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INVOLUTIONS OF COMPACT RIEMANNIAN 4-SYMMETRIC SPACES

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Abstract

Let G/H be a compact 4-symmetric space of inner type such that the dimension of the center $Z(H)$ of H is at most one. In this paper we shall classify involutions of G preserving H for the case where $\dim Z(H) = 0$, or H is a centralizer of a toral subgroup of G.

1. Introduction

It is known that Riemannian k-symmetric spaces is a generalizations of Riemannian symmetric spaces. The definition is as follows:

Let G be a Lie group and H a compact subgroup of G. A homogeneous space $(G/H, \langle\ ,\ \rangle)$ with G-invariant Riemannian metric $\langle\ ,\ \rangle$ is called a *Riemannian k-symmetric space* if there exists an automorphism σ on G such that

1. $G^\sigma_o \subset H \subset G^\sigma$, where G^σ and G^σ_o is the set of fixed points of σ and its identity component, respectively,
2. $\sigma^k = \mathrm{Id}$ and $\sigma^l \neq \mathrm{Id}$ for any $l < k$,
3. The transformation of G/H induced by σ is an isometry.

We denote by $(G/H, \langle\ ,\ \rangle, \sigma)$ a Riemannian k-symmetric space with an automorphism σ. Gray [5] classified Riemannian 3-symmetric spaces (see also Wolf and Gray [15]). Moreover compact Riemannian 4-symmetric spaces is classified by Jeménez [7]. The structure of Riemannian k-symmetric spaces is closely related to the study of finite order automorphisms of Lie groups. Such automorphisms of compact simple Lie groups were classified (cf. Kac [8] and Helgason [6]).

It is known that involutions on k-symmetric spaces are important. For example, the classifications of affine symmetric spaces by Berger [1] are, in essence, the classification of involutions on compact symmetric spaces G/H preserving H. Similarly, such involutions play an important role in the classification of symmetric submanifolds on compact symmetric spaces (cf. Naitoh [11] and [12]).

On a compact 3-symmetric space $(G/H, \langle\ ,\ \rangle, \sigma)$, an involution τ preserving H satisfies $\tau \circ \sigma = \sigma \circ \tau$ or $\tau \circ \sigma = \sigma^{-1} \circ \tau$. The classification of affine 3-symmetric spaces ([15]) was made by classifying involutions τ satisfying $\tau \circ \sigma = \sigma \circ \tau$. Moreover, [13]

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and [14] classify half-dimensional, totally real and totally geodesic submanifold (with respect to the canonical almost complex structures) of compact Riemannian 3-symmetric spaces $(G/H, \langle\ ,\ \rangle, \sigma)$ by classifying involutions τ on G satisfying $\tau \circ \sigma = \sigma^{-1} \circ \tau$.

In general, there exists an involution τ such that $\tau \circ \sigma \circ \tau^{-1} \neq \sigma$ or σ^{-1} for Riemannian 4-symmetric spaces. These automorphisms do not appear in Riemannian symmetric spaces and 3-symmetric spaces. However, if the dimension of the center of H is at most one, each involution τ preserving H satisfies $\tau \circ \sigma \circ \tau^{-1} = \sigma$ or σ^{-1}.

