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Note on the dimension of modules and algebras

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Let Λ be a ring with unit element. The left dimension (notation: l. dim ${}_\Lambda A$), the left injective dimension (l inj. dim. ${}_\Lambda A$) and the left weak dimension (w. l. dim ${}_\Lambda A$) for left Λ-modules and the left global dimension (l. gl. dim Λ) and the global weak dimension (w. gl. dim Λ) of Λ are those defined in [3].

Len Λ and Γ be rings and ψ a ring homomorphism of Λ to Γ. Then each left Γ-module A may be regarded as a left Λ-module, by setting, for $\lambda \in \Lambda$ $a \in A$

$$\lambda \cdot a = \psi \lambda \cdot a$$

If Γ is Λ-projective in this sence, the following inequalities are shown in [3]; l. dim ${}_\Lambda A \leqq$ l. dim ${}_\Gamma A$, w. 1. dim ${}_\Lambda A \leqq$ w. 1. dim ${}_\Gamma A$ and 1. inj. dim ${}_\Lambda A \leqq$ l. inj. dim ${}_\Gamma A$ for left Γ-madules A.

M. Auslander [1] has shown that l. gl. dim $\Lambda =$ sup l. dim $\Lambda/\mathfrak{l}$ where $\mathfrak{l}$ ranges over all left ideals of Λ and obtained some relations among l. gi. dim Λ_1, l. gl. dim Λ_2 and l. gl. dim $\Lambda_1 \otimes \Lambda_2$ in the special cases where Λ_1 and Λ_2 are algebras over a field K.

If $\mathfrak{A}$ is a two-sided ideal in Λ, there is in general very little relation between l. gl. dim Λ and l. gl. dim $(\Lambda/\mathfrak{A})$; it was however proved in Elenberg-Nagao-Nakayama [6] that if l. gl. dim $\Lambda \leqq 1$ and Λ is semi-primary, then gl. dim $(\Lambda/\mathfrak{A}) < \infty$.

Now, we show in section 1 of the present note that for each left Λ-module A we have l. dim ${}_\Lambda A =$ l. dim ${}_{\Lambda_n}A^n$, w. l. dim ${}_\Lambda A =$ w. l. dim ${}_{\Lambda_n}A^n$ and l. inj. dim ${}_\Lambda A =$ l. inj. dim ${}_{\Lambda_n}A^n$ and conversely, for each left Λ_n-module A, l. dim ${}_\Lambda A =$ l. dim ${}_{\Lambda_n}A$ and so on, where Λ_n is the total matrix ring of order n over Λ. Hence, as the special case of $\Lambda_1 \otimes \Lambda_2$ we obtain l. gl. dim $\Lambda =$ l. gl. dim Λ_n and w. gl. dim Λ = w. gl. dim Λ_n for any ring Λ and further if Λ is an algebra over a commutative ring K, we obtain dim $\Lambda =$ dim Λ_n.

In section 2 we show that the analogous theorem to Auslander's is valid for w. gl. dim Λ and some characterization of ring Λ with w. gl. dim $\Lambda \leqq n$ or l. gl. dim $\Lambda \leqq n$ $(n \geqq 1)$. In section 3, we assume that ψ is a ring homomorphism of Λ to Γ and 1. dim ${}_\Lambda\Gamma = 0$ or r. dim ${}_\Lambda\Gamma = 0$, then we obtain some relations between the dimensions of Λ and Γ, regarding Γ-modules A as Λ-modules. In particular, if two sided ideal $\mathfrak{A}$ is equal to Λe or $e\Lambda$ $(e = e^2)$, we obtain l. gl. dim $\Lambda \geqq$ l. gl. dim $(\Lambda/\mathfrak{A})$ and w. gl. dim $\Lambda \geqq$ w. gl. dim $(\Lambda/\mathfrak{A})$.

In section 4 we show that $\mathrm{w.gl.dim}\,\Lambda = 0$ if and only if Λ is regular, hence we obtain an example of the case that $\mathrm{l.gl.dim}\,\Lambda > \mathrm{w.gl.dim}\,\Lambda$. Finally in section 5 we study some relations between the dimensions of Λ and $e\Lambda e$ under some assumptions. The definitions and notions employed in this paper are based on those introduced by H. Cartan and S. Eilenberg [3].

1 Let Λ be a ring with unit element and Λ_n be the total materix ring of order n over Λ. We assume that each Λ-module is unitary and that each ring homomorphism maps unit upon unit. If two rings Λ and Γ and a ring homomorphism ψ of Λ to Γ are given, than each left Γ-module A may be regarded as a left Λ-module, by setting, for $a \in A$, $\lambda \in \Lambda$

$$\lambda a = \psi(\lambda) a . \tag{1}$$

In particular Γ may be regarded as Λ-module.
The following lemma is an immediate consequence of [3; XVI, Exer. 5]

LEMMA 1. *Let Λ, Γ and ψ be as avove. Then*
if $w.\,l.\,dim\ {}_\Lambda\Gamma = 0$, we have $w.\,l.\,dim\ {}_\Lambda A \leqq w.\,l.\,dim\ {}_\Gamma A$,
if $l.\,dim\ {}_\Lambda\Gamma = 0$, we have $l.\,dim\ {}_\Lambda A = l.\,dim\ {}_\Gamma A$, and
if $w.\,r.\,dim\ {}_\Lambda\Gamma = 0$, we have $l.\,inj.\,dim\ {}_\Lambda A \leqq l.\,inj.\,dim\ {}_\Gamma A$, for each left Γ-module A.

Let A be *a* left Λ-module and A^n and A_n be the direct sums of n and n^2 $A's$, respectively. The left operations of Λ_n over A^n and A_n are defined, by setting, for $\lambda = (\lambda_{i\,j}) \in \Lambda_n$ $\quad a = (a_1 \dots a_n) \in A^n$ $\quad \tilde{a} = (a_{ij}) \in A_n$

$$\begin{aligned} \lambda a &= (\sum_j \lambda_{1j}\, a_j\,, \cdots \sum_j \lambda_{nj}\, a_j) \\ \lambda a &= (\sum_k \lambda_{ik}\, a_{kj}). \end{aligned} \tag{2}$$

A^n and A_n become left Λ_n-modules under these operations. We define a ring homomorphism φ of Λ to Λ_n as follows,

$$\varphi(\lambda) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \qquad \text{for } \lambda \in \Lambda . \tag{3}$$

A^n and A_n become left Λ-modules by (1), (2) and (3), and these coincide with natural direct sums of n and n^2 $A's$ as Λ-modules respectively.

