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THE COVERS OF A NOETHERIAN MODULE

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In this paper we define the covers of a module and describe some of their applications.

1. Introduction.

Let R be a commutative ring and A an R-module. A cover of A is defined to be a subset T of $\mathrm{Max}(R)$ satisfying that for any $x \in A$, $x \neq 0$, there is $M \in T$ such that $0 :_R x \subseteq M$. If we denote by J the intersection of all the maximal ideals belonging to T and suppose that $A \neq 0$ is finitely generated, then we have $JA \neq A$. This generalises the Nakayama's lemma; if, in addition, R is Noetherian, then $\bigcap_{n=1}^{\infty} J^n A = 0$. This is a generalization of a well-known result. A key observation for the covers is that, in the case that R is Noetherian and A is finitely generated, there is a cover T of A which is itself a finite set. From this we have the following result: Let R be a Noetherian ring. Then there is a finite number of maximal ideals $M_1, \ldots, M_m$ of R such that $\bigcap_{n=1}^{\infty} J^n = 0$, where $J = \bigcap_{i=1}^{m} M_i$. This generalises the Krull's theorem for Jacobson radicals. Using this result we can embed the Noetherian ring R in the J-adic completion $\widehat{R}$ of R, which is a complete semi-local Noetherian ring; besides, if R is a Cohen-Macaulay (C-M for short) ring, then $\widehat{R}$ is a C-M ring. We also use the covers to deal with the maximal component of a finitely generated module over a Noetherian ring, which was introduced by Matlis in [**3**].

Throughout the paper, R will denote a (non-trivial) commutative ring with identity. Also, if T is a subset of $\mathrm{Max}(R)$ we denote by $\cap T$ (resp. $\cup T$) the intersection (resp. union) of all the maximal ideals belonging to T.

2. The covers.

In this section we define the covers of a module and generalise some known results.

Definition. Let A be an R-module. A subset T of $\mathrm{Max}(R)$ is called a cover of A if for any $x \in A$, $x \neq 0$, there is $M \in T$ such that $0 :_R x \subseteq M$.

Clearly, if T is a cover of A and B is a submodule of A, then T is a cover of B. If T is a cover of A and $T \subseteq T' \subseteq \mathrm{Max}(R)$, then T' is a cover of A. We say that T is a finite cover of A, or A has a finite cover T, if T is a cover of A and T is itself a finite set. If T is a cover of A, we also say that T covers A.

Lemma 2.1. *Let T be a cover of A. Then each $r \in R - \cup T$ is A-regular. Indeed if $a \in A - \{0\}$ and $ra = 0$, then $r \in (0 :_R a) \subseteq M$ for some $M \in T$, a contradiction.*

Proposition 2.2. *Let $A \neq 0$ be a finitely generated R-module and T a cover of A. Then $JA \neq A$, where $J = \cap T$.*

Proof. Suppose that $JA = A$, then there is $r \in J$ such that $(1 + r)A = 0$, which contadicts Lemma 2.1. □

Proposition 2.3. *Let A be an R-module, T a cover of A, and $I \not\subseteq 0 :_R A$ an ideal of R. Set $J = \cap T$. If $A/0 :_A I$ is finitely generated, then $JA + (0 :_A I) \neq A$.*

Proof. Since $I \not\subseteq 0 :_R A$, $A/0 :_A \neq 0$. Let $\bar{x} \in A/0 :_A I$ and $\bar{x} \neq 0$. Then $0 :_R \bar{x} = (0 :_A I) :_R x \subseteq 0 :_R Ix$. Since $x \notin 0 :_A I$, $Ix \neq 0$. Take $r \in I$ such that $rx \neq 0$, then $0 :_R \bar{x} \subseteq 0 :_R rx$. it follows that T is a cover of $A/0 :_A I$. By Proposition 2.2, $J(A/0 :_A I) \neq A/0 :_A I$, hence $JA + (0 :_A I) \neq A$. □

Proposition 2.4. *Let R be a Noetherian ring, A a finitely generated R-module, T a cover of A, and $I \subseteq \cap T$ an ideal of R. Then $\bigcap_{n=1}^{\infty} I^n A = 0$.*

Proof. Set $\bigcap_{n=1}^{\infty} I^n A = B$. By Krull's theorem, there is $r \in I$ such that $(1 + r)B = 0$. From Lemma 2.1, $B = 0$. □

