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UNIQUENESS FOR THE n-DIMENSIONAL HALF SPACE DIRICHLET PROBLEM

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In $\mathbb{R}^n$, we prove uniqueness for the Dirichlet problem in the half space $x_n > 0$, with continuous data, under the growth condition $u = o(|x|\sec^\gamma\theta)$ as $|x| \to \infty$ ($x_n = |x|\cos\theta$, $\gamma \in \mathbb{R}$). Under the natural integral condition for convergence of the Poisson integral with Dirichlet data, the Poisson integral will satisfy this growth condition with $\gamma = n-1$. A Phragmén-Lindelöf principle is established under this same growth condition. We also consider the Dirichlet problem with data of higher order growth, including polynomial growth. In this case, if $u = o(|x|^{N+1}\sec^\gamma\theta)$ ($\gamma \in \mathbb{R}$, $N \geq 1$), we prove solutions are unique up to the addition of a harmonic polynomial of degree N that vanishes when $x_n = 0$.

1. Introduction and notation.

We use the following notation. In $\mathbb{R}^n$ ($n \geq 2$) let Π_+ be the half space $x_n > 0$ and $\partial\Pi_+$ the hyperplane $x_n = 0$. For $x \in \mathbb{R}^n$, let $y \in \mathbb{R}^{n-1}$ be identified with the projection of x onto $\partial\Pi_+$. For $x \in \Pi_+$, write $x_n = |x|\cos\theta$ and $|y| = |x|\sin\theta$ ($0 \leq \theta < \frac{\pi}{2}$). Let B_ρ be the ball of radius ρ, centre the origin in $\mathbb{R}^n$, and $d\,S_{n-1}$ its surface element. A ball with centre $x \neq 0$ is denoted $B_\rho(x)$. The volume of the unit n-ball is $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$. When integrating over regions in $\mathbb{R}^{n-1}$ the integration variable is written y' and the angle between y' and y (for fixed y) is θ_1. Unit vectors are written with a caret, e.g., $\widehat{x} = x/|x|$, and $\widehat{e}_i$ is the unit vector along the ith coordinate axis. Finally, for $k \in \mathbb{Z}$, $\mathcal{P}_k$ is the set of (real) homogeneous harmonic polynomials of degree k and $\mathcal{Y}_k$ the set of (real) spherical harmonics of degree k (see [**3**]) with the proviso that $\mathcal{P}_k = \mathcal{Y}_k = \{0\}$ for $k < 0$. If g is a function on the unit sphere, then $||g||^2 = \int_{\partial B_1} |g(\widehat{x})|^2\,dS_{n-1}$.

The half space Dirichlet problem is to find u satisfying

$$u \in C^2(\Pi_+) \cap C^\circ(\overline{\Pi}_+) \tag{1.1}$$

$$\Delta u = 0, \quad x \in \Pi_+ \tag{1.2}$$

$$u = f, \quad x \in \partial\Pi_+, \tag{1.3}$$

where f is a given continuous function on $\mathbb{R}^{n-1}$. The Poisson integral is defined

$$(1.4) \qquad \mathcal{P}[f](x) = \int_{\mathbb{R}^{n-1}} \mathcal{K}(x,y')\, f(y')\, dy'$$

where the Poisson kernel is

$$(1.5) \qquad \mathcal{K}(x,y') = \frac{2x_n}{n\omega_n} \left[|y'-y|^2 + x_n^2\right]^{-\frac{n}{2}}.$$

The integral will exist if

$$(1.6) \qquad \int_{\mathbb{R}^{n-1}} \frac{|f(y')|\, dy'}{|y'|^n + 1} < \infty.$$

Since the kernel satisfies the mean value property for harmonic functions, $u = \mathcal{P}[f]$ will then define a harmonic function in Π_+. If f is continuous then $u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+)$ and satisfies (1.3) ([**1**], Exercise 16 of Chapter 7).

We will also consider continuous data of higher order growth. Let D_N $(N \geq 0)$ be the set of continuous functions, f, for which

$$\int_{\mathbb{R}^{n-1}} |f(y')| \left(|y'|^{n+N} + 1\right)^{-1} dy' < \infty.$$

In this paper we will consider uniqueness of solutions to (1.1)–(1.3) under pointwise growth conditions and will extend the results of [**12**] from two to n dimensions. It is a classical result that if $u = o(|x|)$ then any solution to (1.1)–(1.3) is unique [**6**]. However, we show below that the Poisson integral behaves as $o(|x| \sec^{n-1}\theta)$ when $|x| \to \infty$ in Π_+ and the order relation is sharp in the sense that the exponents cannot in general be decreased. By an order relation $u = o(|x| \sec^{\gamma}\theta)$ we mean $\mu(r)/r \to 0$ as $r \to \infty$ where $\mu(r)$ is the supremum of $|u(x)| \cos^{\gamma}\theta$ over $x \in \Pi_+, |x| = r$. It is thus desirable to have a uniqueness theorem that allows this behaviour. Using a spherical harmonics expansion we prove that under the growth condition $u = o(|x|^{N+1} \sec^{\gamma}\theta)$ $(\gamma \in \mathbb{R},\ N = 0, 1, 2, \ldots)$, solutions to (1.1)–(1.3) are unique up to the addition of a harmonic polynomial of degree N that vanishes when $x_n = 0$. Hence, if $f \in D_0$ then $u = \mathcal{P}[f]$ is the unique solution to (1.1)–(1.3) under the relaxed growth condition $u = o(|x| \sec^{n-1}\theta)$.

Closely connected with growth conditions for uniqueness are Phragmén-Lindelöf principles. Using classical barriers, we prove a Phragmén-Lindelöf principle for the half space under the growth condition $u = o(|x| \sec^{\gamma}\theta)$ $(\gamma \in \mathbb{R})$. This then gives another proof of the uniqueness theorem under $u = o(|x| \sec^{\gamma}\theta)$. Using other techniques, H. Yoshida ([**16**]) has obtained a more general Phragmén-Lindelöf principle.

