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A REMARK ON NON-LOCAL OPERATORS WITH VARIABLE ORDER

Dedicated to professor Itaru Mitoma on his sixtieth birthday

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Abstract

We reveal a relationship between the non-local operator $\mathcal{L}$ with variable order having n as a Lévy-type kernel and the symmetric quadratic form defined by the kernel n. The relationship is obtained through the carré du champ operator relative to $\mathcal{L}$.

1. Introduction

There are many pure jump Markov processes on $\mathbb{R}^d$ for which the infinitesimal generators are the following form:

$$\mathcal{L}u(x) = \int_{y\neq x}(u(y) - u(x) - \nabla u(x)\cdot(y-x)\mathbf{1}_{B(1)}(y-x))n(x, y)\,dy, \quad x \in \mathbb{R}^d, \tag{1.1}$$

or

$$\mathcal{L}u(x) = \int_{h\neq 0}(u(x+h) - u(x) - \nabla u(x)\cdot h\mathbf{1}_{B(1)}(h))\nu(x, h)\,dh, \quad x \in \mathbb{R}^d, \tag{1.1$'$}$$

for some nonnegative function $n(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d - D$, where D is the diagonal set, $D = \{(x, x)\colon x \in \mathbb{R}^d\}$ (or $\nu(x, h)$ defined on $\mathbb{R}^d \times (\mathbb{R}^d - 0)$). Here $B(r)$ means the open ball at the origin with radius r and we denote by $\mathbf{1}_{B(1)}$ the indicator function for $B(1)$.

Intuitively, the function $n(x, y)$ represents the jump rate of the paths of the associated process from the point x to y, while $\nu(x, h)$ shows the jump size $h = y - x$ at x. So the two expressions are the same if the functions $\nu(x, h)$ and $n(x, y)$ are the following:

$$\nu(x, h) = n(x, x+h) \quad \text{or} \quad n(x, y) = \nu(x, y-x) \quad (\text{for } y = x+h).$$

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In this note, we shall reveal a relation between the integro-differential operator $\mathcal{L}$ and the symmetric quadratic form $\mathcal{E}$, where

$$\mathcal{E}(u,v) := \iint_{\mathbb{R}^d\times\mathbb{R}^d - D} (u(y)-u(x))(v(y)-v(x))n(x,y)\,dy\,dx. \tag{1.2}$$

Defining the so-called "carré du champ" operator (see [15, 16]) $\Gamma(\cdot\,,\,\cdot)$ as follows:

$$\Gamma(u,v)(x) := \mathcal{L}(u\cdot v)(x) - \mathcal{L}u(x)\cdot v(x) - u(x)\cdot\mathcal{L}v(x), \quad x\in\mathbb{R}^d,$$

(see (2.2) in Section 2), we will show that

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d}\Gamma(u,v)(x)\,dx, \quad u, v\in C_0^2(\mathbb{R}^d),$$

under suitable conditions on $n(x,y)$ (or $\nu(x,h)$). If we denote by $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ the L^2-generator of the Dirichlet form $\mathcal{E}$, we also investigate a connection between the two generators $\mathcal{L}$ and $\mathcal{A}$. Further a connection between the (non-symmetric) bilinear form generated by $\mathcal{L}$ and the symmetric one $\mathcal{E}$ will be also discussed. We will examine these relations to the case of stable-like processes in the last section.

If $\mathcal{L}$ is the self-adjoint operator on $L^2(X;m)$ associated to a symmetric Dirichlet form $(\eta,\mathcal{D}[\eta])$, then assuming the existence of some nice "core" $\mathcal{C}$ for both $\mathcal{L}$ and η, we see that

$$\eta(u,v) = \mathcal{E}(u,v) - \frac{1}{2}\int_X \mathcal{L}(u\cdot v)(x)m(dx), \quad u,v\in\mathcal{C},$$

where $(X,\mathfrak{F},m)$ is a σ-finite measure space. Carré du champ operators Γ play a role when we study, for example, the logarithmic Sobolev inequalities for the given quadratic forms in the case of infinite dimensional spaces or the diffusion cases (see e.g., [2, 1]).

2. Carré du champ operator

We first give a sufficient condition in order that the operator $\mathcal{L}$ maps $C_0^2(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for $p\geq 1$.