Proposition 2.5. *Let T be a finite subset of* $\mathrm{Max}(R)$ *and A an R-module. Set $J = \cap T$. If $\bigcap_{n=1}^{\infty} J^n A = 0$, then T is a cover of A.*

Proof. If it were not true, there would be a non- zero element x of A such that for any $M \in T$, $0 :_R x \not\subseteq M$. Thus for any integer $n > 0$ we have $(0 :_R x) + M^n = R$, so $M^n x = Rx$. It then follows that $J^n x = Rx$, and thus $\bigcap_{n=1}^{\infty} J^n A \neq 0$, a contradiction. □

Let R be a Noetherian ring and A a finitely generated R-module. We know that $\mathrm{Ass}(A)$ is a finite set. Let $\mathrm{Ass}(A) = \{P_1, \ldots, P_n\}$. Choose a finite subset T of $\mathrm{Max}(R)$ in such a way that for any P_i, there is $M_i \in T$ such that $P_i \subseteq M_i$. Since for any $x \in A$, $x \neq 0$, there is P_i such that $0 :_R x \subseteq P_i$, it follows that T is a finite cover of A. Hence finite covers exist for any finitely generated module over a Noetherian ring. In particular, any Noetherian ring (as a module over itself) has finite covers.

As a consequence of the above remarks and Proposition 2.4 we have the following theorem.

Theorem 2.6. *Let R be a Noetherian ring and A a finitely generated R-module. Then there is a finite subset T of* $\mathrm{Max}(R)$ *such that* $\bigcap_{n=1}^{\infty} J^n A = 0$, *where $J = \cap T$. In particular, if $A = R$,* $\bigcap_{n=1}^{\infty} J^n = 0$.

It is clear that if R is a Noetherian ring and A is a finitely generated R-module, then for any cover T of A we have $T \supseteq \mathrm{Ass}(A) \cap \mathrm{Max}(R)$.

In general, if T is a cover of the module A and B is a submodule of A, T is not a cover of A/B. For example, if T is a cover of the ring R and $T \neq \mathrm{Max}(R)$, then for any $M \in \mathrm{Max}(R) - T$, T is not a cover of R/M.

Proposition 2.7. *Let R be a Noetherian ring, A a finitely generated R-module, B a submodule of A, and T a finite cover of A. Then T is a cover of A/B if and only if B is a closed submodule of A in the J-adic topology, where $J = \cap T$.*

Proof. Suppose first that B is closed, then we have $\bigcap_{n=1}^{\infty} (J^n A + B) = B$, so $\bigcap_{n=1}^{\infty} J^n(A/B) = 0$. By Proposition 2.5, T is a cover of A/B. Conversely, if T is a cover of A/B, then $\bigcap_{n=1}^{\infty} J^n(A/B) = 0$, by Proposition 2.4. So $\bigcap_{n=1}^{\infty} (J^n A + B) = B$, and hence B is closed. □

Proposition 2.8. *Let R be a Noetherian ring, A a finitely generated R-module, B a submodule of A, and I an ideal of R. Then it is possible to choose a finite subset T of* $\mathrm{Max}(R)$ *such that* $\bigcap_{n=1}^{\infty} (J^n A + I^s B) = I^s B$, *for all* $s \geq 0$, *where $J = \cap T$.*

Proof. By [**5**, Theorem 5.5(1)], the sequence $\mathrm{Ass}(A/I^s B)$ is constant for large s, thus the set $\bigcup_{s=0}^{\infty} \mathrm{Ass}(A/I^s B)$ is finite. Hence it is possible to choose a finite subset T of $\mathrm{Max}(R)$ in such a way that T covers all $A/I^s B$. By Proposition 2.4, the Proposition follows. □

3. The maximal component of a Noetherian module.

Throughout this section and the next section the ring R will be Noetherian and the modules will be finitely generated.

Let A be an R-module and define $X(A) = \{x \in A|$ every prime ideal containing $0:_R x$ is maximal $\}$. Then $X(A)$ is a submodule of A. Matlis [**3**] called $X(A)$ the maximal component of A. By [**3**, Corollary (3)], $X(A)$ is the sum of all Artinian submodules of A, and hence is the largest Artinian submodule of A, since A is Noetherian. Further, $X(A/X(A)) = 0$.

Chatters [**4**] gave a similar discussion for Noetherian rings (not necessary to be commutative).

From [**3**, Corollary (1)] and the fact that $X(A)$ has finite length we have the following result.