In the last section of the paper we consider data $f \in D_N$ ($N \geq 1$). A modified Poisson integral may then be used to give solutions to (1.1)–(1.3) satisfying $u = o(|x|^{N+1} \sec^{n-1}\theta)$. These solutions are unique up to the addition of a harmonic polynomial of degree N, vanishing when $x_n = 0$. The kernels for these modified Poisson integrals are no longer positive. Our final result is the non-existence of positive solutions to (1.1)–(1.3) when $f \geq 0$ such that the integral in (1.6) diverges.

In the conclusion we indicate directions for further work and connections with the integral growth conditions studied by H. Yoshida ([**17**]).

2. Growth estimates.

When u is a solution of (1.2), (1.3) various estimates on the L^p norm of u and of u_{x_n} (where $u_{x_n}(y) = u(x)$) are given in [**1**] and [**2**]. However, as we are concerned with pointwise behaviour of u we give the following estimate of $|u|$. (See [**1**] for estimates when u is in a harmonic Hardy space.)

Theorem 2.1. *Let $a > 1$, $0 < b < a + n - 1$ or $a = 1$, $0 < b \leq n$. If f is measureable such that $\int_{\mathbb{R}^{n-1}} |f(y')|^a(|y'|^b + 1)^{-1}\,dy' < \infty$ then* (1.6) *holds and $u = \mathcal{P}[f]$ satisfies $u = o(|x|^{(b-n+1)/a} \sec^{(n-1)/a}\theta)$ as $|x| \to \infty$ in Π_+.*

Proof. Let $0 < \alpha \leq n/2$ and p, q Hölder conjugate exponents ($p^{-1} + q^{-1} = 1$, $p \geq 1$). Using the notation in the Introduction, the Poisson kernel, (1.5), may be written

$$\begin{aligned}\mathcal{K}(x,y') &= \frac{2x_n}{n\omega_n}(1 - \sin\theta\,\cos\theta_1)^{-\alpha} \\ &\qquad \times \left[\frac{(|y'| - |x|)^2}{1 - \sin\theta\cos\theta_1} + 2|y'|\,|x|\right]^{-\alpha} \left[|y' - y|^2 + x_n^2\right]^{-(\frac{n}{2}-\alpha)} \qquad (2.1) \\ &\leq \frac{2^{\alpha+1}x_n(1+\sin\theta)^\alpha}{n\omega_n\cos^{2\alpha}\theta}\,(|y'| + |x|)^{-2\alpha}\,\left[|y' - y|^2 + x_n^2\right]^{-(\frac{n}{2}-\alpha)}. \qquad (2.2)\end{aligned}$$

Let $|x| \geq 1$. For $p > 1$, $\alpha < n/2$, the Hölder inequality gives

$$\int_{\mathbb{R}^{n-1}} \mathcal{K}(x,y')\,|f(y')|dy' \leq \frac{2^{2\alpha+1}}{n\omega_n}\,|x|\sec^{2\alpha-1}\theta\,I_1^{\frac{1}{p}}\,I_2^{\frac{1}{q}}, \qquad (2.3)$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p\,dy'}{(|y'| + |x|)^{2\alpha p}} \qquad (2.4)\\ &\leq 2\int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p\,dy'}{|y'|^{2\alpha p} + |x|^{2\alpha p}} \qquad (2.5)\end{aligned}$$

and

$$I_2 = \int_{\mathbb{R}^{n-1}} \left[|y' - y|^2 + x_n^2\right]^{-q(\frac{n}{2}-\alpha)} dy'. \tag{2.6}$$

To evaluate I_2, introduce spherical coordinates centred on y, i.e., $\rho = |y' - y|$. Then

$$\begin{aligned} I_2 &= \int_{\rho=0}^{\infty} [\rho^2 + x_n^2]^{-q(\frac{n}{2}-\alpha)} \int_{\partial B_\rho} dS_{n-2}\, d\rho \\ &= (n-1)\omega_{n-1} \int_{\rho=0}^{\infty} [\rho^2 + x_n^2]^{-q(\frac{n}{2}-\alpha)} \rho^{n-2}\, d\rho \\ &= (n-1)\,\omega_{n-1} x_n^{n-1-q(n-2\alpha)} \int_{\rho=0}^{\infty} [\rho^2+1]^{-q(n/2-\alpha)} \rho^{n-2} d\rho. \end{aligned} \tag{2.7}$$

This integral converges whenever $n - q(n - 2\alpha) < 1$ or $2\alpha p < p + n - 1$.

When $p = 1$ $(\alpha < n/2)$, (2.3) holds with $I_2^{\frac{1}{q}}$ replaced by

$$\sup_{y' \in \mathbb{R}^{n-1}} \left[|y' - y|^2 + x_n^2\right]^{-(\frac{n}{2}-\alpha)} = x_n^{-(n-2\alpha)}.$$

And, if $\alpha = n/2$, (2.3) holds with $I_2 = 1$.

Now, put $a = p$, $b = 2\alpha p$. Hence, (1.6) holds and $u = \mathcal{P}[f]$ exists on Π_+. Furthermore, by dominated convergence and (2.5), $I_1 \to 0$ as $|x| \to \infty$. The theorem follows by putting (2.5) and (2.7) into (2.3). □

Corollary 2.1. *If* (1.6) *holds, then* $u = \mathcal{P}[f](x) = o(|x| \sec^{n-1}\theta)$.

Proof. Let $a = 1$, $b = n$. □

Remarks. Corollary 2.1 with $n = 2$ was obtained by F. Wolf ([**15**]). If $a \geq 1$ and $f \in L^a$ then the Hölder inequality shows that (1.6) holds and $|u(x)| \leq c_{n,a}\|f\|_a x_n^{-(n-1)/a}$. The above constant is given in terms of the Beta function,

$$c_{n,a} = \frac{2}{n\omega_n}\left[\frac{(n-1)\,\omega_{n-1}}{2} B\left(\frac{n-1}{2}, \frac{n}{2(a-1)} + \frac{1}{2}\right)\right]^{1-\frac{1}{a}} \quad \text{when } a > 1$$

and $c_{n,1} = 2/(n\omega_n)$. It is obtained by evaluating (2.7) in the case $\alpha = 0$ ([4], 1.5.2). In [**1**], Theorem 7.11, an inequality of the same form is derived by a different method.

