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FOURIER MULTIPLIERS FOR $L_p(\mathbb{R}^n)$ VIA q–VARIATION

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We give a new sufficient condition for a function to be a Fourier multiplier of $L_p(\mathbb{R}^n)$ via its q-variation on dyadic rectangles. This solves a problem posed by Coifman, Rubio de Francia and Semmes, who had considered the one-dimensional case.

1. Introduction.

Let I be an interval of $\mathbb{R}$. For $1 \le q < \infty$ we denote by $V_q(I)$ the space of all the complex-valued functions of bounded q-variation on I, that is, $V_q(I)$ consists of the functions m on I such that

$$\|m\|_{V_q(I)} = \sup \left(|m(x_0)|^q + \sum_{k\ge 0} |m(x_{k+1}) - m(x_k)|^q \right)^{1/q} < \infty,$$

where the supremum is taken over all increasing sequences $\{x_k\}_{k\ge 0}$ in I.

In [**2**], Coifman, Rubio de Francia and Semmes proved the following considerable improvement of the classical Marcinkiewicz multiplier theorem for $L_p(\mathbb{R})$.

Theorem A. *Let $I_k = [2^k, 2^{k+1}]$ and $J_k = [-2^{k+1}, -2^k]$ for every $k \in \mathbb{Z}$. Let $m \in L_\infty(\mathbb{R})$. If $\sup_{k\in\mathbb{Z}}(\|m\|_{V_q(I_k)} + \|m\|_{V_q(J_k)}) < \infty$ for some $1 \le q < \infty$, then m is a Fourier multiplier for $L_p(\mathbb{R})$ for every $1 < p < \infty$ satisfying $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$.*

The ingredient of the proof of Theorem A in [**2**] is Rubio de Francia's generalized Littlewood-Paley inequality for arbitrary families of disjoint intervals (cf. [**7**]). Let us emphasize that the above theorem is one-dimensional, while the classical Marcinkiewicz theorem holds as well in the multiple dimensional case. The problem of extending Theorem A to $\mathbb{R}^n$ was left open in [**2**]. The purpose of this note is to solve it.

Let us define the space of functions of bounded q-variation on a rectangle of $\mathbb{R}^n$. We consider only rectangles with sides parallel to the axes, and also we restrict ourself to finite rectangles. Now let R be such a rectangle. Write $R = \prod_{k=1}^n [a_k, b_k]$. Let m be a function defined on R. Define Δ_R by

$$\Delta_R(m) = \Delta_{h_1}^{(1)} \Delta_{h_2}^{(2)} \cdots \Delta_{h_n}^{(n)} m(a_1, a_2, \cdots, a_n),$$

where $h_k = b_k - a_k$ and where $\Delta^{(k)}$ is the usual difference operator in the k-th variable (with all the others fixed), i.e., for any function f on $\mathbb{R}^n$ and any positive real number h

$$\Delta_h^{(k)}(f)(x_1, \cdots, x_n) = f(x_1, \cdots, x_{k-1}, x_k + h, x_{k+1}, \cdots, x_n) - f(x_1, \cdots, x_n).$$

Now for $1 \leq q < \infty$ we define $V_q(R)$ inductively in n. We already have the definition for the case $n = 1$. Thus suppose $n \geq 2$. Let m be a function on R. We say that m is of bounded q-variation on R if the following properties are satisfied:

(i) for each $1 \leq k \leq n-1$ the function $m(x_1, \cdots, x_k, a_{k+1}, \cdots, a_n)$, considered as a function of the first k variables, is of bounded q-variation on the k-dimensional rectangle $\prod_{j=1}^k [a_j, b_j]$;

(ii) the condition analogous to (i) is valid for each permutation of the variables $x_1, x_2, \cdots, x_n$;

(iii) for the full n variables we have

$$\sup_{\mathcal{Q}} \left(\sum_{Q \in \mathcal{Q}} |\Delta_Q(m)|^q \right)^{1/q} < \infty,$$

where the supremum runs over all decompositions $\mathcal{Q}$ of R into subrectangles of disjoint interior.