PROPOSITION 1. *If a left Λ-module A is projectve, then the left Λ_n-module A^n so is.*

Proof. If $A = \Lambda$, we have $\Lambda_n = \Lambda^n \oplus \dots \oplus \Lambda^n$ as Λ_n-module, hence Λ^n is Λ_n-projective. Thus by a direct sum argument we have proposition.

PROPOSITION 2. *Each left Λ_n-modele A is Λ_n-isomorphic to $(e_{1\,1} A)^n$, where we regard $e_{1\cdot1} A)$ as left Λ-module.*

Proof. Wh have a decomposition of A as follows,

$$A = e_{11}\, A + e_{21}\, A + \cdots + e_{n1}\, A$$

and $e_{i1} A$ is Λ-isomorphic to $e_{11} A$. We obtain a Λ_n-isomorphism of A to $(e_{11} A)^n$ by the following correspondenc, for $a \in A$ $\quad a' \in (e_{11} A)^n$

$$a = e_{11}a_1 + e_{21}a_2 + \cdots e_{n\cdot1}a_n \longleftrightarrow a' = e_{11}a_1 + e_{11}a_2 + \cdots + e_{11}a_n .$$

PROPOSITION 3. *For left Λ-modules A, B and a right Λ-module C, we have isomorphism.*

$$Hom_{\Lambda}(A, B) \approx Hom_{\Lambda_n}(A^n, B^n),\ C \underset{\Lambda}{\otimes} A \approx C^n \underset{\Lambda_n}{\otimes} A^n.$$

Proof. We denote an element $(0 \cdots 0 \overset{(i)}{a} 0 \cdots 0)$ by $a^{(i)}$. Any element f of Hom $_{\Lambda_n}(A^n, B^n)$ is uniquely decided by the image of the first component of A, for $f(a^{(i)}) = f(e_{i1}a^{(1)}) = e_{i1}a^{(1)})$. And since $f(a^{(1)}) = f(e_{11}a^{(1)}) = e_{11}f(a^{(1)})$, f is uniquely determined by a element of Hom $_{\Lambda}(A. B)$.
Next we have

$$(c_1, \ldots, c_n) \otimes (a_1, \ldots, a_n) = \sum_{i \cdot j} c^{(i)} \otimes a^{(j)}.$$

If $i \neq j$, $c^i \otimes a^{(j)} = c^{(i)} e_{ii} \otimes a^{(j)} = c^{(i)} \otimes e_{ii} a^{(j)} = 0$, and $c^{(i)} \otimes a^{(i)} = c^{(1)} e_{1i} \otimes a^{(i)} = c^{(1)} \otimes e_{1i} a^{(i)} = c^{(1)} \otimes a^{(1)}$,
hence $(c_1 \ldots c_n) \otimes (a_1 \ldots a_n) = \sum_i c_i^{(1)} \otimes a_i^{(1)}$.
We define an epimorphic mapping ψ: $C \underset{\Lambda}{\otimes} A \longrightarrow C^n \underset{\Lambda}{\otimes} A^n$ by setting

$$\psi(c \otimes a) = c^{(1)} \otimes a^{(1)}.$$

Coversely we define a mapping φ : $C^n \underset{\Lambda^n}{\otimes} A^n \longrightarrow C \underset{\Lambda}{\otimes} A$ by setting

$$\varphi(c^{(1)} \otimes a^{(1)}) = c \otimes a,$$

this mapping is defined inedpendent on the choice of representatives.
Then φ is epimorphic and $\psi \circ \varphi$ is the identity mapping. Therefore ψ is isomorphic

PROPOSIEION 4. *Let A, B and C be as above, then we have isomorphisms;*

Ext $_{\Lambda}(A, B) \approx$ *Ext* $_{\Lambda_n}(A^n, B^n)$, *Tor* $^{\Lambda}(C, A) \approx$ Tor $^{\Lambda^n}(C^n, A^n)$.

Proof. Let

$$X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} - \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} A \longrightarrow 0$$

be a projective resolution of A. By the natural manner we can extend this sequence to a Λ_n-projective resolution of A^n, using proposition 1, as follows

$$\longrightarrow X_m^n \xrightarrow{d_m^n} X_{m-1}^n \xrightarrow{d_{m-1}^n} \cdots \longrightarrow X_1^n \xrightarrow{d_1^n} X_0^n \xrightarrow{d_0^n} A^n \longrightarrow 0$$

Passing to homology yields the desired results in virtue of the definitions of Ext and Tor.

COROLLARY 1. *For each leaft Λ-module A we have l. dim. $_{\Lambda}A$ = l. dim $_{\Lambda_n}$ A^n, l. inj. dim $_{\Lambda}A$ = l. inj. dim $_{\Lambda_n}A^n$, and w. l. dim $_{\Lambda_n}A$ = w. l. dim $_{\Lambda_n}A^n$.*

Proof. We have immediately the conclusion for l. dim $_{\Lambda}$ by lemma 1 and the consideration in the proof of proposition. Let B be a left Λ_n-module, then we have following isomorphisms from propositions 2 and 4.

$$\text{Ext}_{\Lambda}(e_{11}B, A) \approx \text{Ext}_{\Lambda_n}((e_{11}B)^n, A^n) \approx \text{Ext}_{\Lambda_n}(B, A^n).$$

Hence l. inj. dim $_{\Lambda_n}A^n \geqq$ l. inj. dim $_{\Lambda}A$.
The inverse inequality is obtained from lemma 1, noting that A^n is the direct sum

of n Λ's as Λ module.

It is similar for w. l. dim.

REMARK 1. From corollary 1 and Theorem 18 of Eilenberg-Nakayama [4] we can obtain the well known result that Λ is quasi-Frobenius if and only if Λ_n so is.