According to [7], a compact simply connected Riemannian 4-symmetric space decomposes as a product $M_1 \times \cdots \times M_r$, where M_i $(1 \leq i \leq r)$ is compact, irreducible Riemannian 4-symmetric space. In this paper we treat a compact, irreducible Riemannian 4-symmetric space $(G/H, \langle\ ,\ \rangle, \sigma)$ such that the dimension of the center of H is at most one. In particular we classify involutions of G preserving H for the case where $\dim Z(H) = 0$, or $\dim Z(H) = 1$ and H is a centralizer of a toral subgroup of G. More precisely, let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Then we first prove that there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{h}$ such that $\tau(\mathfrak{t}) = \mathfrak{t}$ for any involution τ preserving $\mathfrak{h}$. Except for the case where $\dim Z(H) = 1$ and $\tau \circ \sigma \circ \tau^{-1} = \sigma^{-1}$, we classify involutions $\bar{\tau}$ of the root system of $\mathfrak{h}$ with respect to $\mathfrak{t}$. Moreover, for each involution $\bar{\tau}$ $(\bar{\tau} \neq \mathrm{Id})$ of the root system of $\mathfrak{h}$, we prove that there exists an involution τ_0 preserving $\mathfrak{h}$ such that $\tau_0|_{\mathfrak{h}} = \bar{\tau}$. Then each involution τ can be written as $\tau = \tau_0 \circ \mathrm{Ad}(\exp \sqrt{-1}h)$ or $\tau = \mathrm{Ad}(\exp \sqrt{-1}h)$ for some $\sqrt{-1}h \in \mathfrak{t}$ since $\tau|_{\mathfrak{t}}$ is an involution of the root system of $\mathfrak{h}$, and we obtain all τ by considering conjugations within automorphisms preserving $\mathfrak{h}$. For the case where $\dim Z(H) = 1$ and $\tau \circ \sigma \circ \tau^{-1} = \sigma^{-1}$, using graded Lie algebras, we classify all τ by an argument similar to that in [13].

According to [14], for 3-symmetric spaces $(G/H, \langle\ ,\ \rangle, \sigma)$ with $\dim Z(H) = 0$, each involution τ with $\tau \circ \sigma \circ \tau^{-1} = \sigma^{-1}$ preserving H is obtained from a grade-reversing Cartan involution of some graded Lie algebra of the third kind. In the case where $(G/H, \langle\ ,\ \rangle, \sigma)$ is 4-symmetric with $\dim Z(H) = 0$ and $\tau \circ \sigma \circ \tau^{-1} = \sigma^{-1}$, we can see that there exists τ which is not obtained from a grade-reversing Cartan involution of any graded Lie algebra of the fourth kind.

The organization of this paper is as follows:

In Section 2, we recall the notions of root systems and graded Lie algebras needed for the remaining part of this paper. Moreover we recall some results on automorphisms of order k $(k \leq 4)$.

In Section 3, we remark on some relation between involutions of 4-symmetric space $(G/H, \langle\ ,\ \rangle, \sigma)$ reserving H and root systems of the Lie algebra of G.

In Section 4, by using the results in Section 3, we describe the restrictions of involutions to the root systems for the case where the dimension of the center is zero.

In Section 5–8, we enumerate all involutions τ of compact 4-symmetric spaces such that $\tau(H) = H$ and the dimension of the center of H is zero, or H is a centralizer of a toral subgroup of G.

In Section 9, we describe some conjugations between involutions.

In Section 10, by making use of the results in Section 5–8 together with conjugations in Section 9, we give the classification theorem of the equivalence classes of involutions.

2. Preliminaries

2.1. Root systems. Let $\mathfrak{g}$ and $\mathfrak{t}$ be a compact semisimple Lie algebra and a maximal abelian subalgebra of $\mathfrak{g}$, respectively. We denote by $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ and $\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\alpha_1, \dots, \alpha_n\}$ the set of fundamental roots of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to a lexicographic order. For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}_{\mathbb{C}};\ [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{t}_{\mathbb{C}}\}. \tag{2.1}$$

Since the Killing form B of $\mathfrak{g}_{\mathbb{C}}$ is nondegenerate, we can define $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ ($\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$) by $\alpha(H) = B(H_\alpha, H)$ for any $H \in \mathfrak{t}_{\mathbb{C}}$. As in [6], we take the Weyl basis $\{E_\alpha \in \mathfrak{g}_\alpha;\ \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\}$ of $\mathfrak{g}_{\mathbb{C}}$ so that

$$\begin{aligned}
&[E_\alpha, E_{-\alpha}] = H_\alpha,\\
&[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad N_{\alpha,\beta} \in \mathbb{R},\\
&N_{\alpha,\beta} = -N_{-\alpha,-\beta},\\
&A_\alpha := E_\alpha - E_{-\alpha},\ B_\alpha := \sqrt{-1}(E_\alpha + E_{-\alpha}) \in \mathfrak{g}.
\end{aligned}$$

We denote by $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ the set of positive roots of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to the order. Then it follows that

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha), \quad \mathfrak{t} = \sum_{i=1}^{n} \mathbb{R}\sqrt{-1}H_{\alpha_i}. \tag{2.2}$$