When f is majorised by a radial function a better estimate of $|u|$ is possible.

Proposition 2.1. *If* $|f(y)| \le F(|y|)$ *for* F *such that* $\int_{\rho=0}^{\infty} F(\rho)(\rho^2+1)^{-1}\,d\rho < \infty$ *then* $u(x) = \mathcal{P}[f](x) = o\left(|x|\sec\theta\right)$.

Proof. From (2.1) and (2.2) (and the binomial theorem),

$$\mathcal{K}(x, y') \le \frac{2^{\frac{n}{2}+1}}{n\omega_n} x_n (1 - \sin\theta\,\cos\theta_1)^{-\frac{n}{2}} (|y'|^n + |y'|^{n-2}|x|^2)^{-1}.$$

We then have

$$|u(x)| \le \frac{2^{\frac{n}{2}+1} x_n}{n\omega_n} \int_{\rho=0}^{\infty} \frac{F(\rho)\,d\rho}{\rho^2 + |x|^2} I_3,$$

where

$$I_3 = \int_{\partial B_1} (1 - \sin\theta\,\cos\theta_1)^{-\frac{n}{2}}\,dS_{n-2}.$$

The integral I_3 is singular when $\theta = \pi/2$ ($x_n = 0$). To determine the nature of the singularity we use the method of spherical means [**10**] to write

$$I_3 = (n-2)\,\omega_{n-2} \int_{\phi=0}^{\pi} (1 + \sin\theta\,\cos\phi)^{-\frac{n}{2}} \sin^{n-3}\phi\,d\phi.$$

Using the substitution $1 - 2t = \cos\phi$, an integral representation of the hypergeometric function and quadratic and linear transformations ([4], 2.12.1, 2.11.4, 2.9.2) we have

$$I_3 = \frac{2\sqrt{\pi}\,\omega_{n-2}\Gamma(\frac{n}{2})\,{}_2F_1(a, b; c; \sin^2\theta)}{\Gamma(\frac{n}{2} - \frac{1}{2})\cos^2\theta},$$

where $a = n/4 - 1$, $b = n/4 - 1/2$ and $c = n/2 - 1/2$. The hypergeometric function, ${}_2F_1$, (with these a, b, c) is bounded above (and below) by positive constants so that

$$|u(x)| \le \frac{A_n|x|}{\cos\theta} \int_{\rho=0}^{\infty} \frac{F(\rho)\,d\rho}{\rho^2 + |x|^2},$$

where A_n is a positive constant. As $|x| \to \infty$ we have $u(x) = o\left(|x|\sec\theta\right)$. □

The following example will show that the estimate on the Poisson integral in the above corollary is sharp. Define continuous data, f, to be zero except on a sequence of balls along the x_1-axis,

$$f(y) = \begin{cases} f_i(1 - \frac{1}{r_i}|y - a_i\hat{e}_1|), & y \in B_{r_i}(a_i\hat{e}_1) \subset \mathbb{R}^{n-1} \\ 0, & \text{otherwise,} \end{cases}$$

where f_i, a_i and r_i are sequences of positive real numbers such that $a_i \to \infty$, $r_i < 1$ and the $B_{r_i}(a_i \hat{e}_1)$ are disjoint. If $u = \mathcal{P}[f]$ then (1.6) is equivalent to convergence of the series

$$\sum_{i=1}^{\infty} \frac{f_i\, r_i^{n-1}}{a_i^n}. \tag{2.8}$$

We can write u as the superposition of translates of the solution to the normalised problem

$$\Delta \widetilde{u} = 0, \quad x \in \Pi_+$$
$$\widetilde{u} = \begin{cases} 1 - |y|, & x \in B_1 \subset \mathbb{R}^{n-1} \\ 0\,; & x_n = 0,\ x \notin B_1. \end{cases}$$

Thus, since $\widetilde{u} \geq 0$,

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} f_i\, \widetilde{u}\left(\frac{x - a_i \hat{e}_1}{r_i}\right) \\ &\geq f_i\, \widetilde{u}\left(\frac{x - a_i \hat{e}_1}{r_i}\right). \end{aligned}$$

Consider the sequence $x^{(m)} = a_m \hat{e}_1 + r_m \hat{e}_n$. We now show that if $\beta + \gamma < n$ or $\beta + \gamma = n$, $\gamma \leq 0$, then $u(x)|x|^{-\beta} \cos^\gamma \theta \not\to 0$ along this sequence. Put $a_i = e^i$, $f_i = e^{ni}$, $r_i = i^{-2}$. Then the series (2.8) converges and yet

$$\begin{aligned} \frac{(x_n^{(m)})^\gamma\, u(x^{(m)})}{|x^{(m)}|^{\beta+\gamma}} &\geq \frac{r_m^\gamma\, f_m\, \widetilde{u}(\hat{e}_n)}{(a_m^2 + r_m^2)^{(\beta+\gamma)/2}} \\ &= \frac{m^{-2\gamma}\, e^{mn}\, \widetilde{u}(\hat{e}_n)}{(e^{2m} + m^{-4})^{(\beta+\gamma)/2}} \\ &\not\to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

For any β and γ such that $\beta + \gamma < n$ or $\beta + \gamma = n$, $\gamma \leq 0$, we then have that $u \neq o(|x|^\beta \sec^\gamma \theta)$ as $|x| \to \infty$ in Π_+. The order estimate in Corollary 2.1 is sharp.

This example provides a solution to (1.1)–(1.3) that does not satisfy the classical uniqueness growth condition $u = o(|x|)$. In the next section we prove uniqueness for the half space problem under the relaxed growth condition $u = o(|x| \sec^\gamma \theta)$, for any $\gamma \in \mathbb{R}$, thus allowing for the behaviour encountered in Corollary 2.1.