CORORRARY 2. *For each left Λ_n-module A we have*

$l.\ dim\ {}_\Lambda A = l.\ dim\ {}_{\Lambda_n}A,\ l.\ inj.\ dim\ {}_\Lambda A = l.\ inj.\ dim\ {}_{\Lambda_n}A$

and $w.\ l.\ dim\ {}_\Lambda A = w.\ l.\ dim\ {}_{\Lambda_n}A$

PROOF. Ovserbing that A is the direct sum of n $(e_{11}A)$'s as a Λ-module, we have by propositions 1 and 2

$$l.\ dim\ {}_\Lambda A = l.\ dim\ {}_{\Lambda_n}(e_{11}A)^n = l.\ dim\ {}_\Lambda e_{11}A = l.\ dim\ {}_\Lambda A.$$

It is similar for the remainders.

From the above two corollaries we have

THEOREM 1. l. gl. dim Λ = l. gl. dim Λ_n, w. gl. dim Λ = w. gl. dim Λ_n.

Now, let Λ be an algebra over a commutatiue ring K. And we have $\Lambda^e = \Lambda \otimes \Lambda^*$ where Λ^* is the inverse algebra. As for two sided Λ-modules A, the standard procedure will be to convert them into left modules over Λ^e. Further we observe that $(\Lambda_n)^e = \Lambda_n \otimes \Lambda_n^*$ is isomrphic to $\Lambda \otimes \Lambda^* \otimes K_{n^2} = (\Lambda^e)_{n^2}$.

Hence from corollary 2 we have l. dim ${}_{\Lambda^e}A$ = l. dim ${}_{(\Lambda)^e{}_n}A$ for each two sided Λ_n-modul A. In partiqular, setting $A = \Lambda_n$ we have

THEOREM 2. $dim\ \Lambda = dim\ \Lambda_n$

PROPOSITION 5. *The following properties are equivalent, respectively*:

a) Λ *is left hereditary,*

b) Λ_n *is left hereditary,*

and

a') Λ *is left semi-hereditary,*

b') Λ_n *is left semi-hereditary,*

The first statements are clear from Theormem 1 and [3, VI, 2·8]. For the proof of the second statements we need the following well know reslut, (cf. [2: 23·15]).

Let $\tilde{\mathfrak{l}}$ be left ideal of Λ_n and $\mathfrak{m}(\tilde{\mathfrak{l}})$ be the left Λ-module consisting of the first row of elements in $\tilde{\mathfrak{l}}$.

Then the correspondence $l \longleftrightarrow \mathfrak{m}(\tilde{\mathfrak{l}})$ gives one to one correspondence between the left ideals of Λ_n and the Λ-submodule of n-dimensional vector space Λ^n over Λ. Moreover, $\mathfrak{m}(\tilde{\mathfrak{l}})$ is finitely generated as a Λ-module if and only if $\tilde{\mathfrak{l}}$ has finite generators as a left ideal.

Now we assume that Λ is left semi-hereditary. If $\tilde{\mathfrak{l}}$ is a finitely generated left ideal of Λ_n, we have from the above remark and corollary 1 of proposition 4

$$\text{l. dim}\ {}_{\Lambda_n}\tilde{\mathfrak{l}} = \text{l. dim}\ {}_{\Lambda_n}\mathfrak{m}(\tilde{\mathfrak{l}})^n = \text{l. dim}\ {}_\Lambda\mathfrak{m}(\tilde{\mathfrak{l}}).$$

From [3; I, 6·2] l. dim ${}_\Lambda\mathfrak{m}(\tilde{\mathfrak{l}}) = 0$, hence Λ_n is left semihereditary. Conversely, let

Λ_n be left semi-hereditary and $\mathfrak{l}$ be a finitely generated left ideal of Λ, then we have $\mathrm{l.\,dim}\ {}_{\Lambda}\mathfrak{l} = \mathrm{l.\,dim}\ {}_{\Lambda_n}\mathfrak{l}_n$ and since $\mathfrak{l}_n$ is finitely generated as a left ideal of Λ_n, $\mathrm{l.\,dim}\ {}_{\Lambda_n}\mathfrak{l}_n = 0$. Therefore Λ is left semi-hereditary.

2. Now we study here some properties of weak dimensions of rings.

LEMMA 2. *Let A be a left Λ-module and consider an exact sequence*

$$0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$$

where $w.\,l.\,dim\ {}_{\Lambda}P = 0$. If $w.\,l.\,dim\ {}_{\Lambda}A \neq 0$, then $w.\,l.\,dim\ {}_{\Lambda}B = w.\,l.\,dim\ {}_{\Lambda}A - 1$, and if $w.\,l.\,dim\ {}_{\Lambda}A = 0$, then $w.\,l.\,dim\ {}_{\Lambda}B = 0$.

It is clear (cf. [3; VI, 2·3]).

The following theorem is analogous to Auslander's theorem in the case of left dimensions.

THEOREM 3.

a) $w.\,gl.\,dim \Lambda = sup\ w.\mathrm{l}.dim\ {}_{\Lambda}B$

b) $\qquad\qquad = sup\ w.\mathrm{l}.dim\ {}_{\Lambda}\Lambda/\mathfrak{l}$

where B ranges over all left Λ-modules generated by a singule element and $\mathfrak{l}$ ranges over all left ideals of Λ.

If further $w.\,gl.\,dim\ \Lambda \neq 0$

c) $w.\,gl.\,dim\ \Lambda = 1 + sup.\,w.\,l.\,dim\ {}_{\Lambda}\mathfrak{l}$

PROOF. a)$\rightarrow$b)$\rightarrow$c) is clear from lemma 2. Hence we prove here only the statement a) of the theorem. This proof is based on

LEMMA 3. *Let A be a left Λ-module, I a non empty well ordered set and $(A_i)_{i \in I}$ a family of sumodules of A such that $\bigcup\limits_{i \in I} A_i = A$ and if $i \in I$ and $i \geqq j$, then $A_i \geqq A_j$.*

If $w.\,l.\,dim\ {}_{\Lambda}(A_i/A_i') \leqq n$ for all $i \in I$ where $A_i' = \bigcup\limits_{j<i} A_j$, $A_1' = (0)$ (1 is the least element of I), then $w.\,l.\,dim\ {}_{\Lambda}A \leqq n$