We define $\|m\|_{V_q(R)}$ as the sum of all the quantities appearing in (i) - (iii) and denote by $V_q(R)$ the space of all functions of bounded q-variation on R. This is a Banach space equipped with the norm $\|\cdot\|_{V_q(R)}$.

Remark. $V_1(R)$ is the usual space of functions of bounded variation on R; moreover, $m \in V_1(R)$ if m is n times continuously differentiable in the interior of R and satisfies the following

(i) for each $1 \leq k \leq n-1$

$$\int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \left| \frac{\partial^k m}{\partial x_1 \cdots \partial x_k}(x_1, \cdots, x_k, a_{k+1}, \cdots, a_n) \right| dx_1 \cdots dx_k < \infty,$$

(ii) the condition analogous to (i) is valid for each permutation of the variables $x_1, x_2, \cdots, x_n$;

(iii) and

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \frac{\partial^n m}{\partial x_1 \cdots \partial x_n}(x_1, \cdots, x_n) \right| dx_1 \cdots dx_n < \infty.$$

Also observe that we can slightly weaken the above condition (iii) by requiring only that m be $n-1$ times continuously differentiable on R and that $\frac{\partial^n m}{\partial x_1 \cdots \partial x_n}(x_1, \cdots, x_n)$ exist (in a reasonable sense) and be absolutely Riemann integrable on R.

For stating the analogue of Theorem A for $\mathbb{R}^n$ we need recall the notion of dyadic rectangles, used in the classical Littlewood-Paley theory (cf. [**8**]). First for $n = 1$ a dyadic interval is an interval of $\mathbb{R}$ from one of the sequences $\{I_k\}$ and $\{J_k\}$ introduced in Theorem A. Then by a dyadic rectangle of $\mathbb{R}^n$ we mean a rectangle R which is a product of n dyadic intervals. Let $\mathcal{D}$ denote the family of the dyadic rectangles. Let us also recall that $A_p^*(\mathbb{R}^n) = A_p(\mathbb{R} \times \cdots \times \mathbb{R})$ denotes the class of the Muckenhoupt A_p-weights on $\mathbb{R}^n$ in the product sense (cf. [**4**]).

For a function $m \in L_\infty(\mathbb{R}^n)$, the corresponding multiplier operator is denoted by T_m, i.e., $\widehat{T_m(f)} = m\hat{f}$, where $\hat{f}$ is the Fourier transform of f. The result of this note is the following theorem, which solves the problem in [**2**] for $\mathbb{R}^n$.

Theorem.

(i) *Let* $1 \le q < \infty$ *and* $m \in L_\infty(\mathbb{R}^n)$. *If* $\sup_{R \in \mathcal{D}} \|m\|_{V_q(R)} < \infty$, *then* m *is a Fourier multiplier for* $L_p(\mathbb{R}^n)$ *for any* $p \in (1, \infty)$ *satisfying* $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$.

(ii) *If* m *satisfies the above condition with* $q = 2$, *then for any* $w \in A_1^*(\mathbb{R}^n)$ *we have*

$$\int_{\mathbb{R}^n} |T_m(f)(x)|^2 w(x)\, dx \le C \sup_{R \in \mathcal{D}} \|m\|^2_{V_2(R)} \int_{\mathbb{R}^n} |f(x)|^2 w(x)\, dx, \quad \forall\, f \in L_2(w),$$

where C *is a constant depending on* w.

Remark. In particular, if $q = 2$, then m is a Fourier multiplier for $L_p(\mathbb{R}^n)$ for all $p \in (1, \infty)$. Recall that the classical Marcinkiewicz multiplier theorem corresponds to the case $q = 1$.

2. Proof.

Now we proceed to prove the theorem. As in [**2**], the ingredient of our proof is again the generalized Littlewood-Paley inequality of Rubio de Francia [**7**]. The new point is a simple observation: a function of bounded q-variation on a rectangle can be regarded as a vector-valued function of bounded q-variation on an interval (see Lemma 1 below). This observation allows us to iterate one-dimensional results.

