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FOUR MANIFOLD TOPOLOGY AND GROUPS OF POLYNOMIAL GROWTH

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In this paper we present a new proof that groups of polynomial growth are "good" in the sense of Freedman and Quinn. It follows from the results of Freedman that the five dimensional s-cobordism theorem and the surgery exact sequence in dimension four hold for $\pi_1(M)$ of polynomial growth. In the processes, we will give a slightly more efficient procedure for grope height raising and a slightly different procedure for using the grope height to kill fundamental group images.

Introduction. The recent advances in four manifold topology can be loosely described as the result of new techniques for finding locally flat topological imbeddings of discs in four manifolds. Basically, Freedman has shown that a regular neighborhood of a certain finite construct, called a 1-story capped tower (with grope height at least four), contains a locally flat topologically imbedded disc (see [3] or [5]). The main difficulty in finding a capped tower in a 4-manifold M comes from the fundamental group $\pi_1(M)$. One says a group G is "good" (defined precisely below) if these difficulties can be overcome for $\pi_1(M) = G$ (and hence the 5-dimensional s-cobordism theorem and the surgery sequence in dimension 4 hold for $\pi_1 = G$). A group G is known to be "good" if it has polynomial growth (and more generally, if G is an elementary group).

M. Freedman has given two proofs that groups of polynomial growth are "good", both of which rely on nontrivial results from group theory. In the first proof [3] or [5], one first shows that finite groups and the integers $\mathbf{Z}$ are "good". One then notes that the class of "good" groups is closed under the four operations: taking subgroups, forming quotient groups, taking direct limits and forming extensions. Therefore all elementary groups are "good". One then applies the result of [6] that groups of polynomial growth are almost nilpotent and hence elementary. In the second proof [4], one shows by a direct argument that if the growth function ρ of a finitely generated group G satisfies $cr^d \leq \rho(r) \leq Cr^d$ for $r > 0$ and some (positive) constants c, C and d then G is "good". One then appeals to the result of [6] and [1] to conclude that such bounds hold for all groups of polynomial growth.

We will give an alternate self-contained proof that groups of polynomial growth are "good" which requires only the bound $\rho(r) \leq Cr^d$. We will actually obtain a formally stronger result that any group whose growth function satisfies $(*)$ $\rho(r) \leq Cr^{(\beta-\varepsilon)\log r}$ for $r < 0$ and some positive C and ε and a specific constant β ($\beta = 0.75965\ldots$) defined below. The author is not aware of a finitely generated group G satisfying $(*)$ that is not also of polynomial growth; however, if such a group did exist it would provide the first known example of a "good" group that is not also an elementary group.

Basic definitions. To avoid presenting a large amount of material, we will assume the reader is familiar with the elementary theory of capped gropes and capped towers as presented in [5, Part 1] and we will follow the notation used there. We will call an immersion split if it is proper and if there is a division of the upper stages (i.e. all but the first stage) into two sides, denoted E_+ and E_-, so that the second stage surfaces on each side are disjoint from one another and the image of the caps of E_+ and the image of the caps of E_- are disjoint.

We will say that a group G is "good" if it has the following property:

(#) Suppose α: $\pi_1(M) \to G$ is a homomorphism and M contains a properly immersed disk-like capped grope of height 3; then there is a properly immersed capped grope with the same body such that the fundamental group of the image is taken to $\{e\}$ by α. This definition agrees in spirit with that of [5, Part II], since [5, Prop. 2.9] implies that a group that is "good" in the above sense is "good" in the sense of [5, Part II].

Let G be a finitely generated group with a fixed generating set. Let $B(r)$ denote the ball of radius r about $e \in G$ in the word metric. Let $\rho(r) = |B(r)|$ be the growth function of G with respect to this generating set. For a fixed homomorphism α: $\pi_1(M) \to G$, we will say that a split immersion of a capped grope in a 4-manifold M has indices $((h_+, r_+), (h_-, r_-))$ if E_+ (resp. E_-) has grope height at least h_+ (resp. h_-) and the group elements corresponding to the cap intersections lie in $B(r_+)$ (resp. $B(r_-)$). We do not assume the heights are maximal or the radii minimal. In fact we will often alter them to achieve neater bounds.

Grope height raising. To obtain the best constant possible in the subsequent treatment we need a slightly different grope height raising procedure than that presented in [5].

PROPOSITION 1. *Suppose that for* $\alpha\colon \pi_1(M) \to G$ *there is a split immersion with indices* $((h_+, r_+), (h_-, r_-))$; *then inside a regular neighborhood of that immersion there is a split immersion with indices* $((h_+ + h_- - 1, 2r_+ + r_-), (h_-, 2r_-))$. *Further the bottom stages of the body of the new immersion agree with the body of the old one.*

Proof. Take a parallel copy of E_-, contract this parallel and push E_- off the contraction. The resulting – side, E'_-, has height h_- and intersections with at most twice the word length of the old – side. Use two copies of the parallel to form an $h_- - 1$ stage transverse sphere-like capped grope, E^t_+, to E_+. Notice that by construction this dual is disjoint from E'_-. Pushing the cap intersections in E_+ down to the bottom stage of E_+ and adding copies of E^t_+ raises the height of E_+ to $h_+ + h_- - 1$. The new immersion is still split and the group elements represented by the new + side have word length at most $2r_+ + r_-$.

