q|}\int_Q f(y)\, dy$. Then

THEOREM 3.2 [3]. *Suppose $S_\gamma f \in L^\infty$. Then*

$$\sup_{Q:\text{cube}} \frac{1}{|Q|}\int_Q \exp\left[c_1 \frac{|f - f_Q|^2}{\|S_\gamma f\|_\infty^2}\right] < c_2$$

where $c_1 > 0$ and $c_2 < \infty$ depend on d and γ.

The purpose of the present paper is to prove the following extension of their result:

THEOREM 1. *If L is as above with $a_{d+1,d+1} \equiv 1$ and $a_{d+1,j} = 0$ for $j \neq d+1$ and surface measure is absolutely continuous with L-harmonic measure and if $S_\gamma u \in L^\infty$ where $u|_{\mathbf{R}^d} = f$, $f \in L^2(\mathbf{R}^d)$, $Lu = 0$ and $\|u(y, t)\|_{L^2(dy)} < c$ as $t \to \infty$, then there are constants c_1 and c_2 not depending on Q or f so that*

$$\frac{1}{|Q|}\int_Q \exp \frac{c_1|f(x) - f_Q|^2}{\|S_\gamma f\|_\infty^2}\, dx < c_2$$

for all cubes Q.

Note. If the function f in Theorem 1 is smooth then the condition that surface measure be absolutely continuous with L-harmonic measure is unnecessary since the identity in Lemma 1 will automatically hold with respect to surface measure. However it does not seem trivial to prove that one can find functions in Schwartz class with uniformly bounded area integrals which converge to any L^2 function whose area integral is bounded.

The proof of Theorem 1 follows the same general outline as the Chang, Wilson, Wolff proof, but differs from it in detail and method—necessarily since the kernel for general L is not translation invariant and several techniques which can be used with the Poisson kernel are not applicable here. The idea is to get a decomposition for $f(x)$ in terms of integrals involving $\partial u/\partial y_i$ over a cone in $\mathbf{R}_+^{d+1}$, split each of these into two parts, one close to the boundary $\Lambda(x) - \Lambda_Q$ and one farther away $\Omega(x) - \Omega_Q$. For technical convenience we replace f_Q by $f_{\tilde{Q}}$, $\tilde{Q} = 9Q$. Then by Jensen's inequality $|f - f_{\tilde{Q}}| \in \exp L^2$ implies $|f - f_Q| \in \exp L^2$. Also by the proof of Lemma 2 $\Lambda_{\tilde{Q}} = 0$ so $f_{\tilde{Q}} = \Omega_{\tilde{Q}}$. Then the following adaptation of Lemma 3.3 from Chang, Wilson, Wolff can be used on $\Lambda(x)$.

LEMMA 3.3$'$ [3]. *If* $\Lambda(x)$ *has the decomposition*

$$\Lambda(x) = \sum_{l(\tilde{Q}) \leq l(Q)} \lambda_{\tilde{Q}}(x)$$

where the $\lambda_{\tilde{Q}}$ *satisfy*

(a) $\lambda_{\tilde{Q}}$ *is supported on* $3\tilde{Q}$,
(b) $\int \lambda_{\tilde{Q}} = 0$,
(c) $\|\lambda_{\tilde{Q}}\|^2_{\mathrm{Lip}\,\alpha} l^{2\alpha}(\tilde{Q}) \leq C \int_{T_{\tilde{Q}}} |\nabla u(y, t)|^2 t^{1-d}\, dy\, dt$ *for some* α, $0 < \alpha < 1$,

then

$$\frac{1}{|Q|} \int_Q \exp \frac{c_1 |\Lambda(x)|^2}{\|S_\gamma f\|^2_\infty} < c_2 < \infty.$$

$\Omega(x) - \Omega_{\tilde{Q}}$ is shown to be in exponential square class separately (Lemma 3).

The proof of Lemma 3.3$'$ is identical for L-harmonic u as the proof of Lemma 3.3 in [3] for harmonic u.

Sketch of proof of Lemma 3.3$'$. Property (a) allows one to write $\sum \lambda_{\tilde{Q}}$ as a finite sum of sums of the form $\sum_{Q'} \lambda_{Q'}$ where each of these sums is such that the supports of $\lambda_{Q'}$ are disjoint for cubes of the same length. Then it suffices to show each $\sum_{Q'} \lambda_{Q'}$ is exponentially square integrable, and writing $\sum_{Q'} \lambda_{Q'}$ as a dyadic martingale, properties (b) and (c) imply the dyadic square function of the martingale is bounded by Sf. The fundamental theorem of sequential analysis can be applied to show that any dyadic martingale whose dyadic square function is in L^∞ is exponentially square integrable [3].