For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, define a Lie subalgebra $\mathfrak{su}_\alpha(2)$ of $\mathfrak{g}$ by

$$\mathfrak{su}_\alpha(2) := \mathbb{R}\sqrt{-1}H_\alpha + \mathbb{R}A_\alpha + \mathbb{R}B_\alpha \cong \mathfrak{su}(2). \tag{2.3}$$

We denote by t_α the root reflection for $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then there exists an extension of t_α to an element of the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$, which is denoted by the same symbol as t_α. Since the root reflection of $\mathfrak{su}_\alpha(2)$ for α coincides with the restriction of t_α to $\mathbb{R}\sqrt{-1}H_\alpha$ and t_α is the identical transformation on the orthogonal complement of $\mathbb{R}\sqrt{-1}H_\alpha$ in $\mathfrak{t}$, the following lemma holds.

Lemma 2.1. *There exists an element $\phi \in \mathrm{Int}(\mathfrak{su}_\alpha(2))$ $(\subset \mathrm{Int}(\mathfrak{g}))$ such that $\phi|_{\mathfrak{t}} = t_\alpha|_{\mathfrak{t}}$.*

Define $K_j \in \mathfrak{t}_{\mathbb{C}}$ $(j = 1, \dots, n)$ by

$$\alpha_i(K_j) = \delta_{ij}, \quad i, j = 1, \dots, n,$$

and denote the highest root δ by

$$\delta := \sum_{j=1}^{n} m_j \alpha_j, \quad m_j \in \mathbb{Z}.$$

We set

$$\tau_H := \mathrm{Ad}(\exp \pi \sqrt{-1} H), \quad H \in \mathfrak{t}_{\mathbb{C}}.$$

Then from (2.1) we have

$$\tau_H(E_\alpha) = e^{\pi\sqrt{-1}\alpha(H)} E_\alpha, \quad \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}). \tag{2.4}$$

Assume that $\mathfrak{g}$ is simple. Then the following is known.

Lemma 2.2 ([10]). *Any inner automorphism of order* 2 *on* $\mathfrak{g}$ *is conjugate within* $\mathrm{Int}(\mathfrak{g})$ *to some* τ_{K_i} *with* $m_i = 1$ *or* 2.

If $h - h' = \sum_{i=1}^{n} a_i K_i$, $a_i \in 2\mathbb{Z}$ for $h, h' \in \mathfrak{t}_{\mathbb{C}}$, we say that h is congruent to h' modulo $2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and it is denoted by $h \equiv h' \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$. It follows from (2.4) that $\tau_h = \tau_{h'}$ if $h \equiv h' \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$.

REMARK 2.1. According to Lemma 2.2, for any inner automorphism τ_H of order 2 on $\mathfrak{g}$, there exists an inner automorphism ν of $\mathfrak{g}$ such that $\nu(H) \equiv K_i$ ($m_i = 1$ or 2) $\pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$.

We write $h \sim k$ if τ_h is conjugate to τ_k within the group of inner automorphism of $\mathfrak{g}$.

Lemma 2.3. (A_n) *If* $\mathfrak{g}$ *is of type* A_n, *then* $K_i \sim K_{n+1-i}$.
(D_n) *If* $\mathfrak{g}$ *is of type* D_n, *then* $K_i \sim K_{n-i}$ $(1 \leq i \leq [n/2])$. *In particular if n is odd, then* $K_{n-1} \sim K_n$.
(E_6) *If* $\mathfrak{g}$ *is of type* E_6, *then* $K_1 \sim K_6$, $K_2 \sim K_3 \sim K_5$.

Proof. (A_n): We identify $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with

$$\{e_i - e_j;\ 1 \leq i \neq j \leq n+1\}$$

(for example, see [6]), where $\{e_1, \dots, e_{n+1}\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$. From [2] there exists an element w of the Weyl group $W(\mathfrak{g}, \mathfrak{t})$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$ such that $w(e_j) = e_{n-j+2}$ $(1 \leq j \leq n+1)$. Set $\alpha_i = e_i - e_{i+1}$. Then we have

$$w(\alpha_i) = w(e_i - e_{i+1}) = e_{n-i+2} - e_{n-i+1} = -\alpha_{n-i+1}.$$

It is easy to see that $w^{-1}(K_i) = -K_{n+1-i} \equiv K_{n+1-i} \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})}$. Hence $\tau_{w^{-1}(K_i)} = w^{-1} \circ \tau_{K_i} \circ w = \tau_{K_{n+1-i}}$.