Proof. If $n = 0$ the then for all $i \in I$ we have $\mathrm{w.\,l.\,dem}\ {}_{\Lambda}A_i/A_i' = 0$. From the exact sequence

$$0 \longrightarrow A_i' \longrightarrow A_i \longrightarrow A_i/A_i' \longrightarrow 0$$

we have for each right Λ-module B and $n \geqq 1$

$$0 = Tor_{n+1}^{\Lambda}(B, A_i/A_i') \longrightarrow Tor_n^{\Lambda}(B, A_i') \longrightarrow Tor_n^{\Lambda}(B, A_i) \longrightarrow Tor_n^{\Lambda}(B, A_i/A_i') = 0\,.$$

Hence $Tor_n^{\Lambda}(B, A_i')$ is isomorphic to $Tor_n^{\Lambda}(B, A_i)$, that is, $\mathrm{w.\,l.\,dim}\ {}_{\Lambda}A_i' = \mathrm{w.\,l, dim}\ {}_{\Lambda}A_i$. By our assumption we have $\mathrm{w.\,l.\,dim}\ {}_{\Lambda}(A_1/A_1') = \mathrm{w.\,l.\,dim}\ {}_{\Lambda}A_1 = 0$. Then we can use the transfinite induction. We assume that all modules A_j such as $j < i$ are those with $\mathrm{w.\,l.\,dim}\ {}_{\Lambda}A_j = 0$. If i is not a limit element, we have $A_i' = A_{i-1}$ and by the above remark $\mathrm{w.\,l.\,dim}\ {}_{\Lambda}A_i = 0$. If i is a limit element, than A_i' is the direct limit of A_j $(j < i)$ and inclusion mappings $\pi_j^{j'}(j \leqq j' < i)$ (see [5; VIII, Exer. B]). Since Tor commutes with the direct limit, we have $Tor_n^{\Lambda}(B, A_i') = 0$ for $n > 0$. Hence by the abve remark we obtain $\mathrm{w.\,l.\,dim}\ {}_{\Lambda}A_i = 0$.

For $n > 0$ we can use the same method as that of proof of [1; pr. 3]. The proof of a) of theorem is also similar to that of [1; Th. 1].

From lemma 2, Theorem 3 and the analogous properties to them in the case of the left dimensions we have the following corollary which is a generalization of [3; I, 5·4].

COROLLARY. *The following properties are equivalent for* $n \geqq 1$, *respectively*;

a) $l.\ gl.\ dim\ \Lambda \leqq n$

b) *For each* Λ-*submodule* A *of a left projective* Λ *module we have* $l.\ dim_{\Lambda} A \leqq n-1$

and

a′) $w.\ gl.\ dim\ \Lambda \leqq n$

b′) *For each* Λ-*submodule* A *of a left* Λ-*module* P *with* $w.\ l.\ dim\ _{\Lambda}P = 0$ *we have* $w.\ l.\ dim\ _{\Lambda} A \leqq n-1$

3. we now consider some relations between dimensions of two ring Λ and Γ which are connected by a ring homorphism ψ of Λ to Γ.

PROPOSITION 6. *Let* Λ, Γ *and* ψ *be as above and we assume that* $l.\ dim\ _{\Lambda}\Gamma = 0$ *and* $l.\ dim\ _{\Gamma}B = 1$ *implies* $l.\ dim\ _{\Lambda}B = 1$ *for left* Γ-*modules* B. *Then we have* $l.\ dim\ _{\Lambda}A = l.\ dim\ _{\Gamma}A$ *for each left* Γ-*module* A *with* $l.\ dim\ _{\Gamma}A < \infty$.

Proof If $\mathrm{l.\,dim}\ _{\Gamma}A = 0$, $\mathrm{l.\,dim}\ _{\Lambda}A = 0$ by lemma 1. Now, we assume that the proposition is proved for left Γ-modules A' with $\mathrm{l.\,dim}\ _{\Lambda}A' < q$. $(1 < q < \infty)$, and that $\mathrm{l.\,dim}\ _{\Gamma}A = q$. There exsits a Γ-exact sequence of A with X projective as

$$\text{(E)} \qquad 0 \longrightarrow Q \longrightarrow X \longrightarrow A \longrightarrow 0.$$

Since $\mathrm{l.\,dim}\ _{\Gamma}A > 1$, we have $\mathrm{l.\,dim}\ _{\Gamma}Q = q-1$, hence by the hypothesis of induction $\mathrm{l.\,dim}\ _{\Lambda}Q = q-1$ and $\mathrm{l.\,dim}\ _{\Lambda}X = 0$. Regarding (E) as Λ-exact sequence, we have $\mathrm{l.\,dim}\ _{\Lambda}A = q$.

If there are the same assumptions for week or injective dsmensions, it is ture for them. In partiqular if ψ is epimorphic, the second condition of proposition is satisfied (cf. cor of pr. 9).

PROPOLTION 7. *Let* Λ, Γ *and* ψ *be as avove and* $\mathfrak{l}$ *be a left ideal of* Λ. *We set* $\mathfrak{l}^* = \Gamma\psi(\mathfrak{l})$. *If* $w.\ r.\ dim\ _{\Lambda}\Gamma = 0$, *then* $l.\ dim_{\Gamma}\ \Gamma/\mathfrak{l}^* \leqq l.\ dim\ _{\Lambda}\ \Lambda/\mathfrak{l}$.

$$w.\ l.\ dim\ _{\Gamma}\ \Gamma/\mathfrak{l}^* \leqq w.\ l.\ dim\ _{\Lambda}\ \Lambda/\mathfrak{l}.$$

Proof. We obtain the following commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varepsilon_{\Lambda}} & \Lambda/\mathfrak{l} \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \Gamma & \xrightarrow{\varepsilon_{\Gamma}} & \Gamma/\mathfrak{l}^* \end{array}$$

where ε_{Λ} is the natural mapping of Λ to $\Lambda/\mathfrak{l}$ and ε_{Γ} is that of Γ to $\Gamma/\mathfrak{l}^*$ and φ: $\Lambda/\mathfrak{l} \longrightarrow \Gamma/\mathfrak{l}^*$ is defined as follows,

for $\bar{\lambda} \in \Lambda/\mathfrak{l}$ ($\bar{\lambda}$ is a residue class of λ mod $\mathfrak{l}$)

$\varphi(\bar{\lambda}) = \widetilde{\psi(\lambda)}$ ($\widetilde{\psi(\lambda)}$ is a residue class of $\psi(\lambda)$ mod $\mathfrak{l}^*$).

