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A UNIQUENESS THEOREM FOR THE MINIMAL SURFACE EQUATION

JENN-FANG HWANG

In 1991, Collin and Krust proved that if u satisfies the minimal surface equation in a strip with linear Dirichlet data on two sides, then u must be a helicoid. In this paper, we give a simpler proof of this result and generalize it.

1. Introduction.

Let $\Omega_\alpha \subset \mathbb{R}^2$ be a sector domain with angle $0 < \alpha < \pi$. Consider the minimal surface equation

$$\operatorname{div} Tu = 0 \tag{1}$$

where $Tu = \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ and ∇u is the gradient of u. In 1965, Nitsche [7] announced the following results:

(1) Given a continuous function f on $\partial\Omega_\alpha$, there always exists a solution u which satisfies the minimal surface equation in Ω_α with Dirichlet data f on $\partial\Omega_\alpha$;

(2) If u satisfies the minimal surface equation with vanishing boundary value in Ω_α, then $u \equiv 0$.

Nitsche thus raised the following question: Let $\Omega \subset \Omega_\alpha$ and let f be an arbitrary continuous function on $\partial\Omega$. If the Dirichlet problem

$$\begin{cases} \operatorname{div} Tu = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a solution, is it unique?

We notice that similar questions for higher dimensions are raised in [6]. Results in this direction were obtained by Miklyukov [5] and Hwang [4] independently, in which the following result was established:

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. For every $R > 0$, set $B_R = \{x \in \mathbb{R}^2 \,|\, |x| < R\}$ and $\Gamma_R = \partial(\Omega \cap B_R) \cap$*

∂B_R. Denote $|\Gamma_R|$ as the length of Γ_R. And suppose that

$$\begin{cases} \text{(i)} \ \operatorname{div} Tu = \operatorname{div} Tv & \textit{in } \Omega, \\ \text{(ii)} \ u = v & \textit{on } \partial\Omega, \\ \text{(iii)} \ \max\limits_{\Omega \cap B_R} |u - v| = O\left(\sqrt{\int_{R_0}^{R} \frac{1}{|\Gamma_r|}\, dr}\right) & \textit{as } R \to \infty, \textit{ for some} \\ & \quad \textit{positive constant } R_0. \end{cases}$$

Then $u \equiv v$ in Ω.

A stronger version of Theorem 1 was discovered by Collin and Krust [**2**] independently, which is the following:

Theorem 1*. *Let $\Omega, u, v, B_R, \Gamma_r$ and $|\Gamma_r|$ as in Theorem 1. And suppose that*

$$\begin{cases} \text{(i)} \ \operatorname{div} Tu = \operatorname{div} Tv & \textit{in } \Omega, \\ \text{(ii)} \ u = v & \textit{on } \partial\Omega, \\ \text{(iii)} \ \max\limits_{\Omega \cap B_R} |u - v| = o\left(\int_{R_0}^{R} \frac{1}{|\Gamma_r|}\, dr\right) & \textit{as } R \to \infty, \textit{ for some} \\ & \quad \textit{positive constant } R_0. \end{cases}$$

Then $u \equiv v$ in Ω.

In fact, for any unbounded domain Ω, we have $|\Gamma_R| = O(R)$, and condition (iii) in Theorem 1* becomes

$$\max_{\Omega \cap B_R} |u - v| = o(\log R) \qquad \text{as } R \to \infty.$$

In the special case when Ω is a strip, then $|\Gamma_R| \leq$ constant, and condition (iii) becomes $\max_{\Omega \cap B_R} |u - v| = o(R)$.

On the other hand, in a strip domain Ω, Collin [**1**] showed that there exist two different solutions for the minimal surface equation such that $u = v$ on $\partial\Omega$ and $\max_{\Omega \cap B_R} |u - v| = O(R)$ as $R \to \infty$. So condition (iii) is necessary.

This counterexample also answers Nitsche's question in the negative.

In contrast, the following result is also given in [**2**].

Theorem 2. *Let $\Omega = (0, 1) \times \mathbb{R}$ be a strip. Suppose that*

$$\begin{cases} \operatorname{div} Tu = 0 & \textit{in } \Omega, \\ u(0, y) = ay + b, \\ u(1, y) = cy + d \end{cases}$$

where a, b, c, d are constant. Then u must be a helicoid.

The following inequality was discovered by Miklyukov [**5**, p. 265], Hwang [**4**, p. 342] and Collin and Krust [**2**, p. 452]:

$$(Tu - Tv) \cdot (\nabla u - \nabla v) \geq \frac{\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}}{2} |Tu - Tv|^2$$

$$\geq |Tu - Tv|^2. \tag{2}$$

Using this inequality, Miklyukov [**5**] and Hwang [**4**] proved Theorem 1 independently, and Collin and Krust [**2**] proved Theorem 1* also based on this inequality.