To be able to use Lemma 3.3$'$ one needs to get the identity for $f(x)$ in terms of integrals of $\partial u/\partial y_i$ over the upper half plane (Lemma 1), then to divide each integral into two parts $\Lambda(x)$ and $\Omega(x)$ and show

Lemma 3.3′ can be applied to $\Lambda(x)$ (Lemma 2) and that $\Omega(x) \in \exp L^2$ (Lemma 3).

So the proof of Theorem 1 depends on the following three lemmas: since $f - f_{\tilde{Q}} = (f + c) - (f + c)_{\tilde{Q}}$ in what follows it suffices to take $f_{\tilde{Q}} = 0$.

LEMMA 1. *For f and u as in Theorem 1 and $K(y)$ a smooth function of compact support in $\mathbf{R}^d$, $K_t(y) = t^{-d}K(y/t)$ then a.e. with respect to L-harmonic measure $d\omega$,*

$$
\begin{aligned}
(1.1) \qquad f(x) = &\int_{\mathbf{R}_+^{d+1}} \frac{\partial u(y\,,\,t)}{\partial y_i} \underset{i\,,\,j\neq d+1}{a^{ij}}(y\,,\,t)\frac{\partial K_t(x-y)}{\partial y_j} t\,dy\,dt \\
&+ \int_{\mathbf{R}_+^{d+1}} \frac{\partial u(y\,,\,t)}{\partial t}\frac{\partial K_t(x-y)}{\partial t} t\,dy\,dt \\
&+ \int_{\mathbf{R}_+^{d+1}} \frac{\partial u(y\,,\,t)}{\partial y_j} \underset{j\neq d+1}{H_t^j}(x-y)\,dy\,dt,
\end{aligned}
$$

where

$$H_{j\,,\,t}(x-y) = \frac{x_j - y_j}{t^{d+1}} K\left[\frac{x-y}{t}\right]$$

and the integrals on the right exist as L^2 functions (see proof of Lemma 1).

Note. Surface measure being absolutely continuous with L-harmonic measure means the identity in Lemma 1 holds a.e. dx.

For future reference the integrals in (1.1) will be labeled:

$$
\begin{aligned}
\mathrm{I} &= \int_{\mathbf{R}_+^{d+1}} \frac{\partial u}{\partial y_i} \underset{i\,,\,j\neq d+1}{a^{ij}} \frac{\partial K_t}{\partial y_j} t\,dy\,dt \\
\mathrm{II} &= \int_{\mathbf{R}_+^{d+1}} \frac{\partial u}{\partial t}\frac{\partial K_t}{\partial t} t\,dy\,dt \\
\mathrm{III} &= \int_{\mathbf{R}_+^{d+1}} \frac{\partial u}{\partial y_j} \underset{j\neq d+1}{H_t^j}\,dy\,dt.
\end{aligned}
$$

Now write each integral I, II, III as $\int_R + \int_{\mathbf{R}_+^{d+1}\setminus R}$ where R is the "rectangle" in $\mathbf{R}_+^{d+1}$ with base $3Q$ in $\mathbf{R}^d$ of height $\frac{4}{\gamma}l(Q)$. Take K supported in $|x| < \frac{\gamma}{4}$. Subdivide R into smaller "rectangles" $T_{Q_n}^{i_n}$ where $Q_n^{i_n}$ are the dyadic cubes in $3Q$ of side length $2^{-n}l(Q)$ and

$$T_{Q_n}^{i_n} = Q_n^{i_n} \times \left[\frac{1}{2^{n+1}}l(Q)\frac{4}{\gamma}\,,\ \frac{1}{2^n}l(Q)\frac{4}{\gamma}\right] \qquad \text{(see Figure 1).}$$

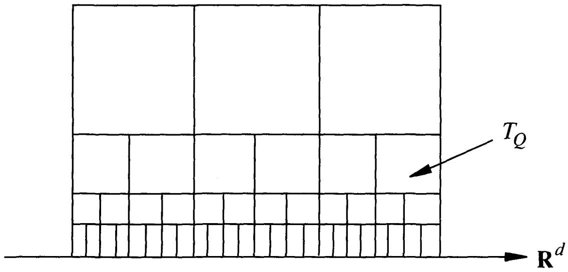

FIGURE 1

Let I_J be the integrand in I, II, III for $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$ respectively. Then