(D_n):

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm e_i \pm e_j;\ 1 \leq i \neq j \leq n\}.$$

Set

$$\alpha_i = e_i - e_{i+1}\ (1 \leq i \leq n-1), \quad \alpha_n = e_{n-1} + e_n.$$

Since there exists $w \in W(\mathfrak{g}, \mathfrak{t})$ such that $w(e_j) = e_{n-j+1}$ $(1 \leq j \leq n)$, we have

$$w(\alpha_i) = w(e_i - e_{i+1}) = e_{n-i+1} - e_{n-i} = -\alpha_{n-i}.$$

Hence we get $w^{-1}(K_i) = -K_{n-i} \equiv K_{n-i} \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})}$. In particular, if n is odd, then there exists a unique $\bar{w} \in W(\mathfrak{g}, \mathfrak{t})$ such that $\{\alpha_1, \ldots, \alpha_n\} \to \{-\alpha_1, \ldots, -\alpha_n\}$. If $\bar{w}(\alpha_i) = -\alpha_i$ for $1 \leq i \leq n$, then $\bar{w} = -\mathrm{Id}$, which is a contradiction (cf, [2]). Thus we get

$$\bar{w}(\alpha_i) = -\alpha_i\ (1 \leq i \leq n-2), \quad \bar{w}(\alpha_{n-1}) = -\alpha_n, \quad \bar{w}(\alpha_n) = -\alpha_{n-1}.$$

Hence we obtain $\bar{w}^{-1}(K_{n-1}) = -K_n \equiv K_n \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})}$.

(E_6): There exists a unique $w \in W(\mathfrak{g}, \mathfrak{t})$ such that $\{\alpha_1, \ldots, \alpha_6\} \to \{-\alpha_1, \ldots, -\alpha_6\}$. Similarly as in the proof of (D_n), we have

$$w(\alpha_1) = -\alpha_6, \quad w(\alpha_2) = -\alpha_2, \quad w(\alpha_3) = -\alpha_5, \quad w(\alpha_4) = -\alpha_4.$$

Hence we obtain $w^{-1}(K_1) = -K_6$ and $w(K_3) = -K_5$. On the other hand, it is easy to see that $t_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} \circ t_{\alpha_2+\alpha_4+\alpha_5}(K_2) = -K_5 + 2K_6 \equiv K_5 \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})}$. Thus we have $K_2 \sim K_5$. □

Let $(G/H, \langle\ ,\ \rangle, \sigma)$ be a compact Riemannian 4-symmetric space such that σ is inner. Then the following holds.

Lemma 2.4 ([7]). *σ is conjugate within* $\mathrm{Int}(\mathfrak{g})$ *to some* $\mathrm{Ad}(\exp(\pi/2)\sqrt{-1}h_a)$ *where either*

$h_0 = K_i$,	$m_i = 4$,
$h_1 = K_i$ *or* $K_j + K_k$,	$m_i = 3,\ m_j = m_k = 2$,
$h_2 = K_i + K_j$,	$m_i = 1,\ m_j = 2$,
$h_3 = K_i + K_j + K_k$,	$m_i = m_j = m_k = 1$,
$h_4 = K_i$,	$m_i = 1$,
$h_5 = K_i,\ K_j + K_k$ *or* $2K_p + K_q$,	$m_i = 2,\ m_j = m_k = m_p = m_q = 1$.

REMARK 2.2. (1) If σ is conjugate to $\tau_{(1/2)h_4}$, then a pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is symmetric. Indeed, for $\alpha = \sum_{r=1}^{n} k_r\alpha_r \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, we have $\alpha(h_4) = k_i$ and

$$\alpha(h_4) \equiv 0 \pmod 4 \Longleftrightarrow \alpha(h_4) \equiv 0 \pmod 2 \Longleftrightarrow k_i = 0$$

since $m_i = 1$. Therefore it follows that $\mathfrak{g}^{\tau_{(1/2)K_i}} = \mathfrak{g}^{\tau_{K_i}}$. Hence $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is a symmetric pair, because τ_{K_i} is an involution.