3. Uniqueness.

The classical Phragmén-Lindelöf principle [**11**] ensures uniqueness to (1.1)–(1.3) under the growth condition $u = o(|x|)$ as $|x| \to \infty$ in Π_+. However, if $f \in D_0$ then $u = \mathcal{P}[f]$ is a solution even though $f(y)$ needn't be $o(|y|)$ for the Poisson integral to exist. In fact, existence of the Poisson integral does not imply any *a priori* pointwise behaviour of u on $\partial\Pi_+$. We now establish a theorem that guarantees a unique solution to (1.1)–(1.3) with a growth condition compatible with any data $f \in D_0$ and gives uniqueness to a harmonic polynomial when $f \in D_N$, $N \geq 1$.

We first prove the following.

Lemma 3.1. *If $h \in \mathcal{P}_k$ $(k \geq 0)$ and $p \geq 0$ is an integer, then there are $h_j \in \mathcal{P}_j$ such that*

$$x_i^p\, h(x) = \sum_{\ell=0}^{p} |x|^{2\ell}\, h_{k+p-2\ell}(x), \tag{3.1}$$

where i is a fixed integer, $1 \leq i \leq n$.

Proof. The proof is by induction on p.

If $p = 0$ the result is immediate.

If (3.1) holds for $0 \leq p \leq q$ then

$$x_i^{q+1} h(x) = x_i \sum_{\ell=0}^{q} |x|^{2\ell} H_{k+q-2\ell}(x) \qquad (H_j \in \mathcal{P}_j). \tag{3.2}$$

Writing $\lambda_j = [n + 2(j-1)]^{-1}$ for $j \geq 1$ and $\lambda_j = 0$ for $j \leq 0$, the function $\widehat{H}_j(x) = x_i\, H_j(x) - \lambda_j |x|^2 \frac{\partial H_j}{\partial x_i}$ is in $\mathcal{P}_{j+1}$ [**3**, p. 534]. Since $\frac{\partial H_j}{\partial x_i} \in \mathcal{P}_{j-1}$, (3.2) may be written

$$\begin{aligned} x_i^{q+1}\, h(x) &= \sum_{\ell=0}^{q} |x|^{2\ell} \left(\widehat{H}_{k+q-2\ell}(x) + \lambda_{k+q-2\ell}\, |x|^2\, \frac{\partial H_{k+q-2\ell}}{\partial x_i} \right) \\ &= \sum_{\ell=0}^{q+1} |x|^{2\ell}\, \widehat{\widehat{H}}_{k+q+1-2\ell}(x) \qquad (\text{for some } \widehat{\widehat{H}}_j \in \mathcal{P}_j) \end{aligned}$$

and the result follows. □

The spherical harmonics of degree k are the restriction of elements of $\mathcal{P}_k$ to the unit sphere. The lemma with $i = n$ may be written

$$\cos^p \theta\, Y_k(\widehat{x}) = \sum_{\ell=0}^{p} \widehat{Y}_{k+p-2\ell}(\widehat{x}) \qquad \text{where } Y_j,\ \widehat{Y}_j \in \mathcal{Y}_j. \tag{3.3}$$

We are now in a position to prove the following uniqueness theorem.

Theorem 3.1. *If $N \geq 0$ ($N \in \mathbb{Z}$) and $\gamma \in \mathbb{R}$ then any solution $u \in C^2(\Pi_+) \cap C^0(\overline{\Pi}_+)$ of*

$$\Delta u = 0, \quad x \in \Pi_+ \tag{3.4}$$

$$u = f, \quad x \in \partial\Pi_+ \tag{3.5}$$

$$u = o\left(|x|^{N+1} \sec^\gamma \theta\right) \quad \text{as } |x| \to \infty \text{ in } \Pi_+ \tag{3.6}$$

is unique to the addition of a harmonic polynomial of degree N that vanishes on $\partial\Pi_+$.

Proof. It suffices to prove the theorem for $\gamma \in \mathbb{Z}_+$. Let v be a solution to the corresponding homogeneous problem ($f \equiv 0$). It is equivalent to prove that $v \in \mathcal{P}_N$ and $v = 0$ on $\partial\Pi_+$. By the Schwarz reflection principle any such v must be harmonic in $\mathbb{R}^n$. The spherical harmonics expansion theorem ([**3**], p. 535) gives

$$v(x) = \sum_{k=1}^{\infty} |x|^k \, Y_k^{(0)}(\widehat{x}) \tag{3.7}$$

where we will write $Y_k^{(i)} \in \mathcal{Y}_k$ and $Y_k^{(0)}$ vanish on $\partial\Pi_+ \cap \partial B_1$.

Using (3.3) we have

$$\cos^\gamma \theta \, Y_k^{(0)}(\widehat{x}) = \sum_{\ell=0}^{\gamma} Y_{k+\gamma-2\ell}^{(k)}(\widehat{x}) \tag{3.8}$$

and

$$\cos^\gamma \theta \, v(x) = \sum_{k=1}^{\infty} |x|^k \sum_{\ell=0}^{\gamma} Y_{k+\gamma-2\ell}^{(k)}(\widehat{x}). \tag{3.9}$$

Let $j \in \mathbb{Z}_+$ and $0 \leq m \leq \gamma$. The series in (3.7) converges uniformly on compact sets and so may be integrated over the unit sphere term by term. With δ_{ab} the Kronecker delta, orthogonality of spherical harmonics gives

$$\begin{aligned} &\int_{\partial B_1} Y_{j+\gamma-2m}^{(j)}(\widehat{x}) \cos^\gamma \theta \, v(|x|\widehat{x}) \, dS_{n-1} \\ &= \sum_{k=1}^{\infty} |x|^k \sum_{\ell=0}^{\gamma} \delta_{j+\gamma-2m,k+\gamma-2\ell} \int_{\partial B_1} Y_{k+\gamma-2\ell}^{(j)}(\widehat{x}) \, Y_{k+\gamma-2\ell}^{(k)}(\widehat{x}) \, dS_{n-1}. \end{aligned} \tag{3.10}$$