Proposition 1. *Set* $j(x,y) = n(x,y)+n(y,x)$, $x\neq y$. *Suppose that*

$$\sup_{x\in\mathbb{R}^d}\int_{y\neq x}(|y-x|^2\wedge 1)j(x,y)\,dy < \infty. \tag{2.1}$$

Then $\mathcal{L}(C_0^2(\mathbb{R}^d))\subset L^p(\mathbb{R}^d)$ *for any* $1\leq p\leq\infty$.

Proof. We denote by M the supremum of the left hand side of (2.1). For any $u\in C_0^2(\mathbb{R}^d)$, take positive numbers r and R so that $\mathrm{supp}[u]\subset B(r)\subset B(R)$, $R-r\geq 1$.

The proof for the case $p = \infty$ is rather easy, so we only show the case $1 \le p < \infty$. Since $|x|^p$ is a convex function on $\mathbb{R}$, we see that

$$\begin{aligned}\|\mathcal{L}u\|_{L^p}^p &= \int_{\mathbb{R}^d} \left| \int_{y \neq x} (u(y) - u(x) - \nabla u(x) \cdot (y-x) \mathbf{1}_{B(1)}(y-x)) n(x,y)\, dy \right|^p dx \\ &\le 2^{p-1} \int_{\mathbb{R}^d} \left| \int_{0<|y-x|<1} (u(y) - u(x) - \nabla u(x) \cdot (y-x)) n(x,y)\, dy \right|^p dx \\ &\quad + 2^{p-1} \int_{\mathbb{R}^d} \left| \int_{|y-x| \ge 1} (u(y) - u(x)) n(x,y)\, dy \right|^p dx \\ &=: 2^{p-1}((\mathrm{I}) + (\mathrm{II})).\end{aligned}$$

Since $\operatorname{supp}[u]$ is contained in $B(r)$ and $R - r \ge 1$, we see that

$$(\mathrm{I}) = \int_{B(R)} \left| \int_{0<|y-x|<1} (u(y) - u(x) - \nabla u(x) \cdot (y-x)) n(x,y)\, dy \right|^p dx.$$

Then by making use of Taylor's expansion for u,

$$(\mathrm{I}) \le C \int_{B(R)} \left(\int_{0<|y-x|<1} |y-x|^2 n(x,y)\, dy \right)^p dx \le C M^p \operatorname{Vol}(B(R)) < \infty.$$

As for (II), divide the integral in (II) into two parts:

$$(\mathrm{II}) \le \left(\int_{B(R)} + \int_{B(R)^c} \right) \left| \int_{|y-x| \ge 1} (u(y) - u(x)) n(x,y)\, dy \right|^p dx =: (\text{II-1}) + (\text{II-2}).$$

It is easy to see that

$$(\text{II-1}) \le (2\|u\|_\infty M)^p \operatorname{Vol}(B(R)) < \infty.$$

Finally we need to see the finiteness of (II-2). Thanks to the inequality

$$|u(y)| \le \|u\|_\infty \mathbf{1}_{B(r)}(y) \quad \text{for} \quad y \in \mathbb{R}^d,$$

we see

$$(\text{II-2}) \le (\|u\|_\infty)^p M^{p-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{B(R)^c}(x) \cdot \mathbf{1}_{B(1)^c}(y-x) \cdot \mathbf{1}_{B(r)}(y) n(x,y)\, dy\, dx.$$

Using the Fubini theorem and then, changing the variables $x \leftrightarrow y$, the right hand side of the inequality is estimated by

$$
\begin{aligned}
&(\|u\|_\infty)^p M^{p-1} \int_{B(r)} \int_{|y-x|\geq 1} n(y, x)\, dy\, dx \\
&\leq (\|u\|_\infty)^p M^{p-1} \int_{B(r)} \int_{|y-x|\geq 1} j(y, x)\, dy\, dx \\
&\leq (\|u\|_\infty M)^p \operatorname{Vol}(B(r)) < \infty.
\end{aligned}
$$

□

REMARK 1. (i) If we want to show $\mathcal{L}u \in L^\infty(\mathbb{R}^d)$ for $u \in C_0^2(\mathbb{R}^d)$, then it is enough for us to assume that

$$\sup_{x\in\mathbb{R}^d} \int_{y\neq x} (|y-x|^2 \wedge 1) n(x, y)\, dy < \infty.$$