It seems that the method of proof of Theorem 1* can not be used to prove Theorem 2, and so Collin and Krust [**2**] resorted to the theory of Gauss maps instead.

In this paper, we will point out that the method of proof of Theorem 1 and Theorem 1* could be use to give a simpler proof of Theorem 2. Moreover, we shall generalize Theorem 1* and Theorem 2 to get the more general results as stated in Theorem 3 and Theorem 4. And we will make a remark after Theorem 3 to point out why Collin and Krust [**2**] could get a better result then Miklyukov [**5**] and Hwang [**4**].

2. A new proof for Theorem 2 and its generalization.

Without loss of generality, we may rephrase Theorem 2 in the following form:

Theorem 2*. *Let $\Omega = (b, a) \times \mathbb{R}$ be a strip domain in $\mathbb{R}^2$ where a, b are two constants with $-\frac{\pi}{2} < b < a < \frac{\pi}{2}$, and let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Suppose that*

$$\begin{cases} \operatorname{div} Tu = 0 & \text{in } \Omega, \\ u = y \tan x & \text{on } \partial\Omega. \end{cases}$$

Then $u \equiv y \tan x$ in Ω; in other words, u must be a helicoid.

Proof. For any $y > 0$, let

$$\begin{aligned} \Omega_y &= (b, a) \times (-y, y), \\ \Gamma_y &= \{(b, a) \times \{y\}\} \cup \{(b, a) \times \{-y\}\} \end{aligned}$$

and, set

$$\begin{aligned} g(y) &= \int_\Gamma (u - v)(Tu - Tv) \cdot \nu \, ds \\ &= \oint_{\partial\Omega_y} (u - v)(Tu - Tv) \cdot \nu \, ds \\ &= \int\int_{\Omega_y} (\nabla u - \nabla v) \cdot (Tu - Tv) \end{aligned}$$

where $v \equiv y \tan x$ and ν is the unit outward normal of Γ_y and $\partial\Omega_y$. Since $(\nabla u - \nabla v) \cdot (Tu - Tv) \geq 0$, Fubini's Theorem yields that the derivative $g'(y)$

exists for almost all $y > 0$ and

$$g'(y) = \int_{\Gamma_y} (\nabla u - \nabla v) \cdot (Tu - Tv)$$

whenever $g'(y)$ exists. Thus, in view of (2), for these y,

$$\begin{aligned} g'(y) &\geq \int_{\Gamma_y} \frac{\sqrt{1+|\nabla u|^2} + \sqrt{1+|\nabla v|^2}}{2} |Tu - Tv|^2 \\ &\geq \left(\min_{\Gamma_y} \frac{\sqrt{1+|\nabla v|^2}}{2} \right) \int_{\Gamma_y} |Tu - Tv|^2, \end{aligned}$$

in which, as $v_x = y \sec^2 x$, we have

$$\frac{\sqrt{1+|\nabla v|^2}}{2} \geq \frac{y \sec^2 x}{2} \geq \frac{y}{2}.$$

Furthermore, by means of Schwarz's inequality,

$$|\Gamma_y| \int_{\Gamma_y} |Tu - Tv|^2 \geq \left(\int_{\Gamma_y} |Tu - Tv| \right)^2,$$

and $|\Gamma_y| = 2(a - b)$ (in virtue of the special geometry of Ω), thus

$$\int_{\Gamma_y} |Tu - Tv|^2 \geq \frac{1}{2(a-b)} \left(\int_{\Gamma_y} |Tu - Tv| \right)^2.$$

Hence, for any y where $g'(y)$ exists,

$$\begin{aligned} g'(y) &\geq \frac{y}{4(a-b)} \left(\int_{\Gamma_y} |Tu - Tv| \right)^2 \\ &\geq \frac{y}{4(a-b)} \left(\frac{1}{\pi} \int_{\Gamma_y} \tan^{-1}(u-v)(Tu - Tv) \cdot \nu \right)^2. \end{aligned} \tag{3}$$

Now, for all $y > 0$, set

$$\begin{aligned} h(y) &= \int_{\Gamma_y} \tan^{-1}(u-v)(Tu - Tv) \cdot \nu \\ &= \int\int_{\Omega_y} \frac{(\nabla u - \nabla v) \cdot (Tu - Tv)}{1 + (u-v)^2}. \end{aligned}$$

We note that $h \geq 0$ and $h(y)$ increases as y increases. Thus, if $h \equiv 0$, it is easy to see that Theorem 2* holds. Hence we may assume that $h \not\equiv 0$ and that there exist two positive constants y_1 and c_1 such that $h(y) \geq c_1$ for all $y \geq y_1$.