The notation $v(|x|\widehat{x})$ indicates $|x|$ remains fixed for the integration. The condition

$j+\gamma-2m = k+\gamma-2\ell$ is satisfied by only a finite number of $k \in \mathbb{Z}_+$, $0 \le \ell \le \gamma$. The right member of (3.10) is then a polynomial in $|x|$ with no constant term. Integrating the order relation (3.6),

$$\int_{\partial B_1} Y^{(j)}_{j+\gamma-2m}(\widehat{x}) \cos^\gamma \theta \, v(|x|\widehat{x})\, dS_{n-1} = o\Big(|x|^{N+1} \int_{\partial B_1} Y^{(j)}_{j+\gamma-2m}(\widehat{x})\, dS_{n-1}\Big)$$
$$= o\left(|x|^{N+1}\right) \qquad (|x| \to \infty),$$

shows the coefficient of $|x|^j$ in (3.10) vanishes when $j > N$, i.e., $\|Y^{(j)}_{j+\gamma-2m}\|^2 = 0$. From (3.8), $Y^{(0)}_k \equiv 0$ for $k > N$. Hence, by (3.7), $v(x) \equiv 0$ if $N = 0$ and if $N \ge 1$,

$$v(x) = \sum_{k=1}^{N} |x|^k Y^{(0)}_k(\widehat{x}) \in \mathcal{P}_N.$$

The theorem follows. □

Corollary 3.1. *If* (1.6) *holds for continuous f then $u = \mathcal{P}[f]$ gives the unique solution to the Dirichlet problem* (1.1)–(1.3) *that satisfies the growth condition $u = o\left(|x| \sec^{n-1}\theta\right)$ $(|x| \to \infty$ in $\Pi_+)$.*

Proof. Use Corollary 2.1 and put $N = 0$, $\gamma = n-1$ in Theorem 3.1. □

4. Phragmén-Lindelöf principle.

In [**12**] a Phragmén-Lindelöf principle was proved in $\mathbb{R}^2$ with growth condition $u = o(|x| \sec\theta)$ (in our present notation). In this section we extend this result to $\mathbb{R}^n$ and to growth $o(|x| \sec^\gamma \theta)$ for any $\gamma \in \mathbb{R}$. Proofs of this type often involve barrier functions on half balls (e.g., [**7**], [**11**]). The weak maximum principle is applied on a half ball of radius ρ and then ρ is allowed to tend to infinity. It may be shown that if a barrier function has growth $|x| \sec^\gamma \theta$ on a half ball, then we must have $\gamma < 2$. In the following proof we employ a barrier on a convex polytope, T_ρ, (isosceles triangle in $\mathbb{R}^2$, pyramid in $\mathbb{R}^3$, see below) all of whose sides make an interior angle less than $\pi/2$ with $\partial\Pi_+$. This allows us to define a barrier function with growth $|x| \sec^\gamma \theta$ on the sides of the polytope, for any $\gamma \in \mathbb{R}$. The maximum possible angular growth of a barrier function defined in the region T_ρ increases as the interior angle that T_ρ makes with $\partial\Pi_+$ decreases. It is not known whether the theorem is true under growth $u = o(|x|/\phi(\cos\theta))$ for arbitrary positive ϕ with $\phi(0) = 0$. For $n = 2$, F. Wolf ([**14**]) proves the theorem holds whenever $\log\phi$ is integrable. H. Yoshida has provided an n-dimensional analogue ([**16**]). Our approach differs in that we use classical barriers.

Theorem 4.1. *Let $\gamma \in \mathbb{R}$. If $u \in C^2(\Pi_+)$ such that*

$$\Delta u \geq 0 \quad \text{in } \Pi_+ \tag{4.1}$$

$$\limsup_{x\in\Pi_+,\ x\to x_0} u(x) \leq 0 \quad \text{for any } x_0 \in \partial\Pi_+ \tag{4.2}$$

$$u = o\left(|x| \sec^\gamma \theta\right) \quad \text{as } |x| \to \infty \text{ in } \Pi_+ \tag{4.3}$$

then $u \leq 0$ in Π_+.

Proof. It suffices to prove the theorem for $\gamma > 2$. Let $\rho > 0$ and $m = \tan(\pi/(2\gamma))$ $(0 < m < 1)$. Define

$$T_\rho = \left\{x \in \mathbb{R}^n \,|\, 0 < x_n < \min_{1\leq i\leq n-1} m(\rho - |x_i|);\ |x_i| < \rho,\ 1 \leq i \leq n-1\right\}.$$

When $n = 2$, $\overline{T}_\rho$ is an isosceles triangle with vertices $(\pm\rho, 0)$ and $(0, m\rho)$ and common angle $\pi/(2\gamma)$. When $n = 3$, $\overline{T}_\rho$ is a square-based pyramid with base corners $(\pm\rho, \pm\rho, 0)$ and apex $(0, 0, m\rho)$. Write $\partial^+ T_\rho = \partial T_\rho \cap \Pi_+$ and $S = \{x \in \overline{T}_\rho |\ x_n = 0,\ |x_i| = \rho \text{ for some } 1 \leq i \leq n-1\}$. Note that $T_\rho \to \Pi_+$ as $\rho \to \infty$. A barrier function is a solution $\psi_\rho \in C^2(T_\rho) \cap C^0(\overline{T}_\rho \backslash S)$ of

$$\Delta\psi_\rho \leq 0, \quad x \in T_\rho \tag{4.4}$$

$$\psi_\rho(x) \geq |x| \sec^\gamma \theta, \quad x \in \partial^+ T_\rho \tag{4.5}$$

$$\psi_\rho \geq 0, \quad x \in \overline{T}_\rho \backslash S \tag{4.6}$$

(ψ_ρ is not defined on S).