Suppose now that we are given a split immersion of a 3 stage capped grope in M (with α fixed). For some $R > 0$ this immersion has indices $((2, R), (2, R))$. Suppose we raise grope height N times alternating the side whose height is raised. A simple recursion shows that afterwards we have a split immersion with indices (after a few simplifying bounds)

$$\left(\left(\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^N, \frac{R}{7}\left(\frac{1+\sqrt{17}}{2}\right)^N + R\right),\right.$$
$$\left.\left(\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^N, \frac{R}{7}\left(\frac{1+\sqrt{17}}{2}\right)^N + R\right)\right).$$

This is an improvement over the grope height raising procedure of **[5]** in the sense their procedure gives word lengths that grow like 5^N. To simplify later formulas we let

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \eta = \frac{1+\sqrt{17}}{2}.$$

Group element control. We now turn to the proof that groups of polynomial growth are "good". This will follow as an easy exercise form the following technical lemma.

LEMMA. *Suppose G has the following property:*
(*) *For all $a > 0$ there is an integer $X > 0$ such that $B(14X)$ can be covered by less than aX^A ($A = \log\phi/\log\eta = 0.51159\ldots$) translates of $B(X)$, then G is "good".*

Proof. Suppose we have a homomorphism $\alpha: \pi_1(M) \to G$ and a properly immersed disk-like 3 stage capped grope in M. Pick an arbitrary division of the upper stages into $+$ and $-$ sides. Raising height on the $+$ side (using the procedure in [5] which does not require a split immersion), contracting the $+$ side and pushing the $-$ side off produces a split immersion with the same body.

Suppose this split immersion has indices $((2, R), (2, R))$. Taking $a \leq R^{-A}/4$ in $(*)$ we obtain an integer X. Let N be the largest integer such that $2X \geq \frac{R}{7}\eta^N + R$, then by $(*)$ we also have $\frac{1}{2}\phi^N > aX^A$. In other words, grope height raising N times produces a split immersion with indices $((h, 2X), (h, 2X))$ where at most $h-1$ translates of $B(X)$ cover $B(14X)$.

Now raise height once on each side (starting with the $+$ side) producing a grope with indices $((2h-1, 12X), (3h-2, 14X))$ and hence with indices $((2h-1, 14X), (3h-2, 14X))$. Cover $B(14X)$ by $h-1$ translates of $B(X)$ and assign each element of $B(14X)$ to one of the translates that covers it. Order the translates. Iteratively contract the kth stage from the top of the grope and push all elements corresponding to the kth translate off. After all $h-1$ steps are complete we are left with a grope with indices $((h, 2X), (2h-1, 2X))$.

Iterating this procedure (alternating starting sides) we can raise grope height (exponentially) while always keeping the group elements in $B(2X)$. Raise height in this way until there are at least $\rho(2X)+2$ stages. Assign each of the top $\rho(2X)-1$ stages to one of the nontrivial elements of $B(2X)$. Contracting each of these stages and pushing off the intersections corresponding to the associated group element produces a 3 stage capped grope with the same body as the original all of whose intersections correspond to $e \in G$. Therefore G is "good".

The condition $(*)$ in the lemma follows from the easier condition $(**)$. For any $b > 0$, there is an integer $Y > 0$ for which $\rho(29Y) < bY^A\rho(Y)$. To see this fix $a > 0$ and let $b = 2^A a$. Let $X = 2Y$. Consider a subset of $B(28Y)$ such that the balls of radius Y about these points are disjoint and that is maximal with respect to this property. By maximality, the balls of radius $2Y = X$ about these points cover $B(14X) = B(28Y)$. Further since the balls of radius Y about them are disjoint and lie in $B(29Y)$, there are at most $bY^A = aX^A$ such balls.

If $(**)$ fails to hold for G then for some $b > 0$ and all Y, $\rho(29Y) \geq bY^A\rho(Y)$. Iterating this we see that for all integers $s \geq 0$, $\rho(29^s) \geq b^s(29)^{s(s-1)A/2}\rho(1)$. Since $\rho(x)$ is monotone in x, we have $\rho(x) \geq Cx^w x^{A\log x/2\log 29}$, for some constants C and w depending on b.

THEOREM. *Suppose G is a group with growth function $\rho(x)$ such that for any constants C, w $(C > 0)$ there is an $x > 0$ with $\rho(x) < Cx^w x^{b\log x}$, where $\beta = A/2\log 29 = 0.75965\ldots$, then G is "good". In particular groups of polynomial growth are "good".*

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