Theorem 3.1. *Let T be a finite cover of A. Set $J = \cap T$. Then* $X(A) = \bigcup_{n=1}^{\infty}(0 :_A J^n)$.

The following result is standard.

Lemma 3.2. *Let I be an ideal of R and $A \neq 0$ an R-module. Then* $\operatorname{dep}_I(A) > 0$ *if and only if* $0 :_A I = 0$.

Theorem 3.3. *Let A be an R-module, not Artinian. Let T be a finite cover of A and set $J = \cap T$. Then $X(A)$ is the least element of the set*

$$S = \{B|B \quad \textit{is a proper submodule of } A \textit{ and} \quad \operatorname{dep}_J(A/B) > 0\}.$$

Proof. Since A is not Artinian, $X(A)$ is a proper submodule of A. By Theorem 3.1, we may assume that $X(A) = 0 :_A J^N$. Now

$$0 :_{A/X(A)} J = (X(A) :_A J)/X(A) = 0 :_A J^{N+1}/0 :_A J^N = 0.$$

From Lemma 3.2, $\operatorname{dep}_J(A/X(A)) > 0$. Hence we have $X(A) \in S$. If B is a proper submodule of A satisfying that $\operatorname{dep}_J(A/B) > 0$, again by Lemma 3.2, $0 :_{A/B} J = (B :_A J)/B = 0$, i.e., $B :_A J = B$. Hence for any integer $n > 0$, $B :_A J^n = B$. Thus we get that $B = B :_A J^N \supseteq 0 :_A J^N = X(A)$, i.e., $X(A)$ is the least element of S. □

Corollary 3.4. *Let A be a non-zero R-module and T a finite cover of A. Set $J = \cap T$. Then* $\operatorname{dep}_J(A) > 0$ *if and only if* $X(A) = 0$.

Let $T = \{M_1, \ldots, M_n\}$ be a finite cover of the R-module A. We want to find the relations between $X(A)$ and $X(A_{M_i})$, $1 \leq i \leq n$. For any $P \in \operatorname{Spec}(R)$, if K is an R_P-submodule of A_P, denote by K^c the contradiction of K to A. We have $(K^c)_P = K$. If B is a submodule of A, then $(B_P)^c = \bigcup_{r \in R-P}(B :_A r)$. It is also easily checked that if B is a submodule of A and

K is an R_P-submodule of B_P, then $(K^c \cap B)_P = K$. It follows that if B is an Artinian submodule of A, then B_P is an Artinian submodule of A_P. In particular, we have $X(A)_P \subseteq X(A_P)$.

Theorem 3.5. *Let A be an R-module and $T = \{M_1, \dots, M_n\}$ be a finite cover of A. Set $J = \cap T$. Then*

$$X(A) = \bigcap_{i=1}^{n} X(A_{M_i})^c .$$

Proof. Since $X(A) \subseteq (X(A)_{M_i})^c \subseteq X(A_{M_i})^c$ for all i, we have $X(A) \subseteq \bigcap_{i=1}^{n} X(A_{M_i})^c$. On the other hand, from Theorem 3.1 we can take a fixed integer $s > 0$ such that $X(A_{M_i}) = 0 :_{A_{M_i}} M_i^s R_{M_i}$ for all i. Hence

$$X(A_{M_i})^c = \left(0 :_{A_{M_i}} M_i^s R_{M_i}\right)^c = \left((0 :_A M_i^s)_{M_i}\right)^c = \bigcup_{r \in R - M_i} ((0 :_A M_i^s) :_A r) .$$

If $x \in \bigcap_{i=1}^{n} X(A_{M_i})^c$, then for each i there is $r_i \in R - M_i$ such that $r_i M_i^s x = 0$. Since $r_i R + M_i = R$, we have $M_i^{s+1} x = M_i^s x$. Thus

$$M_1^{s+1} M_2^{s+1} x = M_1^{s+1} M_2^s x = M_2^s M_1^{s+1} x = M_2^s M_1^s x = M_1^s M_2^s x.$$

Similarly we have $M_1^{s+1} \cdots M_n^{s+1} x = M_1^s \cdots M_n^s x$. So $J^{s+1} x = J^s x$, and hence $J^s x = 0$ by Proposition 2.2. Thus $x \in 0 :_A J^s \subseteq X(A)$, and the proof is complete. □

In the remainder of this section we consider modules over local rings.

Lemma 3.6. *Let (R, M) be a local ring (M is the unique maximal ideal of R) and A an R-module. If A is not Artinian, then* $\dim(A) = \dim(A/X(A))$.