Substituting this into (3), we obtain $g'(y) \geq \frac{c_1^2}{4(a-b)\pi^2} y$ for almost all $y \geq y_1$, which yields $g(y) - g(y_1) \geq \frac{c_1^2}{4(a-b)\pi^2}(y - y_1)^2$. Since $|u| = O(|y|)$ on $\partial\Omega$ as $|y| \to \infty$, by [**7**, p. 256], we have $|u| = O(|y|)$ in Ω as $|y| \to \infty$. Since for all $y > 0$, $g(y) = \int_{\Gamma_y} (u-v)(Tu-Tv)\cdot\nu$ and $|Tu-Tv| \leq 2$, we have $g(y) = O(y)$ as $y \to \infty$, which gives a contradiction and completes our proof. $\square$

By modifying the proof of Theorem 2*, we can derive the following

Theorem 3. *Let $\Omega \subseteq \mathbb{R}^2$ be an unbounded domain and let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Let B_R, Γ_R and $|\Gamma_R|$ be as in Theorem* 1. *Suppose that*

$$\begin{cases} \text{(i)} \ \ \operatorname{div} Tu = \operatorname{div} Tv & \text{in } \Omega, \\ \text{(ii)} \ \ u = v & \text{on } \partial\Omega, \\ \text{(iii)} \ \ \max\limits_{\Omega\cap B_R} |u - v| = o\left(\int_{R_0}^R \frac{1}{|\Gamma_R|} \min\limits_{\Gamma_R} \sqrt{1 + |\nabla v|^2}\, dR\right) & \text{as } R \to \infty, \end{cases}$$

where R_0 is a positive constant. Then we have $u \equiv v$ in Ω.

Remark.

(a) Notice that condition (iii) depends on $|\nabla v|$ only, without assuming any condition on $|\nabla u|$.

(b) In Theorem 2*, since $\operatorname{div} Tu = 0$ in Ω and $u = y \tan x$ on $\partial\Omega$, by [**7**, p. 256], we have $u = O(|y|)$ in Ω as $|y| \to \infty$. And so, condition (iii) of Theorem 3 holds.

Proof of Theorem 3. The proof is similar to that of Theorem 2*. For every $R > 0$, let

$$M(R) = \max_{\Omega\cap B_R} |u - v| = \max_{\Gamma_R} |u - v|,$$

$$Q(R) = \min_{\Gamma_R} \frac{\sqrt{1 + |\nabla v|^2}}{2},$$

$$g(R) = \int_{\Gamma_R} (u - v)(Tu - Tv) \cdot \nu = \int\int_{\Omega_R} (\nabla u - \nabla v) \cdot (Tu - Tv)$$

and

$$h(R) = \int_{\Gamma_R} \tan^{-1}(u - v)(Tu - Tv) \cdot \nu.$$

As in the proof of Theorem 2*, we may assume that $h \not\equiv 0$ and that there exist two positive constants R_1 and C_1 such that $R_1 > R_0$ and

$$h(R) \geq C_1 \qquad \text{for all } R \geq R_1. \tag{4}$$

For almost all $R > 0$, we have

$$g'(R) = \int_{\Gamma_R} (\nabla u - \nabla v) \cdot (Tu - Tv) \geq \int_{\Gamma_R} Q(R)|Tu - Tv|^2 \geq Q(R)|\Gamma_R|^{-1} \left(\int_{\Gamma_R} |Tu - Tv| \right)^2 . \tag{5}$$

Thus $g'(R) \geq (\frac{\pi}{2})^2 C_1^2 |\Gamma_R|^{-1} Q(R)$, for almost all $R > R_1$. Hence, for every R and R_2 such that $R > R_2 \geq R_1$, we have

$$g(R) - g(R_2) \geq \left(\frac{2C_1}{\pi} \right)^2 \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} \, dr. \tag{6}$$

By (4), we have $M(R) > 0$ for all $R \geq R_1$, hence (5) yields, for almost all $R \geq R_1$,

$$g'(R) \geq Q(R)|\Gamma_R|^{-1} \int |Tu - Tv|^2 \geq \frac{g^2(R)Q(R)}{M^2(R)|\Gamma_R|};$$

and so, for every R and R_2 such that $R > R_2 \geq R_1$,

$$- \frac{1}{g} \bigg|_{R_2}^R \geq \int_{R_2}^R \frac{g'}{g^2} \geq \int_{R_2}^R \frac{Q(r)}{M^2(r)|\Gamma_r|} \, dr \geq \frac{1}{M^2(R)} \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} \, dr,$$

and then

$$\frac{1}{g(R_2)} \geq \frac{1}{M^2(R)} \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} \, dr. \tag{7}$$