If σ is conjugate to $\tau_{(1/2)h_5}$, then a pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is 3-symmetric. Indeed, for example, if $h_5 = 2K_p + K_q$, then we have

$$\alpha(h_5) \equiv 0 \pmod 4 \Longleftrightarrow \alpha(K_p + K_q) \equiv 0 \pmod 3 \Longleftrightarrow k_p = k_q = 0,$$

for $\alpha = \sum_{r=1}^{n} k_r\alpha_r \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Therefore, we obtain $\mathfrak{g}^{\tau_{(1/2)h_5}} = \mathfrak{g}^{\tau_{(2/3)(K_p+K_q)}}$, and hence $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is a 3-symmetric pair because $\tau_{(2/3)(K_p+K_q)}$ is of order 3.
(2) Let $\mathfrak{z}$ be the center of $\mathfrak{h}$. If $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}h_a)$ $(a = 0, 1, 2, 3)$, then the dimension of $\mathfrak{z}$ is equal to a ([7]).

2.2. Graded Lie algebras. In this subsection we recall notions and some results on graded Lie algebras.

Let $\mathfrak{g}^*$ be a noncompact semisimple Lie algebra over $\mathbb{R}$. Let τ be a Cartan involution of $\mathfrak{g}^*$ and

$$\mathfrak{g}^* = \mathfrak{k} + p^*, \quad \tau|_{\mathfrak{k}} = \mathrm{Id}_{\mathfrak{k}}, \quad \tau|_{\mathfrak{p}^*} = -\mathrm{Id}_{\mathfrak{p}^*} \tag{2.5}$$

the Cartan decomposition of $\mathfrak{g}^*$ corresponding to τ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}^*$ and Δ the set of restricted roots of $\mathfrak{g}^*$ with respect to $\mathfrak{a}$. We denote by $\Pi = \{\lambda_1, \dots, \lambda_l\}$ the set of fundamental roots of Δ with respect to a lexicographic ordering of $\mathfrak{a}$. We call a collection of subsets $\{\Pi_0, \Pi_1, \dots, \Pi_n\}$ of Π a *partition* of Π if $\Pi_1 \neq \emptyset$, $\Pi_n \neq \emptyset$ and

$$\Pi = \Pi_0 \cup \Pi_1 \cup \cdots \cup \Pi_n \quad \text{(disjoint union)}.$$

Let Π and $\bar{\Pi}$ be fundamental root systems of noncompact semisimple Lie algebras $\mathfrak{g}^*$ and $\bar{\mathfrak{g}}^*$ respectively. Partitions $\{\Pi_0, \Pi_1, \dots, \Pi_m\}$ of Π and $\{\bar{\Pi}_0, \bar{\Pi}_1, \dots, \bar{\Pi}_n\}$ of $\bar{\Pi}$ are said to be *equivalent* if there exists an isomorphism ϕ from Dynkin diagram of Π to that of $\bar{\Pi}$ such that $m = n$ and $\phi(\Pi_i) = \bar{\Pi}_i$ $(i = 0, 1, \dots, n)$.

Take a gradation

$$\begin{aligned} &\mathfrak{g}^* = \mathfrak{g}^*_{-\nu} + \cdots + \mathfrak{g}^*_0 + \cdots + \mathfrak{g}^*_\nu, \\ &[\mathfrak{g}^*_p, \mathfrak{g}^*_q] \subset \mathfrak{g}^*_{p+q}, \ \tau(\mathfrak{g}^*_p) = \mathfrak{g}^*_{-p}, \quad -\nu \leq p,\ q \leq \nu, \end{aligned} \tag{2.6}$$

of ν-th kind on $\mathfrak{g}^*$ so that $\mathfrak{g}^*_1 \neq \{0\}$. We denote by Z the characteristic element of the gradation, i.e. Z is a unique element in $\mathfrak{p}^* \cap \mathfrak{g}_0^*$ such that

$$\mathfrak{g}_p^* = \{X \in \mathfrak{g}^*;\ [Z, X] = pX\}, \quad -\nu \leq p \leq \nu.$$