Let us first extend the definition of q-variation to functions with values in a Banach space B. This is done just by replacing the absolute value by the norm of B. Thus if I is an interval (say, $I = [a, b]$ a finite interval), $V_q(B, I)$ is the space of all functions m from I to B such that

$$\|m\|_{V_q(B,I)} = \sup_{a=x_0<x_1<\cdots\leq b} \left(\|m(a)\|^q + \sum_{k\geq 0} \|m(x_{k+1}) - m(x_k)\|^q \right)^{1/q} < \infty.$$

Note that the above V_q-norm is not equal but equivalent to that introduced at the beginning.

Now let R be a rectangle in $\mathbb{R}^n$. Write $R = \prod_{k=1}^n I_k$ with $I_k = [a_k, b_k]$. Then we can introduce the space $V_q(V_q(\cdots(V_q(I_n)\cdots), I_2), I_1)$ of functions on R. This space is denoted by $\tilde{V}_q(R)$. For example, if $n = 2$, a function m on $R = I_1 \times I_2$ belongs to $\tilde{V}_q(R)$ iff for all $x_1 \in I_1$ the function $m(x_1, \cdot)$ belongs to $V_q(I_2)$ uniformly in $x_1 \in I_1$ and the vector-valued function $x_1 \mapsto m(x_1, \cdot) \in V_q(I_2)$ belongs to $V_q(V_q(I_2), I_1)$.

The following lemma is elementary and almost obvious.

Lemma 1. *For any $1 \leq q < \infty$ and any rectangle R in $\mathbb{R}^n$ we have $V_q(R) \subset \tilde{V}_q(R)$ of inclusion norm ≤ 1.*

By Lemma 1 it suffices to prove the theorem for $\tilde{V}_q$ in place of V_q.

To continue the proof of the theorem we need introduce another space U_q related to V_q. Let B be a Banach space and I an interval of $\mathbb{R}$. Denote by $\mathcal{E}$ the family of all step functions m from I to B which can be written as

$$m = \sum_{k=1}^{K} a_k \chi_{I_k},$$

where $\{a_k\} \subset B$ and $\{I_k\}$ is a finite sequence of disjoint intervals with $I = \bigcup_k I_k$ (χ_e standing for the indicator function of a subset e). For $m \in \mathcal{E}$ as above set

$$[\![m]\!] = \left(\sum_{k=1}^{K} \|a_k\|^q \right)^{1/q}$$

and

$$\|m\|_{U_q(B,I)} = \inf \left\{ \sum_{j=1}^{J} [\![m_j]\!] \ : \ m = \sum_{j=1}^{J} m_j,\ m_j \in \mathcal{E},\ 1 \leq j \leq J,\ J \geq 1 \right\}.$$

Then $\|\cdot\|_{U_q(B,I)}$ is a norm on $\mathcal{E}$. The completion of $\mathcal{E}$ with respect to this norm is denoted by $U_q(B, I)$. If $B = \mathbb{C}$, $U_q(\mathbb{C}, I)$ is simply denoted by $U_q(I)$.

It is trivial that $U_q(B,I) \subset V_q(B,I)$. Clearly, the inverse inclusion is not true. However, we have the following

Lemma 2. *Let B be a Banach space and I an interval. Let $1 \le p < q < \infty$. Then $V_p(B,I) \subset U_q(B,I)$ of inclusion norm bounded by a constant depending only on p and q.*

This lemma is proved in [**2**] for $B = \mathbb{C}$. The same proof works for any B. Note that Lemma 2 was already known to the authors of [**6**]. It follows from more general results on the real interpolation spaces between the V_q-spaces (see [**6**] and [**9**]).

If $R = \prod_{k=1}^n I_k$ is a rectangle in $\mathbb{R}^n$, as for $\tilde{V}_q$ above, let

$$\tilde{U}_q(R) = U_q(U_q(\cdots(U_q(I_n),\cdots),I_2),I_1).$$

This is a space of functions on R. We have the obvious inclusion $\tilde{U}_q(R) \subset \tilde{V}_q(R)$; also Lemma 2 implies that $\tilde{V}_p(R) \subset \tilde{U}_q(R)$ for $1 \le p < q < \infty$. The following lemma is the n-dimensional analogue of a lemma in [**2**].