But this can not guarantee the integrability of $\mathcal{L}u$ in general.
(ii) We can make a bit weaker assumption in order to see that $\mathcal{L}(C_0^2(\mathbb{R}^d))$ is included in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ as follows:

$$x \mapsto \int_{y\neq x} (|x-y|^2 \wedge 1) n(x, y)\, dy \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$$

and for all R, r with $0 < r < R$,

$$x \mapsto \int_{B(r)} 1_{|x-y|>1}(y) n(x, y)\, dy \in L^p(\mathbb{R}^d \setminus B(R)).$$

DEFINITION ("carré du champ" operator). Assume the condition in Proposition 1. Then we define a carré du champ operator Γ relative to $\mathcal{L}$ from $C_0^2(\mathbb{R}^d) \times C_0^2(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ as follows: for $u, v \in C_0^2(\mathbb{R}^d)$

$$\Gamma(u, v)(x) := \mathcal{L}(u\cdot v)(x) - u(x)\cdot \mathcal{L}v(x) - \mathcal{L}u(x)\cdot v(x), \quad x \in \mathbb{R}^d. \tag{2.2}$$

Theorem 1. *Assume* (2.1) *in* Proposition 1 *holds. Then for any* $u, v \in C_0^2(\mathbb{R}^d)$, *we have*

$$\Gamma(u, v)(x) = \int_{y\neq x} (u(y)-u(x))(v(y)-v(x)) n(x, y)\, dy, \quad x \in \mathbb{R}^d. \tag{2.3}$$

This means that the form $\mathcal{E}$ *defined by* (1.2) *is written as*

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \Gamma(u, v)(x)\, dx, \quad u, v \in C_0^2(\mathbb{R}^d).$$

Moreover, $(\mathcal{E}, C_0^2(\mathbb{R}^d))$ *is then a closable symmetric Markovian form on* $L^2(\mathbb{R}^d)$.

Proof. Once we show (2.3), the closability and the Markov property for $(\mathcal{E}, C_0^2(\mathbb{R}^d))$ are easily seen as in the Example 1.2.4 in [6] (see also [19, 20]) under the condition. So we show (2.3). For $u, v \in C_0^2(\mathbb{R}^d)$,

$$\begin{aligned}\Gamma(u, v)(x) &= \mathcal{L}(u \cdot v)(x) - u(x) \cdot \mathcal{L}v(x) - \mathcal{L}u(x) \cdot v(x)\\ &= \int_{y \neq x} (u(y)v(y) - u(x)v(x) - \nabla(u(x) \cdot v(x)) \cdot (y-x)\mathbf{1}_{B(1)}(y-x))n(x, y)\, dy\\ &\quad - v(x) \int_{y \neq x} (u(y) - u(x) - \nabla u(x) \cdot (y-x)\mathbf{1}_{B(1)}(y-x))n(x, y)\, dy\\ &\quad - u(x) \int_{y \neq x} (v(y) - v(x) - \nabla v(x) \cdot (y-x)\mathbf{1}_{B(1)}(y-x))n(x, y)\, dy.\end{aligned}$$

Note that $\nabla(u(x) \cdot v(x)) = v(x)\nabla u(x) + u(x)\nabla v(x)$ for $x \in \mathbb{R}^d$. Therefore, dividing each integral in the above into two parts respectively, one is on the set $\{0 < |y - x| < 1\}$ and the other is on $\{|y - x| \geq 1\}$, and summing up the integrands respective parts, then we easily see

$$\Gamma(u, v)(x) = \int_{y \neq x} (u(y) - u(x))(v(y) - v(x))n(x, y)\, dy. \qquad \square$$

From now on, we always assume (2.1). Let $\mathcal{F}$ be the closure of $C_0^2(\mathbb{R}^d)$ with respect to the norm $\sqrt{\mathcal{E}_1(\,\cdot\,,\,\cdot\,)}$, where

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \int_{\mathbb{R}^d} u(x)v(x)\, dx, \quad u, v \in C_0^2(\mathbb{R}^d).$$

Then $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$ and a Hunt process associated with it is a pure jump Markov process (see [6]). We denote by $(\mathcal{A}, \mathcal{D}[\mathcal{A}])$ the (L^2)-infinitesimal generator corresponding to $(\mathcal{E}, \mathcal{F})$. Now we try to find a relation between $\mathcal{L}$ and $\mathcal{A}$. In order to do so, we may need to know the exact form of $\mathcal{A}u$ for appropriate functions u. As for this question, if we assume a bit stronger condition on $n(x, y)$, we can have an exact form of $\mathcal{A}u$ for $u \in C_0^2(\mathbb{R}^d)$.