Let

$$\mathfrak{g}^* = \sum_{i=-\nu}^{\nu} \mathfrak{g}_i^*, \quad \overline{\mathfrak{g}}^* = \sum_{i=-\overline{\nu}}^{\overline{\nu}} \overline{\mathfrak{g}}_i^*$$

be two graded Lie algebras. These gradations are said to be *isomorphic* if $\nu = \overline{\nu}$ and there exists an isomorphism $\phi : \mathfrak{g}^* \to \overline{\mathfrak{g}}^*$ such that $\phi(\mathfrak{g}_i^*) = \overline{\mathfrak{g}}_i^*$ $(-\nu \leq i \leq \nu)$. Then the following holds.

Theorem 2.1 (Kaneyuki and Asano [9]). *Let $\mathfrak{g}^*$ be a noncompact semisimple Lie algebra over $\mathbb{R}$ and Π a fundamental root system of $\mathfrak{g}^*$. Then there exists a bijection between the set of equivalent classes of partitions of Π and set of isomorphic classes of gradations on $\mathfrak{g}^*$.*

The bijection in the theorem is constructed as follows: Let $\{\Pi_0, \Pi_1, \ldots, \Pi_n\}$ be a partition of Π. Define $h_\Pi : \Delta \to \mathbb{Z}$ by

$$h_\Pi(\lambda) := \sum_{\lambda_i \in \Pi_1} m_i + 2 \sum_{\lambda_j \in \Pi_2} m_j + \cdots + n \sum_{\lambda_k \in \Pi_n} m_k, \quad \lambda = \sum_{i=1}^{l} m_i \lambda_i \in \Delta.$$

Then there is a unique Z in $\mathfrak{a}$ such that $\lambda(Z) = h_\Pi(\lambda)$ for all $\lambda \in \Delta$. For a partition $\{\Pi_0, \Pi_1, \ldots, \Pi_n\}$ we obtain a gradation $\mathfrak{g}^* = \sum_{i=-\nu}^{\nu} \mathfrak{g}_i^*$ whose characteristic element equals Z. This correspondence induces a bijection mentioned in the theorem.

Define $h_i \in \mathfrak{a}$ $(i = 1, 2, \ldots, l)$ by

$$\lambda_j(h_i) = \delta_{ij}.$$

Let $\mathfrak{t}^*$ be a Cartan subalgebra of $\mathfrak{g}^*$ such that $\mathfrak{a} \subset \mathfrak{t}^*$. Take compatible orderings on $\mathfrak{t}^*$ and $\mathfrak{a}$. We clarify the relation between K_i and h_j.

Lemma 2.5. *Let λ_i be any root in Π.*

(1) *If there exists a unique $\alpha_j \in \Pi(\mathfrak{g}_{\mathbb{C}}^*, \mathfrak{t}_{\mathbb{C}}^*)$ such that $\alpha_j|_{\mathfrak{a}} = \lambda_i$, then $h_i = K_j$.*

(2) *If there exist two fundamental roots $\alpha_j, \alpha_k \in \Pi(\mathfrak{g}_{\mathbb{C}}^*, \mathfrak{t}_{\mathbb{C}}^*)$ such that $\alpha_j|_{\mathfrak{a}} = \alpha_k|_{\mathfrak{a}} = \lambda_i$, then $h_i = K_j + K_k$.*

Proof. (1): Considering the classification of the Satake diagrams, for $\alpha_p \in \Pi(\mathfrak{g}_{\mathbb{C}}^*, \mathfrak{t}_{\mathbb{C}}^*)$, $p \neq j$, it follows that $\alpha_p|_{\mathfrak{a}} = 0$ or $\alpha_p|_{\mathfrak{a}} = \lambda_q$ for some q $(q \neq i)$. Thus we have

$$\alpha_p(h_i) = \alpha_p|_{\mathfrak{a}}(h_i) = 0, \quad \alpha_j(h_i) = \lambda_i(h_i) = 1,$$

which implies $h_i = K_j$.