LEMMA 2. *For* $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$,

$$\int_R I_J = \sum_{n=0}^{\infty} \sum_{i_n} \int_{T_{Q_n}^{i_n}} I_J = \sum_{\tilde{Q} \text{ dyadic subdivisions of } Q} \lambda_{\tilde{Q}}$$

where the $\lambda_{\tilde{Q}}$*'s have the following properties*:

(a) *support* $\lambda_{\tilde{Q}} \subseteq 3\tilde{Q}$

(b) $\int_{\mathbf{R}^d} \lambda_{\tilde{Q}} = 0$

(c) $\|\lambda_{\tilde{Q}}\|^2_{\mathrm{Lip}\,\alpha}[l(\tilde{Q})]^{2\alpha} \le C \int_{T_{\tilde{Q}}} |\nabla u(y, t)|^2 t^{1-d}\, dy\, dt$ *for* $0 < \alpha < 1$.

And finally to deal with $\int_{\mathbf{R}^{d+1}_+ \setminus R}$:

LEMMA 3. *There are constants* b_1 *and* b_2 *depending only on* γ, L *and* d *so that if* $\Omega(x) = \int_{\mathbf{R}^{d+1}_+ \setminus R} I_j$ *then for* $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$

$$\frac{1}{|Q|} \int_Q \exp \frac{b_1 |\Omega(x) - \Omega_{\tilde{Q}}|^2}{\|Sf\|^2_\infty}\, dx < b_2.$$

Then Lemmas 2 and 3 imply the theorem.

Proof of Lemma 3. It suffices to show for any cube Q, fixed, with x, $x_0 \in Q$, $\exists c$ not depending on Q or $\Omega(x)$ such that

$$|\Omega(x) - \Omega(x_0)| \le c\|Sf\|_\infty.$$

Then, since $\exp|\Omega(x) - \Omega_Q|^2 \le \exp 2[|\Omega(x) - \Omega(x_0)|^2 + |\Omega(x_0) - \Omega_{\tilde{Q}}|^2]$ and $|\Omega(x_0) - \Omega_{\tilde{Q}}| \le c\|Sf\|_\infty$, Lemma 3 is true.

In the notation for I_J as defined above, let

$$G_j(y) = \begin{cases} \frac{\partial K(y)}{\partial y_j} t & \text{in I}, \quad j \ne d+1, \\ \frac{\partial}{\partial y_j}(y^j K(y)) t & \text{in II}, \quad j \ne d+1, \\ y_j K(y) & \text{in III}, \quad j \ne d+1 \end{cases}$$

(see (1.3) in the proof of Lemma 2 for why $(\partial/\partial y_j)(y^j K(y))$ appears in II). Then $G_j(y)$ is smooth since K is smooth and $\|G_j\|_\infty$, $\|\nabla G_j\|_\infty < \infty$.

Also let $G_{j,t}(y) = t^{-d} G_j(y/t)$.

Then

$$\Omega_J(x) = \int_{\mathbf{R}_+^{d+1}\setminus R} G_{j,t}(x-y) a_j^i(y,t) \frac{\partial u(y,t)}{\partial y_i}\, dy\, dt$$

where

$$a_j^i(y,t) = \begin{bmatrix} a^{ij}(y,t) & \text{in I,} \\ \delta_{ij} & \text{in II and III.} \end{bmatrix}$$

So for $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$

$$\begin{aligned}
&|\Omega_J(x) - \Omega_J(x_0)| \\
&\quad \le c \int_{\mathbf{R}_+^{d+1}\setminus R} |G_{j,t}(x-y) - G_{j,t}(x_0-y)| \left|\frac{\partial u(y,t)}{\partial y_j}\right| dy\, dt \\
&\quad \le c \int_{\mathbf{R}_+^{d+1}\setminus R} t^{-d} \left| G_j\left[\frac{x-y}{t}\right] - G_j\left[\frac{x_0-y}{t}\right]\right| |\nabla u(y,t)|\, dy\, dt \\
&\quad \le c \int_{\mathbf{R}_+^{d+1}\setminus R\cap\{(y,t):|y-x_0|<ct\}} t^{-d} \|\nabla G_j\|_\infty \left|\frac{x-x_0}{t}\right| |\nabla u(y,t)|\, dy\, dt \\
&\quad \le c|x-x_0| \left[\int_{\mathbf{R}_+^{d+1}\setminus R\cap\{(y,t):|y-x_0|<ct\}} |\nabla u(y,t)|^2 t^{1-d}\, dy\, dt\right]^{1/2} \\
&\qquad \times \left[\int_{\mathrm{cl}(Q)}^\infty \|\nabla G\|_\infty^2 t^{-3}\right]^{1/2} \\
&\quad \le c\|Sf\|_\infty t^{-1}|_{\mathrm{cl}(Q)}^\infty |x-x_0| \le c\|Sf\|_\infty
\end{aligned}$$

since $|x-x_0| \le l(Q)$ for x, $x_0 \in Q$. The last constant c depends only on K, d, $\|a_{ij}\|_\infty$ and γ.