Lemma 3. *For any $m \in \tilde{U}_2(\mathbb{R}^n)$ and $w \in A_1^*(\mathbb{R}^n)$ we have*

$$\left(\int_{\mathbb{R}^n} |T_m f(x)|^2 w(x)\,dx\right)^{1/2} \le C\,\|m\|_{\tilde{U}_2(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx\right)^{1/2}, \quad \forall f \in L_2(w),$$

where C is a constant depending on w.

Proof. It is based on Rubio de Francia's inequality. Let us recall this inequality (in its weighted form; see [**7**] and [**2**]). Let $\{I_k\}$ be an arbitrary sequence of disjoint intervals in $\mathbb{R}$. Let S_{I_k} denote the partial sum operator associated to I_k (i.e., S_{I_k} is the multiplier operator with symbol χ_{I_k}). Then for any $w \in A_1(\mathbb{R})(= A_1^*(\mathbb{R}))$

$$(*) \qquad \int_{\mathbb{R}} \left(\sum_k |S_{I_k} f|^2\right) w \le C \int_{\mathbb{R}} |f|^2 w, \quad \forall\, f \in L_2(w),$$

where C is a constant depending on w only.

Now we show Lemma 3 by induction on n. The case $n = 1$ is just $(*)$. Then suppose the lemma is true for $n-1$ ($n \ge 2$). Let $w \in A_1^*(\mathbb{R}^n)$ and $m \in \tilde{U}_2(\mathbb{R}^n)$ of norm 1. By convexity, we may suppose m is a step function as in the definition of $U_2(B,\mathbb{R})$, where $B = \tilde{U}_2(\mathbb{R}^{n-1})$ (in the last $n-1$ variables). Thus m is a finite sum

$$m(x_1, x_2, \cdots, x_n) = \sum_k \lambda_k a_k(x_2,\cdots,x_n)\chi_{I_k}(x_1),$$

where $a_k \in \tilde{U}_2(\mathbb{R}^{n-1})$ of norm 1, $\{I_k\}$ are disjoint intervals and $\lambda_k \geq 0$ are such that $\sum \lambda_k^2 \leq 1$. Let T_k and S_k be the multiplier operators with symbols $\chi_{\mathbb{R}}\, a_k$ and $\chi_{I_k \times \mathbb{R}^{n-1}}$ respectively. Then

$$T_m = \sum_k \lambda_k T_k S_k.$$

Recall that if $w \in A_1^*(\mathbb{R}^n)$, then $w(x_1, \cdot) \in A_1^*(\mathbb{R}^{n-1})$ uniformly for all $x_1 \in \mathbb{R}$ and also $w(\cdot, x_2, \cdots, x_n) \in A_1(\mathbb{R})$ uniformly for all $(x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$. Therefore, by the Fubini theorem, the induction assumption on $n-1$ and $(*)$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |T_m f|^2 w &\leq \sum_k \int_{\mathbb{R}^n} |T_k(S_k f)|^2 w \\
&= \sum_k \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}^{n-1}} |T_k(S_k f)|^2 w \, dx_2 \cdots dx_n \\
&\leq C \sum_k \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}^{n-1}} |S_k f|^2 w \, dx_2 \cdots dx_n \\
&= C \int_{\mathbb{R}^{n-1}} dx_2 \cdots dx_n \int_{\mathbb{R}} \sum_k |S_k f|^2 w \, dx_1 \\
&\leq C\, C' \int_{\mathbb{R}^{n-1}} dx_2 \cdots dx_n \int_{\mathbb{R}} |f|^2 w \, dx_1 \\
&= C\, C' \int_{\mathbb{R}^n} |f|^2 w.
\end{aligned}$$

Thus the lemma is proved. □

Now we can deduce the theorem from Lemmas 2 and 3. Recall that if R is a rectangle, S_R denotes the associated partial sum operator.