(2): Similarly as above, for $\alpha_p \in \Pi(\mathfrak{g}_{\mathbb{C}}^*, \mathfrak{t}_{\mathbb{C}}^*)$, $p \neq j, k$, it follows that $\alpha_p|_{\mathfrak{a}} = 0$ or $\alpha_p|_{\mathfrak{a}} = \lambda_q$ for some q $(q \neq i)$. Therefore

$$\alpha_p(h_i) = \alpha_p|_{\mathfrak{a}}(h_i) = 0, \ \alpha_m(h_i) = \alpha_m|_{\mathfrak{a}}(h_i) = \lambda_i(h_i) = 1, \quad m = j, k,$$

which implies $h_i = K_j + K_k$. □

3. Riemannian 4-symmetric spaces

In this section we use the same notation as in Section 2. Let $(G/H, \langle\ ,\ \rangle, \sigma)$ be a Riemannian 4-symmetric space with an inner automorphism σ of order 4. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Note that $\mathfrak{h}$ coincides with the set $\mathfrak{g}^{\sigma}$ of fixed points of σ. Choose a subspace $\mathfrak{m}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is an $\mathrm{Ad}(H)$- and σ-invariant decomposition. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{h}$, and $\mathfrak{z}$ the center of $\mathfrak{h}$.

Suppose that $\mathfrak{g}$ is a compact simple Lie algebra. Let $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ be the set of automorphisms of $\mathfrak{g}$ preserving $\mathfrak{h}$.

Lemma 3.1. *Assume* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$, $m_i = 3$ *or* 4, *where* $\delta = \sum_{j=1}^{n} m_j\alpha_j$ *is the highest root of* $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *as in* Section 2. *Then for each* $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$, *we have* $\mu \circ \sigma \circ \mu^{-1} = \sigma$ *or* σ^{-1}.

Proof. Since $\mu(\mathfrak{h}) = \mathfrak{h}$, we obtain $\mathfrak{g}^{\tilde{\sigma}} = \mathfrak{h}$, where $\tilde{\sigma} := \mu \circ \sigma \circ \mu^{-1}$. In particular, we have $\tilde{\sigma}|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$. Therefore, it follows from Proposition 5.3 of Chapter IX of [6] that there is $\sqrt{-1}Z \in \mathfrak{t}$ such that

$$\tilde{\sigma} = \mathrm{Ad}\left(\exp \frac{\pi}{2}\sqrt{-1}Z\right). \tag{3.1}$$

Since $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ with $m_i = 3$ or 4, we obtain $E_{\alpha_j} \in \mathfrak{h}_{\mathbb{C}}$ $(j \neq i)$ and $E_{\alpha_i} \notin \mathfrak{h}_{\mathbb{C}}$. Moreover, since $\mathfrak{g}^{\sigma} = \mathfrak{g}^{\tilde{\sigma}} = \mathfrak{h}$, it follows from (3.1) that

$$\tilde{\sigma}(E_{\alpha_j}) = E_{\alpha_j}, \quad \tilde{\sigma}(E_{\alpha_i}) = cE_{\alpha_i}, \tag{3.2}$$

for some $c \in \mathbb{C}$ with $|c| = 1$. Then $c^4 = 1$ and $c^2 \neq 1$, because $\tilde{\sigma}^4 = \mathrm{Id}$ and $\tilde{\sigma}^2 \neq \mathrm{Id}$. From (3.2), we can see that if $c = \sqrt{-1}$, then $\tilde{\sigma} = \sigma$, and if $c = -\sqrt{-1}$, then $\tilde{\sigma} = \sigma^{-1}$. □

REMARK 3.1. Lemma 3.1 dose not hold in general. If σ is conjugate to $\mathrm{Ad}(\exp(\pi/2)\sqrt{-1}(K_i + K_j))$ ($m_i = m_j = 2$), then Lemma 3.1 holds. However in other cases, Lemma 3.1 dose not hold.

REMARK 3.2. If σ is an automorphism of order 2 or 3, then by an argument similar to the proof of Lemma 3.1, it follows that $\mu \circ \sigma \circ \mu^{-1} = \sigma$ for any $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$.