Theorem 2 (c.f. Theorem 2.2 and Proposition 2.5 in [17]). *Assume* (2.1). *Suppose further that there exists a function $b \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d)$ so that*

$$\lim_{\varepsilon \searrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\varepsilon < |h| < 1} h j(x, x + h)\, dh - b(x) \right| = 0. \tag{2.4}$$

Then $C_0^2(\mathbb{R}^d) \subset \mathcal{D}[\mathcal{A}]$ and for $u \in C_0^2(\mathbb{R}^d)$, $\mathcal{A}u$ is given by

$$\mathcal{A}u(x) = \int_{h \neq 0} (u(x + h) - u(x) - \nabla u(x) \cdot h\mathbf{1}_{B(1)}(h)) j(x, x + h)\, dh + b(x) \cdot \nabla u(x)$$

By taking care of the appearance of the function b, similar arguments of the proofs of Theorem 2.2 and Proposition 2.5 of [17] give us the theorem. So we omit it. Note that for a jump rate $n(x, y)$, $j(x, y) = n(x, y) + n(y, x)$, $x \neq y$. Since $(\mathcal{A}, \mathcal{D}[\mathcal{A}])$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$, it is symmetric. But $\mathcal{L}$ is not in general. So, if we want to know a relation between $\mathcal{L}$ and $\mathcal{A}$, we may also need to know an exact form of the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$. We now try to find the form of the adjoint operator $\mathcal{L}^*$ if possible. To this end, first set $\Theta(h) = h\mathbf{1}_{B(1)}(h)$, $h \in \mathbb{R}^d$. For any $u, v \in C_0^2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$ with $x \neq y$, noting the equality:

$$\begin{aligned} & u(x)(v(y) - v(x) - \nabla v(x) \cdot \Theta(y - x)) - v(y)(u(x) - u(y) - \nabla u(y) \cdot \Theta(x - y)) \\ & = u(y)v(y) - u(x)v(x) - \nabla(u \cdot v)(x)\Theta(y - x) + (v(x)\nabla u(x) - u(y)\nabla v(y)) \cdot \Theta(y - x), \end{aligned}$$

we then obtain

$$\begin{aligned} & \iint_{|x-y|>\varepsilon} u(x)(v(y) - v(x) - \nabla v(x) \cdot \Theta(y - x))n(x, y)\,dx\,dy \\ & - \iint_{|x-y|>\varepsilon} v(y)(u(x) - u(y) - \nabla u(y) \cdot \Theta(x - y))n(x, y)\,dx\,dy \\ & = \iint_{|x-y|>\varepsilon} (u(y)v(y) - u(x)v(x) - \nabla(u \cdot v)(x) \cdot \Theta(y - x))n(x, y)\,dx\,dy \\ & \quad + \iint_{|x-y|>\varepsilon} (v(x)\nabla u(x) - u(y)\nabla v(y)) \cdot \Theta(y - x)n(x, y)\,dx\,dy. \end{aligned} \tag{2.5}$$

We also note that, by the condition (2.4),

$$\begin{aligned} & \iint_{|x-y|>\varepsilon} (v(x)(y - x) \cdot \nabla u(x) - u(y)\nabla v(y) \cdot \Theta(y - x))n(x, y)\,dx\,dy \\ & = \iint_{|x-y|>\varepsilon} v(x)\nabla u(x) \cdot \Theta(y - x)j(x, y)\,dx\,dy \\ & \to \int v(x)b(x)\nabla u(x)\,dx \quad \text{as} \quad \varepsilon \downarrow 0. \end{aligned}$$

Set $\mathcal{D} = b(x) \cdot \nabla$ and define an operator $\tilde{\mathcal{L}}$ by

$$\tilde{\mathcal{L}}u(y) = \int_{h\neq 0} (u(x) - u(y) - \nabla u(y) \cdot \Theta(x - y))n(x, y)\,dx, \quad y \in \mathbb{R}^d.$$

From the above calculus, we have the following equality

$$(u, \mathcal{L}v) - (\tilde{\mathcal{L}}u, v) = (1, \mathcal{L}(u \cdot v)) + (v, \mathcal{D}u).$$

Summarizing the discussion above, we have the following theorem:

Theorem 3. *Assume* (2.1) *and* (2.4). *Then we have*

$$(u, \mathcal{L}v) = (v, \tilde{\mathcal{L}}u) + (v, \mathcal{D}u) + (1, \mathcal{L}(u \cdot v)), \quad \textit{for} \quad u, v \in C_0^2(\mathbb{R}^d).$$