Now, since $M(R) > 0$ for all $R \geq R_1$, $M(R)$ is an increasing function of R and, in view of condition (iii),

$$(M(R))^{-1} \int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} \, dr \to \infty \qquad \text{as } R \to \infty,$$

and also

$$\int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} \, dr \to \infty \qquad \text{as } R \to \infty;$$

hence we can choose a constant $R_3 > R_1$ such that

$$(M(R))^{-1} \int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} \geq \sqrt{2}\pi C_1^{-1}, \qquad \text{for every } R \geq R_3,$$

and a constant R_4, $R_4 > R_3$, which depends on R_3, such that

$$\int_{R_1}^{R_4} \frac{Q(r)}{|\Gamma_r|}\, dr = 2 \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|}\, dr.$$

With this choice of R_3 and R_4, we have

$$\begin{aligned}
1 &\geq \frac{g(R_3) - g(R_1)}{g(R_3)} \\
&\geq \left[\left(\frac{2C_1}{\pi}\right)^2 \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|}\right] \left[(M^2(R_4))^{-1} \int_{R_3}^{R_4} \frac{Q(r)}{|\Gamma_r|}\right] \qquad \text{(by (6), (7))} \\
&= \left[\left(\frac{2C_1}{\pi}\right)^2 (M^2(R_4))^{-1}\right] \frac{1}{4} \left(\int_{R_1}^{R_4} \frac{Q(r)}{|\Gamma_r|}\right)^2 \qquad \text{(by the choice of } R_3,\ R_4) \\
&\geq \frac{C_1^2}{\pi^2} (2\pi^2) C_1^{-2} \qquad \text{(again by the choice of } R_3 \text{ and } R_4) \\
&\geq 2,
\end{aligned}$$

which is desired contradiction. □

Remark. The above proof is to show (6), which is the lower bound of $g(R)$, and (7), which is the upper bound of $g(R)$. And from (6) and (7), we get contradiction and so prove the theorem. Miklyukov [**5**] and Hwang [**4**] only observed the upper bound of $g(R)$, and so could not derive the better result as in Collin and Krust [**2**].

Let Ω be a domain in $\mathbb{R}^2$. Consider the following equation in divergence form

$$\operatorname{div} A(x, u, \nabla u) = f(x, u, \nabla u),$$

where

$$A = (A_1, A_2),\ A_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \quad i = 1, 2,$$
$$f \colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R},$$

and

$$A_i \in C^0\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2\right) \cap C^1(\Omega \times \mathbb{R} \times \mathbb{R}^2), \quad i = 1, 2,\ f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^2).$$

We rewrite $A(x, u, \nabla u)$ briefly as Au.

Suppose that Au satisfies the following structural condition:

$$(8) \begin{cases} (Au - Av) \cdot (\nabla u - \nabla v) \geq |Au - Av|^2 Q(R), \\ \qquad\qquad \text{where } R = \sqrt{x^2 + y^2} \text{ and } Q(R) \text{ is a positive function,} \\ (\nabla u - \nabla v) \cdot (Au - Av) = 0, \qquad \text{iff } \nabla u = \nabla v. \end{cases}$$

Now we have the following result:

Theorem 4. *Let $\partial\Omega = \Sigma^\alpha + \Sigma^\beta$ be a decomposition of $\partial\Omega$ such that $\Sigma^\beta \in C^1$. Let $u, v \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma^\beta) \cap C^0(\overline{\Omega})$ and let $M(R) = \max_{\Omega \cap B_R}(u - v, 0)$. Suppose that*

$$\begin{cases} \text{(i)} \quad A & \textit{satisfies the structural condition } (8) \\ \text{(ii)} \quad \operatorname{div} Au \geq \operatorname{div} Av & \textit{in } \Omega \\ \text{(iii)} \quad u \leq v & \textit{on } \Sigma^\alpha \\ \text{(iv)} \quad Au \cdot \nu \leq Av \cdot \nu & \textit{on } \Sigma^\beta \\ \text{(v)} \quad M(R) = o\left(\int_{R_0}^R \frac{|Q_r|}{|\Gamma_r|}\, dr\right) & \textit{as } R \to \infty, \textit{ where } R_0 \textit{ is} \\ & \qquad\qquad \textit{a positive constant.} \end{cases}$$

Then, if $\partial\Omega = \Sigma^\beta$, we have either $u(x) \equiv v(x) +$ a positive constant or else $u(x) \leq v(x)$. Otherwise, $u(x) \equiv v(x)$.

The proof of Theorem 4 is exactly the same as that of Theorem 3. The interested readers may consult [**4**].

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Academia Sinica
Taipei, Taiwan
R.O.C. 11529