We define a homorphism g of $\Gamma\underset{\Lambda}{\otimes}\Lambda/\mathfrak{l}$ to $\Gamma/\mathfrak{l}^*$ as follows; for $\gamma\in\Gamma$, $\bar{\lambda}\in\Lambda/\mathfrak{l}$ $g(\gamma\otimes\bar{\lambda})=\gamma\varphi(\bar{\lambda})=\gamma\widetilde{\psi(\lambda)}$. Observe that $\Gamma\underset{\Lambda}{\otimes}\Lambda/\mathfrak{l}=\Gamma\underset{\Lambda}{\otimes}\bar{1}$ and the kernel of g is $\mathfrak{l}^*\otimes\bar{1}$. For $x\in\mathfrak{l}^*\otimes\bar{1}$ we have $x=\sum\gamma_i\psi(l_i)\otimes\bar{1}=\sum\gamma_i\otimes\bar{l}_i=0$ where $\gamma_i\in\Gamma$, $l_i\in\mathfrak{l}$ Hence g is isomorphic. Since by the assumption $Tor_n^\Lambda(\Gamma,\Lambda/\mathfrak{l})=0$ for $n>0$ we have from the mapping theorem [3; VIII, 3·1]

$$Tor_\Lambda(A,\Lambda/\mathfrak{l})\approx Tor^\Lambda(A,\Gamma/\mathfrak{l}^*),$$
$$Ext_\Lambda(\Lambda/\mathfrak{l},C)\approx Ext_\Lambda(\Lambda/\mathfrak{l},C)$$

for right Γ-modules A and left Γ-modules C. This proves the first half. For the second half we have the same thorem as the mapping theorem and we can prove the last statements.

COROLLARY. *Let ψ be epimorphic and N be its kernel. If $w.r.dim\ \Lambda/N=0$, then $l.gl.dim\ \Lambda/N\leqq l.gl.dim.\ \Lambda$. And if $w.r.dim_\Lambda\Lambda/N=0$ or $w\cdot l.dim_\Lambda\Lambda/N=0$, then*

$$w.gl.dim\ \Lambda/N\leqq w.gl.dim\ \Lambda.$$

PROPOSITION 8. *Let Λ, Γ be semi-primary*(1) *and a ring homomorphism ψ of Λ to Γ be given. And let N_Λ be the radical of Λ and we assume that $N_\Gamma=\Gamma\psi(N_\Lambda)$ be the redical of Γ and that $r.dim_\Lambda\Gamma=0$. Then we have for each right Γ-module A and left Γ-module B*

$$r.dim_\Gamma A=r.dim_\Lambda A,\quad l.inj.dim_\Gamma B=l.inj.dim_\Lambda B.$$

Proof. From the consideration in proposition 7 we obtain the following isomorphism,

$$Tor_\Lambda(A,\ \Lambda/N_\Lambda)\approx Tor^\Gamma(A,\ \Gamma/N_\Gamma).$$

We have from the analogous properties of [1; pr. 7] such equivalent relations as

$$r.dim_\Lambda A<n\longleftrightarrow Tor_n^\Lambda(A,\ \Lambda/N_\Lambda)=0\longleftrightarrow Tor_n^\Gamma(A,\ \Gamma/N_\Gamma)=0$$
$$\longleftrightarrow r.dim_\Gamma A<n.$$

It is similar for left injective dimension .

PROPOSITION 9. *Let $\mathfrak{A}$ be a two sided ideal of Λ and we assume that $w.r.dim_\Lambda\Lambda/\mathfrak{A}=0$ or $w.l.dim_\Lambda\Lambda/\mathfrak{A}=0$, then we have for each left $\Lambda/\mathfrak{A}$-module B and right $\Lambda/\mathfrak{A}$-module C $Tor_\Lambda(C,B)\approx Tor_{\Lambda/\mathfrak{A}}(C,B)$. And if $l.dim_\Lambda\Lambda/\mathfrak{A}=0$ or $r.dim_\Lambda\Lambda/\mathfrak{A}=0$, we have $Ext_{\Lambda/\mathfrak{A}}(A,B)$ for each left $\Lambda/\mathfrak{A}$-modules A and B.*

Proof.

It is easily seen that $\mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A},B)$ is isomorphic to B. We define a homomorphism ψ of $\Lambda/\mathfrak{A}\otimes A$ to $\Lambda/\mathfrak{A}A$ by setting, for $\bar{1}\otimes a\in\Lambda/\mathfrak{A}\otimes A(\bar{1}$ is a residue class of 1 mod $\mathfrak{A})$

$$\psi(\bar{1}\otimes a)=\tilde{a}\quad(\tilde{a}\text{ is a residue class of } a \bmod \mathfrak{A}A).$$

Then it is clear that ψ is isdmorphic. From [3, VI. pr. 4·1·2·3·4] we obtain isomorpnisms.

(1) A ring Λ is called semi-primary if it cantains a nilpotent two-dided idal N such that the residue ring Λ/N is semi-simple. It does not coincide with "half primar" of Deuring, Algebren, Ergebn. Math.

COROLLARY. *If* $l.\,dim_{\Lambda}\,\Lambda/\mathfrak{A} = 0$ *we have for left* $\Lambda/\mathfrak{A}$*-modules* A $l.\,dim_{\Lambda}A = l.\,dim_{\Lambda/\mathfrak{A}}A$, $w.\,l.\,dim_{\Lambda}A = w.\,l.\,dim_{\Lambda/\mathfrak{A}}A$. *If* $w.\,r.\,dim_{\Lambda}\,\Lambda/\mathfrak{A} = 0$, *then* $l.\,inj.\,dim_{\Lambda}A = l.\,inj.\,dim_{\Lambda/\mathfrak{A}}A$.

Proof. For each left $\Lambda/\mathfrak{A}$-module B we have a isomorphism: $Ext_{\Lambda}(A, B) \approx Ext_{\Lambda/\mathfrak{A}}(A, B)$, hence we obtain $\mathrm{l.\,m.\,dim}_{\Lambda/\mathfrak{A}}A \leqq \mathrm{l.\,dim}_{\Lambda}A$. The inverse inequality is obtained from lemma 1. It is similar for the remainders.