Proof of the Theorem. Let us first prove (ii). Fix $w \in A_1^*(\mathbb{R}^n)$. Then by the reverse Hölder inequality $w^\alpha \in A_1^*(\mathbb{R}^n)$ for some $\alpha > 1$. Let $1/\theta = \alpha$ and $q = 2\alpha$. Then $0 < \theta < 1$ and $q > 2$. We claim that for any $m \in \tilde{U}_q(\mathbb{R}^n)$, T_m is bounded on $L_2(w)$. This follows from Lemma 3 by interpolation. Indeed, let us consider the bilinear operator $\mathcal{B}$ defined by $\mathcal{B}(m, f) = T_m f$. By Lemma 3, $\mathcal{B}$ is bounded from $\tilde{U}_2(\mathbb{R}^n) \times L_2(w^\alpha)$ to $L_2(w^\alpha)$; on the other hand, $\mathcal{B}$ is obviously bounded from $L_\infty(\mathbb{R}^n) \times L_2$ to L_2. Therefore, by the complex interpolation (see [**1**]), $\mathcal{B}$ is bounded from $(L_\infty(\mathbb{R}^n), U_2(\mathbb{R}^n))_\theta \times L_2(w)$ to $L_2(w)$. Thus if $m \in \left(L_\infty(\mathbb{R}^n), \tilde{U}_2(\mathbb{R}^n)\right)_\theta$, then T_m is bounded on $L_2(w)$; it is however clear that $\tilde{U}_q(\mathbb{R}^n) \subset \left(L_\infty(\mathbb{R}^n), \tilde{U}_2(\mathbb{R}^n)\right)_\theta$, from which follows our claim.

Now with the claim and Lemma 2, the proof of (ii) can be finished by a standard argument. Let $m \in L_\infty(\mathbb{R}^n)$ be such that $\sup_{R\in\mathcal{D}} \|m\|_{\tilde{V}_2(R)} \leq 1$. Set $m_R = m\chi_R$. For $f \in L_2(w)$ choose $g \in L_2(w)$ of unit norm such that

$$\|T_m(f)\|_{L_2(w)} = \int_{\mathbb{R}^n} T_m(f)gw.$$

Note that

$$T_m = \sum_{R\in\mathcal{D}} S_R \circ T_m \circ S_R.$$

Then using the self-adjointness of S_R we get

$$\int_{\mathbb{R}^n} T_m(f)gw = \int_{\mathbb{R}^n} \sum_{R\in\mathcal{D}} T_{m_R}(S_Rf)S_R(\bar{g}w);$$

and so

$$\|T_m(f)\|^2_{L_2(w)} \leq \left(\int \sum_R |T_{m_R}(S_Rf)|^2 w\right)\left(\int \sum_R |S_R(\bar{g}w)|^2 w^{-1}\right).$$

Observe that $\bar{g}w \in L_2(w^{-1})$ and $w^{-1} \in A_2^*(\mathbb{R}^n)$. By the weighted Littlewood-Paley inequality (this can be obtained by the standard argument as in the un-weighted case, see [4] or [**8**]),

$$\left(\int \sum_R |S_R(\bar{g}w)|^2 w^{-1}\right)^{1/2} \leq C\|\bar{g}w\|_{L_2(w^{-1})} \leq C.$$

Also, by the condition on m, $m_R \in \tilde{V}_2(R)$ and is of norm ≤ 1; so by Lemma 2, $m_R \in \tilde{U}_q(\mathbb{R}^n)$ for the above $q = 2\alpha$. Thus it follows from the previous claim that

$$\int |T_{m_R}(S_Rf)|^2 w \leq C \int |S_Rf|^2 w, \quad \forall\, R \in \mathcal{D}.$$

Putting together the preceding inequalities, we obtain the boundedness of T_m on $L_2(w)$. This is (ii) of the theorem.

The proof of (i) is now easy. First we show (i) for $q = 2$. This is done by (ii) and the elementary fact that for any $g \in L_r(\mathbb{R}^n)$ $(r > 1)$ there exists $w \in L_r(\mathbb{R}^n)$ such that $w \in A_1^*(\mathbb{R}^n)$, $|g| \leq w$ and $\|w\|_r \leq 2\|g\|_r$ (see [4]). In particular, if $\sup_{R\in\mathcal{D}} \|m\|_{V_2(R)} < \infty$, m is a Fourier multiplier of $L_p(\mathbb{R}^n)$ for all $1 < p < \infty$. Then interpolating this with the trivial case where $m \in L_\infty(\mathbb{R}^n)$ and by an argument similar to that at the beginning of the present proof, we deduce that (i) is true with $\tilde{U}_q$ instead of V_q. Therefore, by Lemma 2, we finally get (i) in its full generality. □

Remarks.