Moreover, noticing that $\mathcal{E}(u, v) = -(u, \mathcal{A}v) = \int \Gamma(u, v)\, dx$, $u \in \mathcal{F}$, $v \in \mathcal{D}[\mathcal{A}]$ *and* $\Gamma(u, v)(x) = \mathcal{L}(u \cdot v)(x) - \mathcal{L}u(x) \cdot v(x) - u(x) \cdot \mathcal{L}v(x)$, $x \in \mathbb{R}^d$, $u, v \in C_0^2(\mathbb{R}^d)$, *we alternatively have that for* $u \in C_0^2(\mathbb{R}^d)$,

$$\mathcal{A}u(x) = \mathcal{L}u(x) + \tilde{\mathcal{L}}u(x) + \mathcal{D}u(x), \quad x \in \mathbb{R}^d.$$

REMARK 2. (i) Denote by $\mathcal{L}^*$ the (formal) adjoint operator of $\mathcal{L}$ on $L^2(\mathbb{R}^d)$. If we are able to justify "$\mathcal{L}^*1(x) =: k(x)$", then the adjoint operator $\mathcal{L}^*$ has the following form: for $x \in \mathbb{R}^d$,

$$\begin{aligned}\mathcal{L}^*u(x) &= \tilde{\mathcal{L}}u(x) + \mathcal{D}u(x) + u(x) \cdot k(x)\\ &= \int_{y \neq x} (u(y) - u(x) - \nabla u(x) \cdot \Theta(y - x))n(y, x)\, dy + \mathcal{D}u(x) + u(x) \cdot k(x).\end{aligned}$$

(ii) Carré du champ operators are known as the operators that take out "the higher order terms". For example, consider the so-called *operator of non-divergence form*:

$$\mathcal{L}u(x) = \sum_{ij} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_j}(x), \quad x \in \mathbb{R}^d,$$

for some positive function $a = (a_{ij}(x))$ satisfying uniformly elliptic condition. In this case, the carré du champ operator Γ is given by

$$\Gamma(u, v)(x) = 2 \sum_{ij} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x), \quad x \in \mathbb{R}^d.$$

This gives us an *operator of divergence form* (divided by 2).

Finally, we define a bilinear form relative to the operator $\mathcal{L}$:

$$\eta(u, v) := -(u, \mathcal{L}v), \quad u, v \in C_0^2(\mathbb{R}^d).$$

Under the condition (2.1), this quadratic form $(\eta, C_0^2(\mathbb{R}^d))$ is a densely defined quadratic form on $L^2(\mathbb{R}^d)$, but is not necessarily symmetric nor positive definite in general. So we do not know this becomes a (quasi-)regular Dirichlet form ([13]). Imitating the theory of non-symmetric Dirichlet form, denote by $\tilde{\eta}$ and $\check{\eta}$ the symmetric part and the anti-symmetric part of η respectively:

$$\tilde{\eta}(u, v) = \frac{1}{2}(\eta(u, v) + \eta(v, u)), \quad \check{\eta}(u, v) = \frac{1}{2}(\eta(u, v) - \eta(v, u)), \quad u, v \in \mathcal{D}[\eta].$$

We now show a connection between $\mathcal{E}(u, v)$ and $\eta(u, v)$ for the functions $u, v \in C_0^2(\mathbb{R}^d)$:

Proposition 2. *Assume* (2.1). *Then we have*

$$\mathcal{E}(u, v) = 2\tilde{\eta}(u, v) + \int_{\mathbb{R}^d} \mathcal{L}(u \cdot v)(x)\, dx, \quad u, v \in C_0^2(\mathbb{R}^d).$$

Proof. This is an easy consequence of Theorem 1. In fact, since for $u, v \in C_0^2(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{E}(u, v) &= \int_{\mathbb{R}^d} \Gamma(u, v)(x)\, dx \\ &= \int_{\mathbb{R}^d} \mathcal{L}(u \cdot v)(x)\, dx - (u, \mathcal{L}v) - (v, \mathcal{L}u) \\ &= 2\tilde{\eta}(u, v) + \int_{\mathbb{R}^d} \mathcal{L}(u \cdot v)(x)\, dx. \qquad \square\end{aligned}$$