Define ψ_ρ by writing

$$\psi_{i,\pm}(\rho; x) = \frac{(n-1)^{(\gamma+1)/2}\rho^{\gamma+1} \sin(\gamma\, \theta_{i,\pm})}{\sin^\gamma(\pi/(2\gamma))\, r_{i,\pm}^\gamma}$$

and $\psi_\rho = \sum_{i=1}^{n-1}(\psi_{i,+} + \psi_{i,-})$. For $x \in T_\rho$, $\theta_{i,\pm}$ is the angle by which x is above the $x_n = 0$ hyperplane measured from the edge of $\overline{T}_\rho$ on $\partial\Pi_+$ through $x_i = \pm\rho$. Explicitly,

$$\cos\theta_{i,\pm} = \frac{\rho \mp x_i}{[(x_i \pm \rho)^2 + x_n^2]^{1/2}}, \quad 0 \leq \theta_{i,\pm} \leq \frac{\pi}{2\gamma}$$

(the angle between the vectors $(x_i \mp \rho)\hat{e}_i + x_n\hat{e}_n$ and $\mp\hat{e}_i$). And, $r_{i,\pm} = [(x_i \pm \rho)^2 + x_n^2]^{1/2}$, the distance from x to the edge of $\overline{T}_\rho$ on $\partial\Pi_+$ through $x_i = \pm\rho$.

With the usual polar coordinates in $\mathbb{R}^2$, $x = r\cos\phi$, $y = r\sin\phi$, the function $r^{-\gamma}\sin(\gamma\phi)$ is harmonic. Identifying $x \mapsto \rho \mp x_i$, $y \mapsto x_n$, we see that each $\psi_{i,\pm}$ is harmonic in T_ρ. Each $\psi_{i,\pm}$ is non-negative in T_ρ and although $\psi_{i,\pm}$ does not have a limit as $x \to x_0 \in S$ it is true that as $x \to x_0$ in T_ρ, $\liminf \psi_{i,\pm}(x) \geq 0$ for any $x_0 \in S$. Also, $\psi_{i,\pm}$ vanishes when $x_n = 0$, $|x_j| < \rho$ $(1 \leq j \leq n-1)$. To show ψ_ρ is a barrier function we need only prove (4.5). Let $x \in \partial^+ T_\rho$ such that $x_n = m(\rho - x_1)$, $0 \leq x_1 < \rho$, i.e., x is on the face through $x_1 = \rho$. We have a right triangle with vertices $P = x$, $Q = (\rho, x_2, x_3, \cdots, x_{n-1}, 0)$, $R = (x_1, x_2, \cdots, x_{n-1}, 0)$, hypotenuse $r_{1,+}$, side $PR = x_n = |x| \cos\theta$, $\angle PQR = \theta_{1,+}$, $\angle PRQ = \pi/2$. Therefore, $r_{1,+} = |x| \cos\theta \csc\theta_{1,+}$. Also, $\theta_{1,+} = \pi/(2\gamma)$ and $|x| \leq \sqrt{n-1}\,\rho$. Hence,

$$\begin{aligned}
\psi_\rho(x) &\geq \psi_{1,+}(x) \\
&= \frac{(n-1)^{(\gamma+1)/2}\,\rho^{\gamma+1}\,\sin(\pi/2)}{\sin^\gamma(\pi/(2\gamma))\; r_{1,+}^\gamma} \\
&\geq |x| \sec^\gamma \theta.
\end{aligned}$$

Similarly when x is on one of the other $2n-3$ faces of $\partial^+ T_\rho$ $(x_n = m(\rho \pm x_i),\ 0 \leq \pm x_i < \rho)$. Hence, ψ_ρ satisfies (4.4)–(4.6) and is a barrier function.

Fix x in T_ρ. We have

$$\theta_{i,\pm} = \arctan\left(\frac{x_n}{\rho \mp x_i}\right) \sim \frac{x_n}{\rho} \quad \text{as } \rho \to \infty,$$

$$r_{i,\pm} = \sqrt{(x_i \pm \rho)^2 + x_n^2} \sim \rho \quad \text{as } \rho \to \infty.$$

Therefore, $\psi_{i,\pm}(x) \sim (n-1)^{(\gamma+1)/2} \rho \sin(\gamma x_n/\rho) \csc^\gamma(\pi/(2\gamma))$ and

$$\lim_{\rho\to\infty} \psi_\rho(x) = \frac{2(n-1)^{(\gamma+3)/2}\gamma\, x_n}{\sin^\gamma(\pi/(2\gamma))}. \tag{4.7}$$

Now, let $\epsilon > 0$. Since $u = o(|x| \sec^\gamma \theta)$ it follows that $u \leq \epsilon\,\psi_\rho$ on $\partial^+ T_\rho$ for sufficiently large ρ. Write $w = u - \epsilon\,\psi_\rho$. With ρ as above,

$$\Delta w \geq 0, \quad x \in T_\rho \tag{4.8}$$

$$w \leq 0, \quad x \in \partial^+ T_\rho \tag{4.9}$$

$$\limsup_{x\in T_\rho, x \to x_0} w \leq 0 \quad \text{for any } x_0 \in \overline{T}_\rho \cap \partial\Pi_+. \tag{4.10}$$

Note that (4.10) holds in particular when x_0 is in the singular set S (on an edge in the $x_n = 0$ hyperplane). For as $x \to x_0$ in T_ρ

$$\begin{aligned}
\limsup w(x) &\leq \limsup u(x) - \epsilon \liminf \psi_\rho(x) \\
&\leq 0.
\end{aligned}$$

The weak maximum principle ([**8**], §3.1) applied to w shows that $w \le 0$ in T_ρ. Finally, given $x \in \Pi_+$, let ρ be large enough so that $x \in T_\rho$ ($\rho > |x|\csc(\pi/(2\gamma))$ suffices). Then, using (4.7)

$$\begin{aligned} 0 &\ge \lim_{\rho\to\infty}[u(x) - \epsilon\,\psi_\rho(x)] \\ &= u(x) - \frac{2\,\epsilon\,(n-1)^{(\gamma+3)/2}\gamma\, x_n}{\sin^\gamma(\pi/(2\gamma))} \end{aligned}$$

and ϵ was arbitrary so $u(x) \le 0$. Hence, $u \le 0$ in Π_+. □

Remarks. Condition (4.3) may be replaced with the weaker condition

$$\limsup_{r\to\infty}\left\{\sup_{\substack{|x|=r\\ x\in\Pi_+}}[u(x)|x|^{-1}\cos^\gamma\theta]\right\} \le 0.$$

Also, if $u \in C^0(\overline{\Pi}_+)$ then (4.2) may be replaced by $u \le 0$ on $\partial\Pi_+$.