Proof of Lemma 2. Wlog $\Lambda_Q = 0$. To prove: each of λ_Q^J, $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$ has properties (a), (b) and (c):

Property (a): K has compact support in $\mathbf{R}^d \Rightarrow$ support in y variable of $K_t(x-y)$ lies inside a cone of aperature $\frac{\gamma}{4}$ (since $\operatorname{supp} K(y) \subseteq \{|y| \le \frac{\gamma}{4}\}$), so support in x variable for $G_{j,t}(x-y)$ lies inside $3Q_n$ if $(y,t) \in T_{Q_n}$. Thus support (in x variable) for $\lambda_Q^J \subseteq 3Q$ for $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$ (see Figures 2 and 3).

Property (b): $J = I$, then

$$\lambda_Q^I = \int_{T_Q} \frac{\partial u}{\partial y_i} \underset{i,j\ne d+1}{a^{ij}} \frac{\partial K_t}{\partial y_j} t\, dy\, dt.$$

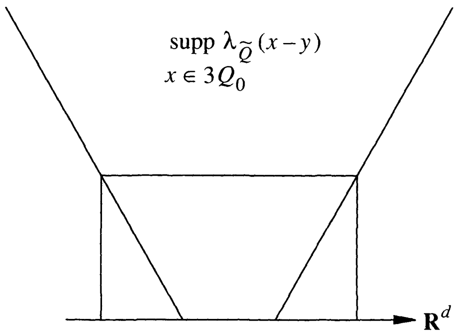

FIGURE 2

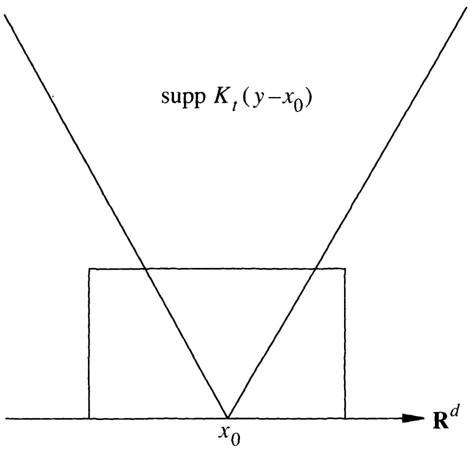

FIGURE 3

So

$$\begin{aligned}\int_{\mathbf{R}^d} \lambda_Q^I dx &= \int_{\mathbf{R}^d}\int_{T_Q} \frac{\partial u(y,t)}{\partial y_i} \underset{i,j\neq d+1}{a^{ij}}(y,t)\frac{\partial K_t(x-y)}{\partial(y_j)} t\, dy\, dt\, dx \\ &= \int_{T_Q} \frac{\partial u(y,t)}{\partial y_i} \underset{i,j\neq d+1}{a^{ij}}(y,t)\int_{\mathbf{R}^d}(-1)\frac{\partial K_t(x-y)}{\partial(x_j-y_j)}\, dx\, t\, dy\, dt\end{aligned}$$

by Fubini, and for $j \neq d+1$

$$\int_{\mathbf{R}^d} \frac{\partial K_t(x-y)}{\partial(x_j-y_j)}\, dx = \int_{\mathbf{R}^d} \frac{\partial K_t(x-y)}{\partial(x_j-y_j)} d(x-y) = 0$$

since K is of compact support.

$J = \mathrm{II}$:

$$\lambda_Q^{\mathrm{II}} = \int_{T_Q} \frac{\partial u}{\partial t}\frac{\partial K_t(x-y)}{\partial t} t\, dy\, dt.$$