THEOREM 4. *If a two-sided ideal* $\mathfrak{A}$ *of* Λ *is generated by an idempotent element* e *as a left ideal or a right, then* $l.\,gl.\,dim\,\Lambda \geqq l.\,gl.\,dim\,\Lambda/\mathfrak{A}$, $w.\,gl.\,dim\,\Lambda \geqq w.\,gl\,dim\,\Lambda/\mathfrak{A}$.

REMARK 2. If Γ is a corossed product over Λ with a finite complete outer automorphisms $\mathfrak{G}$ of Λ, then all the assumptions of propositions 7 and 8 are satisfied.

If Γ is a commutative semi-primary ring and $\mathfrak{G}$ is a finite complete automorphisms of Γ and Λ is the $\mathfrak{G}$-invariant subring of Γ, then Γ ahd Λ satisfy all assumptions of propositions 7 and 8.

PROPOSITION 10. *Let* Γ *be a crossed product over* Λ *as above, then*

$$gl.\,dim\,\Lambda = gl.\,dim\,\Gamma.$$

Proof. Let A be *a* left Λ-module. We defind a Γ-module $p(A)$ as follows,

$$p(A) = \sum_{\sigma\in\mathfrak{G}} \otimes\, V_{\sigma}A \qquad (\{V_{\sigma}\} \text{ is a base of } p(A))$$

for $x \in \Lambda$, $V_{\sigma}a \in V_{\sigma}A$

$$x(V_{\sigma}a) = V_{\sigma}x^{\sigma}a \qquad u_{\tau}(V_{\sigma}a) = V_{\tau\sigma}a_{\tau\cdot\sigma}a$$

where $\{u_{\tau}\}$ is a base of Γ over Λ and $\{a_{\tau\cdot\sigma}\}$ is a factor set of Γ over Λ. Since u_1A is a direct sumand of $p(A)$ as left Λ module we obtain by lemma $\mathrm{l.\,dim}_{\Gamma}p(A) \geqq \mathrm{l.\,dim}_{\Lambda}A$. Which proves propotion.

Observing that we can obtain naturally a Γ projective resolttion of $p(A)$ from Λ-projective one of A. we have $\mathrm{l.\,dim}_{\Lambda}A = \mathrm{l.\,dim}_{\Gamma}\,p(A)$.

If Λ is semi-primary, from proposition 8 we obtain,

COROLLARY 1. *If* Λ *is semi-primary, then* $gl.\,dim\,\Lambda = gl.\,dim\,\Gamma$.

We obtain a similar result for the second example of remark 2 as follows.

COROLLARY 2. *Let* Λ *and* Γ *be the same as the second example, then*

$$gl.\,dim\,\Lambda = gl.\,dim\,\Gamma.$$

4. We now characterize rings Λ with $\mathrm{w.\,gl.\,dim}\,\Lambda = 0$

PROPOSITION 11. *Let* $\mathfrak{l}$ *be a left ideal of* Λ. *Then*

$w.\,l.\,dim_{\Lambda}\,\Lambda/\mathfrak{l} = 0$ *if and only if, for each right* Λ*-moule* A *and each right* Λ*-snbmodule* A' *of* A $A' \frown A\mathfrak{l} = A'\mathfrak{l}$ *holds.*

Proof. We assume $\mathrm{w.\,l.\,dim}_{\Lambda}\Lambda/\mathfrak{l} = 0$ and we obtain a exact sequence as follows

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A/A' \longrightarrow 0$$

From our assumption we obtain the exact sequence; $0 \longrightarrow A \underset{\Lambda}{\otimes} \Lambda/\mathfrak{l} \longrightarrow A \underset{\Lambda}{\otimes} \Lambda/\mathfrak{l} \longrightarrow A/A' \underset{\Lambda}{\otimes} \Lambda/\mathfrak{l} \longrightarrow 0$. By the isomorphism ψ in the proof of proposition 9 $A' \frown A\mathfrak{l} = A'\mathfrak{l}$ holds. Conversely if $A' \frown A\mathfrak{l} = A'\mathfrak{l}$ we obtain $\mathrm{w.\,l.\,dim}_{\Lambda}\Lambda/\mathfrak{l} = 0$ by the above consideration

We call an element a of a ring Λ regular if there exists such an element x as

$axa=a$ and a left ideal $\mathfrak{l}$ regular if all elements of $\mathfrak{l}$ are regular.

PROMOTION 12 *If a left ideal* $\mathfrak{l}$ *is regular then*

$$w.\,l.\,dim\ {}_{\Lambda}\Lambda/\mathfrak{l}=0$$

Proof. For a right Λ-module A and its submodule A' we prove the equality $A'\frown A\mathfrak{l}=A'\mathfrak{l}$. For $x\in A'\frown A\mathfrak{l}$, we have $x=\sum a_i y_i$, $a_i\in A$, $y_i\in\mathfrak{l}$.

Since $\mathfrak{l}$ is regular, the left ideal generated by $\{y_i\}$ is generated by an idempotent e. Honce $x\cdot e=\sum a_i y_i e=\sum a_i y_i x\in A'\mathfrak{l}$

LEMMA 4 *For each left Λ-module B we obtain* $w.\,l.\,dem\ {}_{\Lambda}B\leqq n$ *if and only if* $Tor_{n+1}^{\Lambda}(x\Lambda B)=0$, *where $x\Lambda$ is a right Λ-module generated by a single element x.*

Proof. The "if part" is trivial. It is sufficient to show $Tor_{n+1}^{\Lambda}(A,B)=0$ for each finitely generated right Λ-module A, since Tor commutes with the direct limits. We assume that it is true for right Λ-module A' generated by $(n-1)$ elements. Let A be generated by $x_1\ldots x_n$ and A' by $x_1\ldots x_{n-1}$, then we obtain $0\longrightarrow A'\longrightarrow A\longrightarrow A/A'\longrightarrow 0$. Then $\longrightarrow 0=Tor_{n+1}^{\Lambda}(A',B)\longrightarrow Tor_{n+1}^{\Lambda}(A,B)\longrightarrow Tor_{n+1}^{\Lambda}(A/A',B)=0\longrightarrow$ is exact, that is, $Tor_{n+1}^{\Lambda}(A,B)=0$. We have the lemma by the incuction.