For data $f \in D_0$, Theorem 4.1 provides an alternate proof of the uniqueness result, Corollary 3.1: if u_1 and u_2 are solutions of (1.1)–(1.3) that satisfy (4.3) then let $v = u_1 - u_2$ and apply Theorem 4.1 to v and $-v$.

5. A modified Poisson integral.

When the integral in (1.6) diverges and $f \in D_M$ for some $M \ge 1$ it is possible to solve (1.1)–(1.3) with an appropriately modified Poisson integral. Following [**5**] and [**12**] we define the modified Poisson kernel

(5.1)
$$\mathcal{K}_M(x,y') = \begin{cases} \mathcal{K}(x,y'), & |y'| < 1 \\ \mathcal{K}(x,y') - \dfrac{2x_n}{n\omega_n}\displaystyle\sum_{m=0}^{M-1}\frac{|x|^m}{|y'|^{m+n}}C_m^{\frac{n}{2}}(\sin\theta\,\cos\theta_1), & |y'| > 1, \end{cases}$$

where $C_m^{\frac{n}{2}}$ are Gegenbauer polynomials (see [**13**] for results concerning Gegenbauer polynomials used in this section). Write

$$U_M(x) = \int_{\mathbb{R}^{n-1}} \mathcal{K}_M(x,y')f(y')\,dy'. \tag{5.2}$$

A generating function for Gegenbauer polynomials is

$$(1 - 2tz + z^2)^{-\lambda} = \sum_{m=0}^{\infty} z^m\,C_m^\lambda(t), \qquad |z| < 1,\ \lambda \ne 0. \tag{5.3}$$

Thus, in $\mathcal{K}_M$ the first M terms of the Taylor expansion of $\mathcal{K}$ in $|x|/|y'|$ are removed (for $|y'| > 1$). Using the method in **[12]** we see that U_M satisfies (1.1)–(1.3).

We now show that U_M satisfies the growth condition of Theorem 2.1 with $a = 1$, $b = n + M$.

Theorem 5.1. *If $f \in D_M$ $(M \geq 1)$ then $U_M = o(|x|^{M+1}\sec^{n-1}\theta)$.*

Proof. In order to determine the behaviour of $\mathcal{K}_M$ we consider the series $S_{M-1}(s) = \sum_{m=0}^{M-1} s^m C_m^\lambda(t)$. The Gegenbauer polynomials satisfy the recurrence relation

$$(m+2)\,C_{m+2}^\lambda(t) - 2(\lambda+m+1)\,t\,C_{m+1}^\lambda(t) + (2\lambda+m)\,C_m^\lambda(t) = 0. \tag{5.4}$$

Following the method in **[13]** used to derive (5.3), we sum (5.4) from $m = 0$ to $m = M - 1$. This yields a first order linear ordinary differential equation for $S_{M-1}(s)$. Solving this for $S_{M-1}(s)$, we find (for $|y'| > 1$)

$$\begin{aligned}\mathcal{K}_M(x,y') = \mathcal{K}(x,y')\Big[&MC_M^{\frac{n}{2}}(\sin\theta\cos\theta_1)I_{M-1}\left(\frac{|x|}{|y'|},\sin\theta\cos\theta_1\right)\\ &-(n+M-1)\,C_{M-1}^{\frac{n}{2}}(\sin\theta\cos\theta_1)I_M\left(\frac{|x|}{|y'|},\sin\theta\cos\theta_1\right)\Big],\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}I_M(s,t) &= \int_{\zeta=0}^{s}(1-2t\zeta+\zeta^2)^{\frac{n}{2}-1}\zeta^M\,d\zeta\\ &\leq \int_{\zeta=0}^{s}(1+\zeta)^{n-2}\zeta^M\,d\zeta \quad \text{if } s>0 \text{ and } |t|\leq 1.\end{aligned}$$

Using the bound

$$\begin{aligned}|C_m^\lambda(t)| &\leq C_m^\lambda(1)\\ &= \binom{m+2\lambda-1}{m} \qquad (|t|\leq 1,\ \ m\geq 0)\end{aligned}$$

it follows that

$$|\mathcal{K}_M(x,y')| \leq n\binom{M+n-1}{n}\mathcal{K}(x,y')I\left(\frac{|x|}{|y'|}\right), \tag{5.6}$$

where $I(s) = \int_{\zeta=0}^{s}(1+\zeta)^{n-2}(\zeta^{M-1}+\zeta^M)\,d\zeta$.

Let $E > 1$ and $|x| > 2E$. Let $\mathcal{J}_1 = [0,1]$, $\mathcal{J}_2 = [1,E]$, $\mathcal{J}_3 = [E,|x|/2]$ and $\mathcal{J}_4 = [|x|/2,\infty)$ and define $J_i = \int_{|y'|\in\mathcal{J}_i}\mathcal{K}_M(x,y')f(y')\,dy'$ $(i = 1,2,3,4)$ so that $U_M = J_1 + J_2 + J_3 + J_4$.

The integral J_1 is equivalent to the Poisson integral with data having compact support. Therefore, $J_1 = \mathcal{O}(x_n|x|^{-1})$.