But

$$\frac{\partial K_t(x-y)}{\partial t} = \frac{\partial}{\partial t}\left[t^{-d}K\left[\frac{x-y}{t}\right]\right] \tag{1.3}$$
$$= -dt^{-d-1}K\left[\frac{x-y}{t}\right] + t^{-d}\frac{\partial K[x-y/t]}{\partial t}$$
$$= -dt^{-d-1}K\left[\frac{x-y}{t}\right] - t^{-d-1}\sum_{j=1}^{d}\frac{x_j-y_j}{t}\frac{\partial K}{\partial v_j}(v)$$
$$= -t^{-d-1}\frac{\partial}{\partial v_j}(v^jK(v)) = -t^{-d-1}\frac{\partial}{\partial v_j}H^j(v)$$

where $v_j = (x_j - y_j)/t$ and $H_j(v) = v_jK(v)$. Then H_j is of compact support so

$$\int_{\mathbf{R}^d}\frac{\partial}{\partial v_j}H^j(v)\,dx = \int_{\mathbf{R}^d}t^d\frac{\partial}{\partial v_j}H^j(v)\,dv = 0$$
$$\Rightarrow \int_{\mathbf{R}^d}\lambda_Q^{\mathrm{II}}\,dx = \int_{T_Q}\int\frac{\partial u}{\partial t}(-t^{-d})\int_{\mathbf{R}^d}\frac{\partial}{\partial v_j}H^j(v)\,dx\,dy\,dt = 0$$

again using Fubini.

$J = \mathrm{III}$:

$$\lambda_Q^{\mathrm{III}} = \int_{T_Q}\frac{\partial u}{\partial y_j}\underset{j\neq d+1}{H_t^j}(x-y)\,dy\,dt$$

and

$$H_{j,t}(x-y) = t^{-d}\frac{x_j-y_j}{t}K\left[\frac{x-y}{t}\right] \Rightarrow \int_{\mathbf{R}^d}\frac{x_j-y_j}{t}K\left[\frac{x-y}{t}\right]\,dx = 0$$

since K is radial $\Rightarrow H_{j,t}(x-y)$ is an odd function in the x variable for y and t fixed. The proof of (c) is a straightforward computation of the Lipschitz norm.

Proof of Lemma 1. Wlog $f_{\tilde{Q}} = 0$. On a domain Ω whose boundary is given by a C^∞ function if

$$L = \frac{\partial}{\partial y_i}\left[a_s^{ij}\frac{\partial}{\partial y_j}\right]$$

where $a_{ij,s}$ are smooth, then the following form of Green's theorem holds:

$$\int_\Omega (Lu)v - \int_\Omega uLv = \int_{\partial\Omega}v\frac{\partial u}{\partial n_a} - \int_{\partial\Omega}u\frac{\partial v}{\partial n_a} \tag{1.4}$$

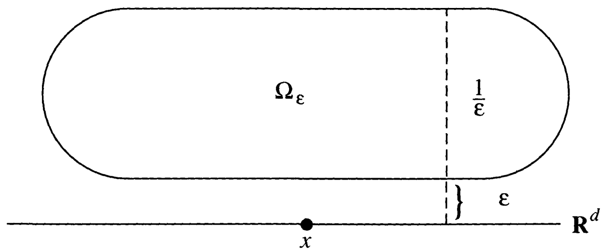

FIGURE 4

when u and v are C^2 functions on $\overline{\Omega}$. Here

$$\frac{\partial}{\partial n_a} = \left[a_1^j \frac{\partial}{\partial y_j}, a_2^j \frac{\partial}{\partial y_j}, \dots, a_{d+1}^j \frac{\partial}{\partial y_j}\right] \cdot \vec{n}$$

where $\vec{n}$ is the normal vector to $\partial\Omega$. $\partial/\partial n_a$ is the co-normal derivative associated to L.

So taking

$$L = \frac{\partial}{\partial y_i}\left[a_s^{ij} \frac{\partial}{\partial y_j}\right]$$

where $a_{ij,s}$ are smooth approximations to the coefficients a_{ij} of L in Theorem 1 and $u_r = u * h_r$ smooth approximations to the solution u, then $(t-\varepsilon)$ and $K_t(x-y)$ being smooth in $\mathbf{R}_+^{d+1}$, Green's formula (2.7) gives that

$$(1.5) \qquad \int_{\Omega_\varepsilon} [L(u_r K_t)](t-\varepsilon) - \int_{\Omega_\varepsilon} u_r K_t L(t-\varepsilon) = \int_{\partial\Omega_\varepsilon} \frac{\partial(u_r K_t)}{\partial n_a}(t-\varepsilon) - \int_{\partial\Omega_\varepsilon} u_r K_t \frac{\partial(t-\varepsilon)}{\partial n_a}$$

where Ω_ε is taken to be a smooth approximation to the rectangle in $\mathbf{R}_+^{d+1}$ of height $\frac{1}{\varepsilon}$, centered at $(x, \varepsilon + \frac{1}{2\varepsilon})$, which is wide enough (width $\sim \frac{1}{\varepsilon}$) so the cone $\Gamma(x)$ intersects the flat part of the boundary of Ω_ε (see Figure 4).