COROLLARY *We have for each left Λ-modul B*

$$w.\,l.\,dim\ {}_{\Lambda}B\leqq n \qquad \textit{if and only if}\ \ Tor_{n+1}^{\Lambda}(\Lambda/\mathfrak{A},B)=0$$

for each right ideal $\mathfrak{A}$ *of* Λ.

PROPOSITION 13 *Let* $\mathfrak{l}$ *be a left ideal of* Λ. *Then* $w.\,l.\,dim\ {}_{\Lambda}\Lambda/\mathfrak{l}=0$ *if and only if* $\mathfrak{l}\frown\mathfrak{A}=\mathfrak{A}\mathfrak{l}$ *holds for each right ideal* $\mathfrak{A}$ *of* Λ.

Proof. If we replace A by Λ and A' by $\mathfrak{A}$ in proposition 11, we obtain the first half. Conversely, we assume $\mathfrak{l}\frown\mathfrak{A}=\mathfrak{A}\mathfrak{l}$. From the exact sequence: $0\longrightarrow\mathfrak{A}\longrightarrow\Lambda\longrightarrow\Lambda/\mathfrak{A}\longrightarrow 0$, we obtain the following exact one: $0\longrightarrow Tor_1^{\Lambda}(\Lambda/\mathfrak{A},\Lambda/\mathfrak{l})\longrightarrow\mathfrak{A}\otimes_{\Lambda}\Lambda/\mathfrak{l}\longrightarrow\Lambda\otimes_{\Lambda}\Lambda\mathfrak{l}\longrightarrow$. By our assumption we see that the third arrow is monomorphic and $Tor_1^{\Lambda}(\Lambda/\mathfrak{A},\Lambda/\mathfrak{l}=0$. Hence we obtain the proposition by lemma 4.

COROLLARY *If* $w.\,l.\,dim\ {}_{\Lambda}\Lambda/\mathfrak{l}=0$, *then for any element x of* $\mathfrak{l}$ $x\mathfrak{l}$ *contains x and* $\mathfrak{l}$ *is idempotent*: $\mathfrak{l}^2=\mathfrak{l}$. *In particular if* $\mathfrak{l}$ *is principal* $(\mathfrak{l}=\Lambda a)$ *then* $w.\,l.\,dim\ {}_{\Lambda}\Lambda/\mathfrak{l}=0$ *if and only if there exists some element x in Λa as* $a\cdot x=a$.

From propositions 12 and 13 and theorm 3 we obtain

THEOREM 5 *For each ring Λ, the following conditions are equivalent*:

a) $w.\,gl.\,dim\ \Lambda=0$

b) $A'\frown A\mathfrak{l}=A'\mathfrak{l}$ *for each right Λ module A, each right Λ-sub-module A' of A and each left ideal* $\mathfrak{l}$ *of* Λ.

c) Λ is regular

From corollary of proposition 7 and proposition 12

THEOREM 6 *If* $\mathfrak{A}$ *is a regular two-sided ideal of Λ, then*

$$l.\,gl.\,dim\ \Lambda\geqq l.\,gl.\,dim.\ \Lambda/\mathfrak{A}\ \textit{and}\ w.\,gl.\,dim\ \Lambda\geqq w.\,gl.\,dim\ \Lambda/\mathfrak{A}.$$

If Λ is regular without minimal conditions, for instance a direct product of infinite number of fiels, the w. gl. dim Λ is smaller then gl. dim Λ. We note that from theorems 1 and 5 we obtain that Λ is regular if and only if Λ_n so is, which was obtained by Neumann [7] and that if Λ is regullar, then Λ is semi-hereditary.

5. We consider now some relations between dimensions of Λ and $e\Lambda e(e=e^2)$ under particular assumptions.

Let A and B be left $e\Lambda e$-modules.

Since Λe is a direct sum of $e\Lambda e$ and $(1-e)\Lambda e$, we may regard B as a sub-module of $\Lambda e \underset{e\Lambda e}{\otimes} B$. Hence we obtain an isomorphism: $\mathrm{Hom}_\Lambda(\Lambda e \underset{e\Lambda e}{\otimes} A, \Lambda e \underset{e\Lambda e}{\otimes} B) \approx \mathrm{Hom}_{e\Lambda e}(A, B)$ by the following mappings φ ann ψ: for $f(\in \mathrm{Hom}_\Lambda(\Lambda e \underset{e\Lambda e}{\otimes} A, \Lambda e \underset{e\Lambda e}{\otimes} B))$

$$\varphi f(a) = f(e\otimes a) = e\cdot f(e\otimes a), \quad \psi g(\lambda e\otimes a) = \lambda e\cdot g(a).$$

PROPOSITION 14 *If* $\mathrm{Tor}_n^{e\Lambda e}(\Lambda e, A)=0$ *for* $n>0$ *and a left* $e\Lambda e$ *module* A, *then* $\mathrm{Ext}_\Lambda(\Lambda e \underset{e\Lambda e}{\otimes} A, \Lambda e \underset{e\Lambda e}{\otimes} B) \approx \mathrm{Ext}_{e\Lambda e}(A, B)$ *for each left* $e\Lambda e$*-module* B.

Proof. Since Λe is left Λ-projective, we obtain the proposition by the same consideration as that of the change of rings in [3, VI].

We can obtain the analogous proposition to the above one for Tor

PROPOSITION 14a *If* $\mathrm{Tor}_n^{e\Lambda e}(A, e\Lambda)=0$ *or* $\mathrm{Tor}_n^{e\Lambda e}(\Lambda e, B)=0$ *for* $n>0$ *and a right* $e\Lambda e$ *module* A *and a left* $e\Lambda e$ *module* B. *Then* $\mathrm{Tor}_\Lambda(A \underset{e\Lambda e}{\otimes} e\Lambda, \Lambda e \underset{e\Lambda e}{\otimes} B) \approx \mathrm{Tor}^{e\Lambda e}(A, B)$.

Proof We only note that since $e\Lambda\otimes\Lambda e$ is isomorphic to $e\Lambda e$ as a two sided $e\Lambda e$ module by the mapping: $e\lambda_1\otimes\lambda_2 e \longrightarrow e\lambda_1\lambda_2 e$, we obtain $(A \underset{e\Lambda e}{\otimes} e\Lambda) \underset{\Lambda}{\otimes} (\Lambda e \underset{e\Lambda e}{\otimes} B) \approx A\otimes B$.