When $|y'| \in \mathcal{J}_2$, $I(|x|/|y'|) \leq 2|x|^{M+n-1}$ for sufficiently large $|x|$. Also,

$$\begin{aligned}[|y'-y|^2 + x_n^2]^{-\frac{n}{2}} &\leq |x|^{-n}\left(1 - \frac{|y'|}{|x|}\right)^{-n} \\ &\leq 2^n|x|^{-n} \qquad \text{for } |x| \geq 2E.\end{aligned}$$

Hence, from (5.6)

$$|J_2| \leq \frac{2^{n+1}x_n}{\omega_n}\binom{M+n-1}{n}|x|^{M-1}\left|\int_{\mathcal{J}_2} f(y')\,dy'\right|$$

and $J_2 = \mathcal{O}(x_n|x|^{M-1})$.

When $|y'| \in \mathcal{J}_3$,

$$\begin{aligned}I\left(\frac{|x|}{|y'|}\right) &\leq \left(\frac{|x|}{|y'|}\right)^M \left(\frac{|x|}{2|y'|} + \frac{|x|}{|y'|}\right)^{n-1} \\ &\leq \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}\left(\frac{|x|}{|y'|}\right)^{M+n}.\end{aligned}$$

Also, $[|y'-y|^2 + x_n^2]^{-\frac{n}{2}} \leq 2^n|x|^{-n}$. Given $\epsilon > 0$ we now take $E > 0$ so that $|\int_{\mathcal{J}_3} f(y')|y'|^{-(n+M)}\,dy'| \leq \epsilon$ whenever $|x| \geq 2E$. Then $J_3 = o(x_n|x|^M)$.

When $|y'| \in \mathcal{J}_4$,

$$\begin{aligned}I\left(\frac{|x|}{|y'|}\right) &\leq \left(1 + \frac{|x|}{|y'|}\right)^{n-1}\left(\frac{|x|}{|y'|}\right)^M \\ &\leq 3^{n-1}\left(\frac{|x|}{|y'|}\right)^M.\end{aligned}$$

Therefore,

$$|J_4| \leq 3^{n-1}n\binom{M+n-1}{n}|x|^M\left|\int_{\mathcal{J}_4} f(y')|y'|^{-M}\mathcal{K}(x,y')\,dy'\right|$$

and $J_4 = o(|x|^{M+1}\sec^{n-1}\theta)$ by Corollary 2.1.

Hence, $U_M = o(|x|^{M+1}\sec^{n-1}\theta)$ as $|x| \to \infty$ in Π_+. □

Corollary 5.1. *If $|f(y)| \leq F(|y|)$ and $F \in D_M$ then $U_M = o(|x|^{M+1}\sec\theta)$.*

Proof. The radial majorisation estimate of Proposition 2.1 is used with J_4 above. □

The modified kernels, $\mathcal{K}_M$ $(M \geq 1)$, are not positive. In fact we have the following result.

Theorem 5.2. *If $f \geq 0$ such that $\int_{\mathbb{R}^{n-1}} f(y')\left(|y'|^n + 1\right)^{-1} dy' = \infty$ then there are no positive solutions to* (1.1)–(1.3).

Proof. Introduce a cutoff function, ξ_N, such that

$$\xi_N = \begin{cases} 1, & |x| \leq N \\ 0, & |x| \geq N+1, \end{cases}$$

$0 \leq \xi_N \leq 1$ and ξ_N is continuous.

Suppose $u \geq 0$ and satisfies (1.1)–(1.3). Let $u_N = \mathcal{P}[f\xi_N]$. Given $\epsilon > 0$, we claim that $u \geq u_N - \epsilon$ on $\partial B_\rho \cap \Pi_+$ for large enough ρ. Indeed, we have $\Delta u_N = 0$ in Π_+, $u_N = f\xi_N$ on $\partial\Pi_+$ and $u_N = \mathcal{O}(x_n|x|^{-1})$ as $|x| \to \infty$ (since $f\xi_N$ has compact support). So, $u \geq u_N$ on $\partial\Pi_+$ and $|u_N| < \epsilon$ on $\partial B_\rho \cap \Pi_+$ for large enough ρ. Therefore, $u \geq u_N - \epsilon$ on ∂B_ρ^+ $(\partial B_\rho^+ = \{x \in \mathbb{R}^n|\, |x| = \rho,\ x_n > 0\})$. Since ϵ is arbitrary, $u \geq u_N$ on ∂B_ρ^+. By the weak maximum principle, $u \geq u_N$ in B_ρ^+. But,

$$\begin{aligned} u_N(x) &\geq \int_{|y'| \leq N} \mathcal{K}(x, y') f(y')\, dy' \\ &\to \infty \quad \text{as } N \to \infty. \end{aligned}$$

Hence, there can be no such u. □

This theorem can also be deduced from the general representation of non-negative harmonic functions on Π_+ (Theorem 7.24 in [**1**]).

6. Conclusion.

We propose three directions for further work in this area. Using the known integral representation of solutions of the half space Neumann and Robin problems it should be possible to obtain analogous results to those in this paper. As per the remarks at the beginning of §4, a Phragmén-Lindelöf principle with maximum angular growth is desirable. Also, in [**7**] and [**9**], the classical Phragmén-Lindelöf principle is extended to uniformly elliptic operators in Π_+. Their methods do not rely on explicit representations of solutions and it is possible their results may be expanded to include a growth condition that has angular dependence, as in (4.3).

Work of H. Yoshida, [**17**], gives related results to ours using an integral condition,

$$\int_{\partial^+ B_1} u(r\hat{x}) \cos\theta\, dS_{n-1} = o(r) \quad \text{as } r \to \infty, \tag{7.1}$$

rather than a pointwise one. Combining Theorem 3 and Lemma 3 of [**17**], we have the result that if f satisfies (1.6), then $u = \mathcal{P}[f]$ is the unique solution of the Dirichlet problem (1.1)–(1.3) that satisfies (7.1). From Theorem 2 of [**17**], a Phragmén-Lindelöf principle holds with condition (4.3) replaced by (7.1).

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