Using integration by parts, the fact that the boundary terms in the y_j variables $j \neq d+1$ (horizontal variables) vanish since $K_t(x-y)$ has compact support in y for x and t fixed (see Figure 5), and

$$(1.6) \qquad \int_{\Omega_\varepsilon} \frac{\partial K_t(x-y)}{\partial t} u_r(y, t) = -\int_{\Omega_\varepsilon} \frac{\partial u_r}{\partial y_j} \underset{j\neq d+1}{H_t^j}(x-y)$$

where $H_{j,t}(x-y) = t^{-d} \frac{x_j - y_j}{t} K\left[\frac{x-y}{t}\right]$.

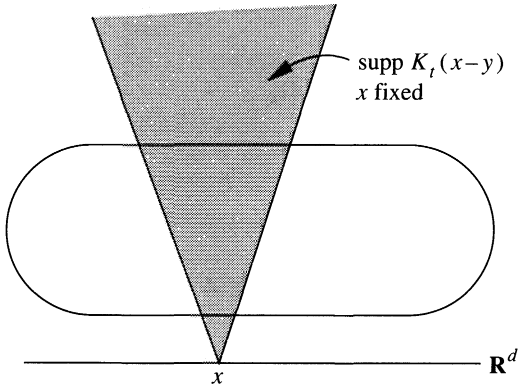

FIGURE 5

One obtains

$$(1.7)\qquad -\int_{\Omega_\varepsilon} \frac{\partial u_r}{\partial y_j} a_s^{ij} \frac{\partial[K_t(t-\varepsilon)]}{\partial y_j} + \int_{\Omega_\varepsilon} \frac{\partial u_r}{\partial y_i} a_s^{ij} \frac{\partial K_t}{\partial y_j}(t-\varepsilon) + \int_{\Omega_\varepsilon} \frac{\partial u_r}{\partial y_j} \underset{j\neq d+1}{H_t^j} (x-y) = -\int_{\partial\Omega_\varepsilon} u_r K_t \frac{\partial(t-\varepsilon)}{\partial n_a}.$$

Letting $a_{ij,s} \to a_{ij}$ and $u_r \to u$ (1.7) holds with $a_{ij,s}$ replaced by a_{ij} and u_r replaced by u. Now let $\varepsilon \to 0$. The 2nd and 3rd integrals in the left in (1.7) converge in the sense of $L^2(Q)$ since by the argument in the proof of Lemma 2,

$$\int_{R\cap\{t>\varepsilon\}} \frac{\partial u}{\partial y_i} a^{ik} G_t^j(x-y)$$

can be written as a dyadic martingale whose dyadic square function is bounded independent of ε. This implies these integrals converge as $\varepsilon \to 0$ in the sense of $L^2(Q)$. The upper part, $\int_{\mathbf{R}_+^{d+1}\setminus R}$, was shown in the proof of Lemma 3 to be bounded by $c\|Sf\|_\infty$.

Now wherever $\lim_{\varepsilon\to 0}$ exists it equals $\lim_{\varepsilon_k\to 0}$ for any subsequence $\{\varepsilon_k\}$. To handle the 1st term on the left in (1.7) we need

SUBLEMMA. (i) *For a.a.* ε,

$$\left|\int_{\Omega_\varepsilon} \frac{\partial u}{\partial y_i} a^{ij} \frac{\partial[K_t(t-\varepsilon)]}{\partial y_j}\right| \lesssim \int_{\partial\Omega_\varepsilon} \left|\frac{\partial u}{\partial n_a} K_t(t-\varepsilon)\right|.$$

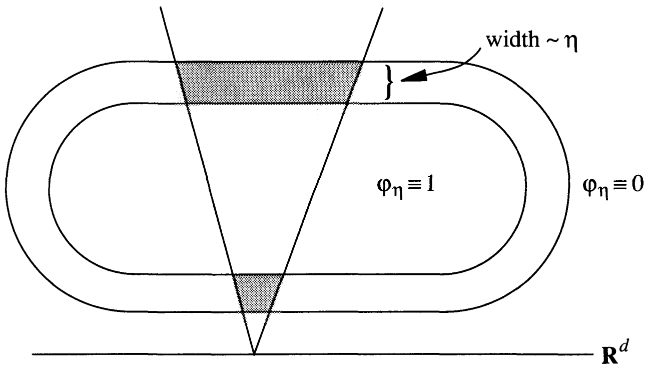

FIGURE 6

(ii) $\exists \varepsilon_k \to 0$ *such that*

$$\lim_{\varepsilon_k \to 0} \int_{\partial \Omega_{\varepsilon_k}} \left| \frac{\partial u}{\partial n_a} K_t(t-\varepsilon_k) \right| = 0.$$