(i) The preceding proof also yields the following vector-valued version of the theorem. Let $1 \le q < \infty$ and $\{m_k\}$ be a sequence of functions in $L_\infty(\mathbb{R}^n)$ such that $M = \sup_k \sup_{R \in \mathcal{D}} \|m_k\|_{V_q(R)} < \infty$. If $1 < p < \infty$ and $|1/p - 1/2| < 1/q$, then

$$\left\| \left(\sum_k |T_{m_k} f_k|^2 \right)^{1/2} \right\|_p \le C M \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_p, \quad \{f_k\} \subset L_p(\mathbb{R}^n),$$

where C is a consatnt depending on p and q.

(ii) As the reader may observe, the preceding proof is essentially the iteration of a one-dimensional argument. Its pattern is the same as that of [**2**]. This is not surprising if one remembers that the Marcinkiewicz multiplier theorem in $\mathbb{R}^n$ is an iteration of one-dimensional results as well.

(iii) Let us point out that the generalized Littlewood-Paley inequality of Rubio de Francia was extended to $\mathbb{R}^n$ by Journé [**5**]. At first glance, one might think that Journé's extention should suit to the proof of our theorem better than the preceding iteration argument. However, this apparently cannot lead to our final goal (at least, the author has not succeeded in proving the theorem by Journé's inequality).

(iv) The case $q = 2$ in our theorem is of special importance (the result in the general case then follows by interpolation, as the above proof shows). If we just interpolate the case $q = 1$, to which the classical Marcinkiewicz theorem applies, with the trivial case $q = \infty$ as above, we can only obtain the following weaker fact: under the condition of the theorem (i), m is a Fourier multiplier of $L_p(\mathbb{R}^n)$ for $1 < p < \infty$ such that $|1/p - 1/2| < 1/(2q)$.

3. An example.

The following example is classical. Let

$$m(x) = \frac{e^{i|x|^a}}{(1 + |x|^2)^b} \qquad (a > 0, b > 0).$$

Then m is a Fourier multiplier for $L_p(\mathbb{R}^n)$ whenever $1 < p < \infty$ and $|1/p - 1/2| < \frac{2b}{na}$. We will obtain this by our theorem. To that end, it suffices to show that m belongs to $V_q(R)$ uniformly in all dyadic rectangles R for any $q > na/(2b)$. For that let us allow b to take complex values. Thus let b be a complex number with $\mathrm{Re}(b) > na/2$. Then it is clear that for any $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_k = 0$ or 1

$$\left| x^\alpha \frac{\partial^\alpha m}{\partial x^\alpha}(x_1, \cdots, x_n) \right| \leq C\,(1 + |\mathrm{Im}(b)|)^{|\alpha|}, \quad \forall\, x \in \mathbb{R}^n,$$

where $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ and $x^\alpha = x_1^{\alpha_1} \cdots x_1^{\alpha_1}$. It follows that for any dyadic rectangle R, m belongs to $V_1(R)$ and is of norm bounded by $C(1+|\mathrm{Im}(b)|)^n$. Multiplying m by a function like $e^{b^2-\theta^2}$ (for a suitable $\theta \in (0,1)$), one sees that the new function is in $V_1(R)$ uniformly in dyadic R for all complex b with $\mathrm{Re}(b) > na/2$. Then interpolating this with the trivial case where $\mathrm{Re}(b) = 0$ (then $m \in L_\infty$), we obtain the desired result.

Remark. Recall that the above result is optimal, that is, if $|1/p - 1/2| > \frac{2b}{na}$, then m is not a multiplier for $L_p(\mathbb{R}^n)$. This shows that our theorem is also optimal. However, it does not cover the critical case where $|1/p - 1/2| = \frac{2b}{na}$. In this case, m is still a Fourier multiplier for $L_p(\mathbb{R}^n)$ (cf. [**3**]).

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