Proof. By the definitions of $\dim(A)$ and $\dim(A/X(A))$ we need to show that $\mathrm{rad}(0 :_R A) = \mathrm{rad}(0 :_R (A/X(A)))$. Clearly, we need only to show that $0 :_R (A/X(A)) \subseteq \mathrm{rad}(0 :_R A)$. This follows from the fact that if $r \in R$ such that $rA \subseteq X(A)$, then $rM^s A \subseteq M^s X(A) = 0$ for some integer $s > 0$, hence $r^{s+1} \in 0 :_R A$. □

Lemma 3.7. [**6**, p. 105]. *Let R be a local ring and A an R-module. If $r_1, \dots, r_n$ is an A-sequence, then*

$$\dim(A/(r_1, \dots, r_n)A) = \dim(A) - n.$$

Theorem 3.8. *Let (R, M) be a local ring and $A \neq 0$ an R-module. Then there is a strictly ascending chain $A_1 \subset \cdots \subset A_s$ of submodules of A such that*

$$\sum_{i=1}^{s} \mathrm{dep}(A/A_i) = \dim(A).$$

Proof. We use induction on $d = \dim(A)$. If $d = 0$, then $R/(0 :_R A)$ is Artinian. It follows that $0 :_R A$ is M-primary, and hence $M^r \subseteq 0 :_R A$ for some integer $r > 0$. It is clear that $\mathrm{dep}(A) = 0$, and we can take $s = 1$ and $A_1 = 0$ in this case. If $d > 0$, then $0 :_R A$ is not M-primary, and thus $M^n \not\subseteq 0 :_R A$ for any integer $n > 0$. It then follows that $A \neq X(A)$, by Theorem 3.1. Since $X(A/X(A)) = 0$, $\mathrm{dep}(A/X(A)) > 0$, by Corollary 3.4. Take a maximal $A/X(A)$-sequence $x_1, \dots, x_n$ and set $B = (x_1, \dots, x_n)A + X(A)$. Further, set $A' = A/X(A)$. From Lemma 3.7 and Lemma 3.6, $\dim(A/B) = \dim(A'/(x_1, \dots, x_n)A') = \dim(A') - n = \dim(A) - n < \dim(A)$. By induction there is a strictly ascending chain $A_2/B \subset \cdots \subset A_s/B$ of submodules of A/B such that $\sum_{i=2}^{s} \mathrm{dep}(A/A_i) = \dim(A/B)$. Set $A_1 = X(A)$, then the submodules $A_1, \dots, A_s$ satisfy the required conditions. □

4. The completions and embeddings.

Proposition 4.1. *Let T be a finite cover of the Noetherian ring R, I an ideal of R. If we consider R with the I-adic topology, the following conditions are equivalent:*

(1) $I \subseteq \cap T$;

(2) *the zero ideal and every prime ideal contained in $\cup T$ is closed;*

(3) $f^{-1}\left(M\widehat{R}\right) = M$ *for all $M \in T$, where $\widehat{R}$ is the I-adic completion of R and $f : R \to \widehat{R}$ is the natural map.*

Proof. (1) ⇒ (2). Since $\bigcap_{m=1}^{\infty} I^m = 0$ the zero ideal is closed. If $P \subseteq \cup T$ is a prime ideal, then $P \subseteq M$ for some $M \in T$. Since $\mathrm{Ass}_R(R/P) = \{P\}$, we see that T is a cover of R/P. By Proposition 2.4, $\bigcap_{m=1}^{\infty} (I^m + P) = P$, i.e., P is closed.

(2) ⇒ (3). Since $\{0\}$ is closed, we can assume that $R \subseteq \widehat{R}$. Let $M \in T$. By [**2**, Theorem 21; p. 421], $M\widehat{R}$ is the closure of M in $\widehat{R}$, hence $M\widehat{R} \cap R$ consists of elements of R which are limits of elements contained in M. Since M is closed we get that $M\widehat{R} \cap R = M$.

(3) ⇒ (1). Since $M\widehat{R}$ is closed in $\widehat{R}$ and since the map $f : R \to \widehat{R}$ is continuous, M is closed in R for all $M \in T$. If $I \not\subseteq \cap T$, then $I \not\subseteq M$ for some $M \in T$. But then we have $I^m + M = R$ for all integer $m > 0$, contradicting the fact that M is closed. □

Let T be a finite cover of R and set $J = \cap T$ and $S = R - \cup T$. It is immediate from Lemma 2.1 that the map $A \to A_S$ is injective. Also, the J-adic completion of R is the same as the JR_S-adic completion of R_S. So we have the following result.