Proof. (i) Multiply $K_t(x-y)(t-\varepsilon)$ by a smooth bump function φ_η of compact support in Ω_ε such that $\varphi_\eta \equiv 1$ on the region interior to Ω_ε of distance η away from $\partial\Omega_\varepsilon$, $\varphi_\eta \equiv 0$ on $(\operatorname{int}\Omega_\varepsilon)^c$ and

$$\left|\frac{\partial \varphi_\eta}{\partial t}\right| \precsim \frac{1}{\eta}, \quad \frac{\partial \varphi}{\partial y_j} \underset{j \neq d+1}{=} 0 \quad \text{on } \operatorname{supp} K_t \cap \operatorname{supp} \frac{\partial \varphi_\eta}{\partial t}$$

(the shaded region in Figure 6).

Then by definition of $Lu = 0$ on $\mathbf{R}_+^{d+1}$, for any smooth Ψ of compact support in $\mathbf{R}_+^{d+1}$,

$$\int_{\mathbf{R}_+^{d+1}} \frac{\partial u}{\partial y_i} a^{ij} \frac{\partial \Psi}{\partial y_j} = 0.$$

So taking $\Psi = \varphi_\eta K_t(t-\varepsilon)$, then

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} \frac{\partial u}{\partial y_j} a^{ij} \frac{\partial}{\partial y_j}[K_t(t-\varepsilon)\varphi_\eta] \\ &= \int_{\Omega_\varepsilon} \frac{\partial u}{\partial y_i} a^{ij} \frac{\partial [K_t(t-\varepsilon)]}{\partial y_j} \varphi_\eta + \int_{\Omega_\varepsilon} \frac{\partial u}{\partial y_i} a^{ij} K_t(t-\varepsilon) \frac{\partial \varphi_n}{\partial y_j}. \end{aligned}$$

Then

$$(1.8) \qquad \left| - \int_{\Omega_\varepsilon} \frac{\partial u}{\partial y_i} a^{ij} \frac{\partial [K_t(t-\varepsilon)]}{\partial y_j} \varphi_\eta \right| \le \int_{\Omega_\varepsilon} \left| \frac{\partial u}{\partial y_i} a^{ij} K_t(t-\varepsilon) \right| \left| \frac{\partial \varphi_\eta}{\partial y_j} \right|.$$

Now take the limit as $\eta \to 0$. Since

$$\left|\frac{\partial \varphi_\eta}{\partial t}\right| \le c\left|\frac{1}{\eta}\right| \quad \text{and} \quad \frac{\partial \varphi_\eta}{\partial y_j} = 0 \quad \text{for } j \ne d+1$$

on $\operatorname{supp} K_t$ the integral on the right approaches the boundary integral a.e. (this means for a.a. ε). One can see this since

$$\iint_{\left[\operatorname{supp}\frac{\partial\varphi_\eta}{\partial t}\right]} \left|\frac{\partial u}{\partial y_i}\right| |a^{ij}|\, |K_t(t-\varepsilon)| \left|\frac{\partial \varphi_\eta}{\partial t}\right| dy\, dt < \infty$$

$\Rightarrow$ as a function of t the inner integral

$$\int_{R^d} \left|\frac{\partial u}{\partial y_i}\right| |a^{ij}|\, |K_t(t-\varepsilon)| \left|\frac{\partial \varphi_\eta}{\partial t}\right| dy \in L^1_{\mathrm{loc}}(\mathbf{R})$$

so by the Lebesgue Differentiation Theorem

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_{t_\varepsilon-\eta}^{t_\varepsilon} \int_{\mathbf{R}^d} \left|\frac{\partial u}{\partial y_i}\right| |a^{ij}|\, |K_t(t-\varepsilon)|\, dy\, dt$$

exists a.e. t_ε, and equals

$$\int_{t=t_\varepsilon} \left|\frac{\partial u}{\partial y_i}\right| |a^{ij}|\, |K_t(t-\varepsilon)|\, dy.$$

For a.a. ε

$$\left|\int_{\Omega_\varepsilon} \frac{\partial u}{\partial y_i} a^{ij} \frac{\partial [K_t(t-\varepsilon)]}{\partial y_j}\right| \lesssim \int_{\partial\Omega_\varepsilon} \left|\frac{\partial u}{\partial y_i}\right| |a^i_d|\, |K_t(t-\varepsilon)| \tag{1.9}$$

is obtained by putting the above into (1.8) and taking $\lim_{\eta\to 0}$ since $\varphi_\eta \to \chi_{\Omega_\varepsilon}$.