Note that, even if $(\eta, C_0^2(\mathbb{R}^d))$ does not produce a (quasi-)regular Dirichlet form, we can always construct a symmetric Hunt process associated to $\mathcal{E}$ whenever $n(x, y)$ satisfies the condition (2.1). But the process is not corresponding to $\mathcal{L}$ directly. Though one possibility to construct a process associated with $\mathcal{L}$ is to show the (quasi-)regularity of the quadratic form η ([13]), we do not know, as we said, that the positive definiteness of η in general. Now we only give a sufficient condition that the quadratic form $(\eta, C_0^2(\mathbb{R}^d))$ relative to $\mathcal{L}$ becomes a positive definite one:

$$\int_{\mathbb{R}^d} \mathcal{L} f(x)\, dx \le 0, \quad f \ge 0, \ f \in C_0^2(\mathbb{R}^d). \tag{2.6}$$

Under this condition, we see, from Proposition 2, that

$$0 \le \mathcal{E}(u, u) = 2\tilde{\eta}(u, u) + \int_{\mathbb{R}^d} \mathcal{L}(u^2)(x)\, dx \le 2\tilde{\eta}(u, u) = 2\eta(u, u), \quad u \in C_0^2(\mathbb{R}^d).$$

So this implies that the form $(\eta, C_0^2(\mathbb{R}^d))$ is positive definite.

REMARK 3. Assume there exists a strong Markov process $\mathbf{M} = (X_t, \mathbb{P}_x)$ for which $\mathbb{P}_x$ solves the $\mathcal{L}$-martingale problem for each $x \in \mathbb{R}^d$: for $f \in C_b^2(\mathbb{R}^d)$,

$$\mathbb{P}_x(X_0 = x) = 1, \quad f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s)\, ds \quad \text{is a } \mathbb{P}_x\text{-local martingale.}$$

Let $\{p_t\}$ be the transition function of $\mathbf{M}$. If the Lebesgue measure dx is $\{p_t\}$-excessive in the sense that,

$$\int_{\mathbb{R}^d} p_t(x, B)\, dx \le \mathrm{Vol}(B) \quad \text{for} \quad t > 0 \quad \text{and} \quad B \in \mathcal{B}(\mathbb{R}^d),$$

then the condition (2.6) is satisfied. In fact, for any $f \in C_0^2(\mathbb{R}^d)$ with $f \geq 0$, we see that

$$0 \geq \int_{\mathbb{R}^d} p_t f(x)\, dx - \int_{\mathbb{R}^d} f(x)\, dx = \int_0^t \int_{\mathbb{R}^d} p_s \mathcal{L} f(x)\, dx\, ds.$$

Hence, as taking $t \to 0$ after dividing by t in the both sides, we have (2.6)

3. Stable-like processes

In this section, we examine the results obtained in the preceding section to the case of "stable-like" processes. Stable-like processes are defined as variants of symmetric stable processes by Bass [3, 4]. For a measurable function α defined on $\mathbb{R}^d$, he introduced the following integro-differential operator: for $u \in C_0^2(\mathbb{R}^d)$,

$$\mathcal{L}u(x) := w(x) \int_{h \neq 0} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)) |h|^{-d-\alpha(x)}\, dh, \quad x \in \mathbb{R}^d,$$

where w is a function chonsen so that $\mathcal{L}e^{iux} = -|u|^{\alpha(x)} e^{iux}$. If α is Lipschitz continuous, bounded below by a constant α_1 which is greater than 0, and bounded above by a constant α_2 which is less than 2, then he showed that there exist a unique strong Markov process $\mathbf{M} = (X_t, \mathbb{P}_x)$ for which $\mathbb{P}_x$ solves the martingale problem for $\mathcal{L}$ at each point $x \in \mathbb{R}^d$. After that, many authors have studied this type of operators to construct stochastic processes by using various techniques including SDE with jumps, pseudo differential operators or Dirichlet form theory (see e.g. [18, 14, 7, 12, 11, 8, 19, 20] and also see [5, 10], for related topics and the references). If α satisfies the condition mentioned above, it is known that w is a bounded continuous function satisfying $\alpha_1 \leq w(x) \leq \alpha_2$. Though the Lévy kernel is indeed given by $w(x)|x-y|^{-d-\alpha(x)}$ for the stable-like process, we consider $n(x, y) = |x-y|^{-d-\alpha(x)}$ as our kernel in the sequel for simplicity.