Theorem 4.2. *Any Noetherian ring R can be embedded in a complete semi-local Noetherian ring; moreover, if R is irreducible, then R can be embedded in a complete local Noetherian ring.*

If I is an ideal of R, we write $\operatorname{dep}(I)$ to stand for $\operatorname{dep}_I(R)$.

Theorem 4.3. *Let $T = \{M_1, \dots, M_n\}$ be a finite cover of the Noetherian ring R and set $J = \cap T$ and $S = R - \cup T$. Then the J-adic completion $\widehat{R}$ of R is a C-M ring if and only if* $\operatorname{dep}(M_i) = \operatorname{ht}(M_i)$, $i = 1, 2, \dots, n$.

Proof. To prove the theorem, it suffices to show that $\operatorname{ht}(M_i) = \operatorname{ht}\left(M_i\widehat{R}\right)$ and $\operatorname{dep}(M_i) = \operatorname{dep}\left(M_i\widehat{R}\right)$, $i = 1, 2, \dots, n$.

(1). The proof of $\operatorname{ht}(M_i) = \operatorname{ht}\left(M_i\widehat{R}\right)$. Let $B = R_S$, $Q_i = M_iR_S$, and $R_i = B_{Q_i}$. We now regard $\widehat{R}$ as the JB-adic completion of B. From [**1**, Theorem 8.15], $\widehat{R} = \widehat{R}_1 \times \cdots \times \widehat{R}_n$, where $\widehat{R}_i$ is the completion of the local ring R_i. By [**2**, Theorem 30; p. 433] we have

$$\operatorname{ht}\left(\left(Q_iR_i\right)\widehat{R}_i\right) = \dim\left(\widehat{R}_i\right) = \operatorname{ht}\left(Q_i\right) = \operatorname{ht}\left(M_i\right).$$

Thus

$$\operatorname{ht}\left(M_i\widehat{R}\right) = \operatorname{ht}\left(\left(Q_iR_i\right)\widehat{R}_i\right) = \operatorname{ht}\left(M_i\right).$$

(2). The proof of $\operatorname{dep}(M_i) = \operatorname{dep}\left(M_i\widehat{R}\right)$. We may view R as a subring of $\widehat{R}$. If A is an R-module, let $\widehat{A}$ be the J-adic completion of $A, z(A)$ and $z\left(\widehat{A}\right)$ the sets of annihilators of A and $\widehat{A}$ respectively. First we have that if $x \notin z(A)$, then $x \notin z(\widehat{A})$. This is because tensoring $\widehat{R}$ over R preserves the monomorphism $A \xrightarrow{x} A$, for $\widehat{R}$ is R-flat. Let $\operatorname{dep}(M_i) = s$ and $x_1, \dots, x_s$ be a maximal regular sequence (on R) contained in M_i. Since $x_{j+1} \notin z(R/(x_1, \dots, x_j))$ implies $x_{j+1} \notin z\left(\widehat{R}/(x_1, \dots, x_j)\widehat{R}\right)$, we have that $x_1 \dots, x_s$ is a regular sequence on $\widehat{R}$ contained in $M_i\widehat{R}$, so $\operatorname{dep}\left(M_i\widehat{R}\right) \geq s$.

On the other hand, since $M_i \subseteq z(R/(x_1, \dots, x_s))$ and since M_i is maximal, there is $x \in R$ such that $M_i = (x_1, \dots, x_s) :_R x$. Thus we have $M_i\widehat{R} = (x_1, \dots, x_s)\widehat{R} :_{\widehat{R}} x$, by [**2**, Lemma 7; p. 424]. So $M_i\widehat{R} \subseteq z\left(\widehat{R}/(x_1, \dots, x_s)\widehat{R}\right)$ and hence $\operatorname{dep}\left(M_i\widehat{R}\right) = s = \operatorname{dep}(M_i)$. The proof is complete. □

Corollary 4.4. *Let R be a semi-local Noetherian ring and J the Jacobson radical of R. Then the J-adic completion $\widehat{R}$ of R is a C-M ring if and only* *if R is a C-M ring.*

Corollary 4.5. *Any C-M ring can be embedded in a complete semi-local C-M ring.*

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