Proof of (ii). On the upper part of $\partial\Omega_\varepsilon$ $t = \varepsilon + \frac{1}{\varepsilon}$ and for each region W,

$$W = \Gamma(x) \cap \left[\frac{1}{2\varepsilon_k} \le t \le \frac{3}{2\varepsilon_k}\right],$$

there exists a set of values for t $(t_k \sim 1/\varepsilon_k + \varepsilon_k)$ of non-zero measure such that

$$\begin{aligned}
\frac{1}{\varepsilon_k} \int_{\partial\Omega_{\varepsilon_k} \cap \{t=t_k\}} \left|\frac{\partial u}{\partial n_a}\right| |K_t(t-\varepsilon_k)| &\le \int_{1/2\varepsilon_k}^{3/2\varepsilon_k} \int_{\mathbf{R}^d} \left|\frac{\partial u}{\partial n_a}\right| |K_t(t-\varepsilon_k)| \\
&\le c\left[\int_W |\nabla u|^2\right]^{1/2} \left[\int_W t^{2-2d}\right]^{1/2} \\
&\le c\left[\varepsilon_k^2 \int_{W^*} |u|^2\right]^{1/2} (e_k^{d-3})^{1/2}
\end{aligned}$$

where the third inequality is by an inequality due to Di Giorgi, Nash, and Moser which states for any ball B of radius t, B^* concentric with B of radius $(1+\xi)t$, then

$$\text{(A)} \qquad \frac{1}{|B|}\int_B |\nabla u|^2 \le c_\xi \frac{t^{-2}}{|B^*|}\int_{B^*}|u|^2 \qquad \text{([4], [7])}.$$

And because $|W| \sim [1/\varepsilon_k]^{d+1}$ and $(y, t) \in W \Rightarrow t \sim 1/\varepsilon_k$.
Then

$$\begin{aligned} &c\left[\varepsilon_k^2 \int_{W^*}|u|^2\right]^{1/2}(e_k^{d-3})^{1/2} \\ &\le c\left[\varepsilon_k^2\frac{1}{\varepsilon_k}\sup_{\frac{1}{2\varepsilon_k}<t}\left[\left[\int_{\mathbf{R}^d}|u(y, t)|^2\,dy\right]\right]\right]^{1/2}\varepsilon_k^{(d-3)/2} \\ &= c\varepsilon_k^{(d/2)-1}\sup_{\frac{1}{2\varepsilon_k}<t}\left[\left[\int_{\mathbf{R}^d}|u|^2\right]\right]^{1/2} \Rightarrow \int_{\Omega_{\varepsilon_k}\cap\{t=t_k\}}\left|\frac{\partial u}{\partial n_a}\right||K_t(t-\varepsilon_k)| \\ &\le c\varepsilon_k^{d/2} \end{aligned}$$

where c depends only on L, d, K and the constant in (A). So since $c\varepsilon_k^{d/2} \to 0$ as $\varepsilon_k \to 0$, the boundary integral on $\{t = t_k\} \to 0$ as $t_k \to \infty$.

One can easily pick a sequence of Ω_{ε_k} such that (i) holds on the boundary and (ii) holds as $\varepsilon_k \to 0$ by the above estimate on the upper boundary. On the lower boundary $t = \varepsilon_k$ so the factor $t - \varepsilon_k = 0$ which means the integral over the lower boundary disappears. □

So (1.7) becomes

$$\text{(1.10)} \quad \begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\frac{\partial u}{\partial y_i}a^{ij}\frac{\partial K_t}{\partial y_j}(t-\varepsilon) + \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\frac{\partial u}{\partial y_j}\underset{j\neq d+1}{H_t^i}(x-y) \\ &\quad = \lim_{\varepsilon\to 0} - \int_{\partial\Omega_\varepsilon} uK_t\frac{\partial(t-\varepsilon)}{\partial n_a} = f(x). \end{aligned}$$

As can be easily seen

$$\lim_{\varepsilon\to 0} - \int_{\partial\Omega_\varepsilon} u(y, \varepsilon)K_\varepsilon(x-y)\frac{\partial(t-\varepsilon)}{\partial n_a} = f(x).$$

Finally writing the first integral in (1.10) as the sum of two integrals (to distinguish $\partial K_t/\partial y_j$, $j \neq d+1$, from $\partial K_t/\partial t$) and writing $\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}$ as $\int_{\mathbf{R}^{d+1}}$ (1.10) becomes (1.1). □

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