Then the symmetric stable-like processes ([19]) can be also constructed by using the "carré du champ" operator $\Gamma(u, v) = \mathcal{L}(uv)(x) - \mathcal{L}u(x)v(x) - u(x)\mathcal{L}v(x)$:

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \Gamma(u, v)(x)\, dx = \iint_{x \neq y} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+\alpha(x)}}\, dx\, dy.$$

In order to justify the results mentioned in the preceding section, we give a sufficient condition that the function α satisfies (2.1) and (2.4):

Proposition 3. *Suppose that the function α satisfies the following conditions*:

- *there exists positive constants α and β such that, for any $x \in \mathbb{R}^d$,*

$$0 < \alpha \leq \alpha(x) \leq \beta < 2,$$

- *there exists positive constants* M *and* δ *satisfying* $(\beta - 1) \vee 0 < \delta \leq 1$ *so that*

$$|\alpha(x) - \alpha(y)| \leq M|x - y|^{\delta}, \quad x, y \in \mathbb{R}^d.$$

Then the conditions (2.1) *and* (2.4) *hold.*

Proof. (2.1) is easily calculated. So we show (2.4). Note that $j(x, y) = |x - y|^{-d-\alpha(x)} + |x - y|^{-d-\alpha(y)}$. Then, for any $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{0<|h|<1} |h| \cdot |j(x, x+h) - j(x, x-h)|\, dh \\
&= \int_{0<|h|<1} |h| \cdot \left| |h|^{-d-\alpha(x+h)} - |h|^{-d-\alpha(x-h)} \right| dh \\
&= \int_{0<|h|<1} |h|^{1-d} \cdot \left| |h|^{-\alpha(x+h)} - |h|^{-\alpha(x-h)} \right| dh.
\end{aligned}$$

Thanks for the formula $|t^{-a} - t^{-b}| = \left| \int_a^b t^{-u} \log t\, du \right|$, we see that, for $0 < |h| < 1$

$$\begin{aligned}
\left| |h|^{-\alpha(x+h)} - |h|^{-\alpha(x-h)} \right| &\leq |\alpha(x+h) - \alpha(x-h)| \cdot \log \frac{1}{|h|} \cdot |h|^{-\max\{\alpha(x+h), \alpha(x-h)\}} \\
&\leq M|(x+h) - (x-h)|^{\delta} \log \frac{1}{|h|} \cdot |h|^{-\beta} \\
&= 2^{\delta} M |h|^{\delta-\beta} \cdot \log \frac{1}{|h|}.
\end{aligned}$$

Thus, since $(\beta - 1) \vee 0 < \delta \leq 1$,

$$\begin{aligned}
\int_{0<|h|<1} |h| \cdot \left| j(x, x+h) - j(x, x-h) \right| dh &\leq 2^{\delta} M \int_{0<|h|<1} |h|^{1-d+\delta-\beta} \log \frac{1}{|h|}\, dh \\
&\leq 2^{\delta} M c_d \int_0^1 u^{\delta-\beta} \log \frac{1}{u}\, du < \infty.
\end{aligned}$$

Set

$$b(x) = \int_{0<|h|<1} h(|h|^{-d-\alpha(x+h)} - |h|^{-d-\alpha(x-h)})\, dh,$$

then we see that b satisfies (2.4). □

REMARK 4. In general, it is difficult to write down the adjoint operator $\mathcal{L}^*$ as an exact form (see Remark 2 (ii)). But, as was pointed out in Remark 3.1 of [9], we are able to do it when the function α satisfies the following stronger conditions: There exists positive constants $\underline{\alpha}$, β, δ and M such that, $0 < \delta \leq 1$,

(3.1) $\quad 0 < \underline{\alpha} \leq \alpha(x) \leq \beta < 1 \quad$ and $\quad |\alpha(x) - \alpha(y)| \leq M|x - y|^{\delta}, \quad$ for $\quad x, y \in \mathbb{R}^d$.