PROPOSITION 15 *If* $w.r.\dim_{e\Lambda e}\Lambda e=0$, *we obtain*

$$l.\dim_\Lambda \Lambda e \underset{e\Lambda e}{\otimes} A = l.\dim_{e\Lambda e} A \quad w.l.\dim_\Lambda \Lambda e \underset{e\Lambda e}{\otimes} A = w.l.\dim_{e\Lambda e} A$$

for each left $e\Lambda e$ *module* A.

Proof. If $\mathrm{l.dim}_{e\Lambda e} A$ is infinite, proposition is clear from the above. We prove it by induction with respect to the dimension n of A. It is clear for $n=0$ We assume the proposition for each module A' with $\mathrm{l.dim}_{e\Lambda e} A' \leqq n-1$. We take an exact sequence of a left $e\Lambda e$ module A with $\mathrm{l.dim}_{e\Lambda e} A=n$: $0\longrightarrow Q\longrightarrow P\longrightarrow A\longrightarrow 0$, where P is $e\Lambda e$-projective. By the hypothesis we obtain $\mathrm{l.dim}_\Lambda \Lambda e \underset{e\Lambda e}{\otimes} Q = n-1$ and $\mathrm{l.dim}_\Lambda \Lambda e \underset{e\Lambda e}{\otimes} P=0$. Furthermore we can obtain the exact sequence of $\Lambda e \underset{e\Lambda e}{\otimes} A$: $0\longrightarrow \Lambda e \underset{e\Lambda e}{\otimes} Q \longrightarrow \Lambda e \underset{e\Lambda e}{\otimes} P \longrightarrow \Lambda e \underset{e\Lambda e}{\otimes} A \longrightarrow 0$ from the above one. Hence we have $\mathrm{l.dim}_\Lambda \Lambda e \underset{e\Lambda e}{\otimes} A = n$ for $\mathrm{l.dim}_\Lambda \Lambda e \underset{e\Lambda e}{\otimes} A \neq 0$. For the weak dimension we only observe that we can obtain the exact sequence: $0\longrightarrow B \underset{e\Lambda e}{\otimes} \Lambda e \longrightarrow C\otimes\Lambda e$ from a Λ-exact one: $0\longrightarrow B\longrightarrow C$ and further if $\mathrm{w.l.dim}\, e\Lambda e\ A=0$ we have finally the exact one: $0\longrightarrow B \underset{\Lambda}{\otimes} \Lambda e \underset{e\Lambda e}{\otimes} A \longrightarrow C \underset{\Lambda}{\otimes} \Lambda e \underset{e\Lambda e}{\otimes} A$.

From the proposition 15 we can obtain

THEOREM 7 *If* $w.r.\dim_{e\Lambda e}\Lambda e=0$ *then we obtain*

$$l.gl.\dim\Lambda \geqq l.gl.\dim e\Lambda e \text{ and } w.gl.\dim\Lambda \geqq w.gl.\dim e\Lambda e$$

In order to obtain an analogous theorem to this we need the following lemma

LEMMA 5 *If* $l.\dim_{e\Lambda e} e\Lambda e=0$, *we have for each left* Λ*-module* A $l.\dim_\Lambda A$

$\geqq l.\ dim_{\ e\Lambda e}\ e\Lambda$.

Proof Let $\longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \Lambda \longrightarrow 0$ be a projective resolution of Λ. Then $\longrightarrow eX_2 \longrightarrow eX_1 \longrightarrow eX_0 \longrightarrow e\Lambda \longrightarrow 0$ is clearly a $e\Lambda e$-projective resolution of $e\Lambda$ from our assumption. This proves proposition.

THEOREM 8 *If* $l.\ dim_{\ e\Lambda e}\ e\Lambda = 0$ *then*

$$l.\ gl.\ dim\ \Lambda \geqq l.\ gl.\ dim\ e\Lambda e\,.$$

Proof Let $\mathfrak{l}'$ be a left ideal of $e\Lambda e$. then $\mathfrak{l} = \Lambda\mathfrak{l}'$ is a left ideal of Λ contained in Λe and further $\Lambda/\mathfrak{l}$ is isomorphic to $\Lambda e/\mathfrak{l} \otimes \Lambda_{(1-e)}$. From lemma 5 we obtain $\mathrm{l.\ dim}_{\ \Lambda}\Lambda/\mathfrak{l} = \mathrm{l.\ dim}_{\ \Lambda}\Lambda e/\mathfrak{l} \geqq \mathrm{l.\ dim}_{\ e\Lambda e}\, e(\Lambda e/\mathfrak{l}) = \mathrm{l.\ dim}_{\ e\Lambda e}\, e\Lambda e/e\mathfrak{l} = \mathrm{l.\ dim}_{\ e\Lambda e} e\Lambda e/\mathfrak{l}'$.

Next we consider algebras over a commutative ring K.

PROPOSITION 16 *If* $l.\ dim_{\ e\Lambda e}\, e\Lambda = r.\ dim_{\ e\Lambda e}\, \Lambda e = 0$,

then $$dim\ \Lambda \geqq dim\ e\Lambda e$$

Proof It is easily seen that $(e\Lambda e)^e$ is isomorphic to $(e \otimes e^*)\Lambda^e(e \otimes e^*)$ and $\mathrm{l.\ dim.}_{(e\Lambda e)^*} e^*\Lambda^*$ is equal to $\mathrm{r.\ dim}_{\ e\Lambda e}\, \Lambda e$. Hence from lemma 5 and [3, IX, 2·5] we obtain $\mathrm{l.\ dim}_{\ \Lambda^e}\Lambda \geqq \mathrm{l.\ dim}\ (e\otimes e^*)\ (e\otimes e^*)\Lambda = \mathrm{l.\ dim}_{\ (e\Lambda e)^e}\, e\Lambda e$.

REMARK 3 If we take the total matrix ring of order n over Λ instead of Λ and e_{11} instead of e, then our hypotheses are satisfied and propositions 14 and 14a coincide with proposition 4.

We can eaily obtain isomorphisms of propositions 4, 14 and 14a by using the formulas (4) and (4a) of [3, XVI, 4].

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