In fact, under the conditions, the operator $\mathcal{L}$ has the following form for $u \in C_0^1(\mathbb{R}^d)$:

$$\mathcal{L}u(x) = \int_{h\neq 0} (u(x+h) - u(x)) \cdot |h|^{-d-\alpha(x)}\, dh, \quad x \in \mathbb{R}^d.$$

As in the discussion developed after Theorem 2, we see that for any $u, v \in C_0^1(\mathbb{R}^d)$,

$$\begin{aligned}(u, \mathcal{L}v) &= \int_{\mathbb{R}^d} u(x) \int_{h\neq 0} \frac{(v(x+h) - v(x))}{|h|^{d+\alpha(x)}}\, dh\, dx \\ &= \int_{\mathbb{R}^d} v(x) \int_{h\neq 0} \frac{\big(u(x+h) - u(x)\big)}{|h|^{d+\alpha(x+h)}}\, dh \\ &\quad + \int_{\mathbb{R}^d}\int_{h\neq 0} \left(\frac{v(x)u(x)}{|h|^{d+\alpha(x+h)}} - \frac{v(x)u(x)}{|h|^{d+\alpha(x)}} \right) dh\, dx.\end{aligned}$$

This shows that

$$\mathcal{L}^* u(x) = \int_{h\neq 0} \frac{(u(x+h) - u(x))}{|h|^{d+\alpha(x+h)}}\, dh + u(x) \cdot \int_{h\neq 0} (|h|^{-d-\alpha(x+h)} - |h|^{-d-\alpha(x)})\, dh, \quad x \in \mathbb{R}^d.$$

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References

[1] S. Aida, T. Masuda and I. Shigekawa: *Logarithmic Sobolev inequalities and exponential integrability*, J. Funct. Anal. **126** (1994), 83–101.

[2] D. Bakry: *L'hypercontractivité et son utilisation en théorie des semigroupes*; in Lectures on Probability Theory (Saint-Flour, 1992), Lecture Notes in Math. **1581**, Springer, Berlin, 1994, 1–114.

[3] R.F. Bass: *Occupation time densities for stable-like processes and other pure jump Markov processes*, Stochastic Process. Appl. **29** (1988), 65–83.

[4] R.F. Bass: *Uniqueness in law for pure jump Markov processes*, Probab. Theory Related Fields **79** (1988), 271–287.

[5] R.F. Bass: *Stochastic differential equations with jumps*, Probab. Surv. **1** (2004), 1–19.

[6] M. Fukushima, Y. Ōshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes, de Gruyter Studies in Mathematics **19**, de Gruyter, Berlin, 1994.

[7] W. Hoh: *The martingale problem for a class of pseudo-differential operators*, Math. Ann. **300** (1994), 121–147.

[8] W. Hoh: *Pseudo differential operators with negative definite symbols of variable order*, Rev. Mat. Iberoamericana **16** (2000), 219–241.

[9] Y. Isozaki and T. Uemura: *A family of symmetric stable-like processes and its global path properties*, Probab. Math. Statist. **24** (2004), 145–164.

[10] N. Jacob: Pseudo Differential Operators and Markov Processes, I–III, Imp. Coll. Press, London, 2001–2005.

[11] V. Kolokoltsov: *Symmetric stable laws and stable-like jump-diffusions*, Proc. London Math. Soc. (3) **80** (2000), 725–768.

[12] T. Komatsu: *On stable-like processes*; in Probability Theory and Mathematical Statistics (Tokyo, 1995), World Sci. Publ., River Edge, NJ, 1996, 210–219.

[13] Z.M. Ma and M. Röckner: Introduction to the Theory of (Nonsymmetric) Dirichlet Forms, Universitext, Springer, Berlin, 1992.

[14] A. Negoro: *Stable-like processes*: *construction of the transition density and the behavior of sample paths near* $t = 0$, Osaka J. Math. **31** (1994), 189–214.

[15] J.-P. Roth: *Opérateurs carré du champ et formule de Lévy-Kinchine sur les espaces localement compacts*, C.R. Acad. Sci. Paris Sér. A **278** (1974), 1103–1106.

[16] J.-P. Roth: *Opérateurs dissipatifs et semi-groupes dans les espaces de fonctions continues*, Ann. Inst. Fourier (Grenoble) **26** (1976), 1–97.

[17] R.L. Schilling and T. Uemura: *On the Feller property of Dirichlet forms generated by pseudo differential operators*, Tohoku Math. J. (2) **59** (2007), 401–422.

[18] M. Tsuchiya: *Lévy measure with generalized polar decomposition and the associated SDE with jumps*, Stochastics Stochastics Rep. **38** (1992), 95–117.

[19] T. Uemura: *On some path properties of symmetric stable-like processes for one dimension*, Potential Anal. **16** (2002), 79–91.

[20] T. Uemura: *On symmetric stable-like processes*: *some path properties and generators*, J. Theoret. Probab. **17** (2004), 541–555.

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