Lemma 3.2. *Suppose that* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ *with* $m_i = 3$. *Let* τ *be an involutive automorphism of* $\mathfrak{g}$ *such that* $\tau(\mathfrak{h}) = \mathfrak{h}$. *Then*
(i) $\tau \circ \sigma = \sigma \circ \tau$ *if and only if the coefficient of* α_i *in* $\tau(\delta)$ *is equal to* 3.
(ii) $\tau \circ \sigma = \sigma^{-1} \circ \tau$ *if and only if the coefficient of* α_i *in* $\tau(\delta)$ *is equal to* -3.

Proof. It is easy to see that $\mathfrak{z} = \mathbb{R}\sqrt{-1}K_i$ for some i with $m_i = 3$. Since $\tau(\mathfrak{h}) = \mathfrak{h}$, we have $\tau(\sqrt{-1}K_i) = \pm\sqrt{-1}K_i$, and therefore

$$\tau \circ \sigma \circ \tau^{-1} = \mathrm{Ad}\left(\exp \frac{\pi}{2}\tau(\sqrt{-1}K_i)\right) = \begin{cases} \sigma & \text{if } \tau(\sqrt{-1}K_i) = \sqrt{-1}K_i, \\ \sigma^{-1} & \text{if } \tau(\sqrt{-1}K_i) = -\sqrt{-1}K_i, \end{cases}$$

and

$$\tau(\delta)(K_i) = \delta(\tau(K_i)) = \delta(\pm K_i) = \begin{cases} 3 & \text{if } \tau(\sqrt{-1}K_i) = \sqrt{-1}K_i, \\ -3 & \text{if } \tau(\sqrt{-1}K_i) = -\sqrt{-1}K_i. \end{cases}$$

This completes the proof of the lemma. □

Lemma 3.3. *Suppose that* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ *with* $m_i = 3$ *or* 4.
(i) *Let* τ_1, τ_2 *be involutive automorphisms of* $\mathfrak{g}$ *such that* $\tau_i(\mathfrak{h}) = \mathfrak{h}$, ($i = 1, 2$). *If there exists* $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *such that* $\mu \circ \tau_1 \circ \mu^{-1} = \tau_2$. *Then*

$$\mathfrak{g}^{\tau_1} \cong \mathfrak{g}^{\tau_2}, \quad \mathfrak{h} \cap \mathfrak{g}^{\tau_1} \cong \mathfrak{h} \cap \mathfrak{g}^{\tau_2}. \tag{3.3}$$

(ii) *Put* $\tau' := \mu \circ \tau \circ \mu^{-1}$, $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$. *If* $\tau \circ \sigma = \sigma^{\pm 1} \circ \tau$, *then* $\tau' \circ \sigma = \sigma^{\pm 1} \circ \tau'$, *respectively.*

Proof. (i) is trivial.
(ii) We have

$$\begin{aligned} \tau \circ \sigma = \sigma^{\pm 1} \circ \tau &\iff \mu \circ \tau \circ \sigma \circ \mu^{-1} = \mu \circ \sigma^{\pm 1} \circ \tau \circ \mu^{-1} \\ &\iff \tau' \circ \mu \circ \sigma \circ \mu^{-1} = \mu \circ \sigma^{\pm 1} \circ \mu^{-1} \circ \tau' \end{aligned}$$

Hence, it follows from Lemma 3.1 that if $\tau \circ \sigma = \sigma^{\pm 1} \circ \tau$, then $\tau' \circ \sigma = \sigma^{\pm 1} \circ \tau'$. □

In the remaining part of this paper, we suppose that $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ for some $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_i = 3$ or 4. If $m_i = 3$, the Dynkin diagram of $\mathfrak{h}$ is isomorphic to the extended Dynkin diagram of $\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ except α_i and α_0, and if $m_i = 4$, it is isomorphic to that of $\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ except α_i (cf. Theorem 5.15 of Chapter X of [6]). We denote by $\Pi(\mathfrak{h})$ the fundamental root system of $\mathfrak{h}$ corresponding to the Dynkin diagram of $\mathfrak{h}$.

Lemma 3.4. *For any involutive automorphism τ of $\mathfrak{g}$ satisfying $\tau(\mathfrak{h}) = \mathfrak{h}$, there exists $\mu \in \mathrm{Int}(\mathfrak{h})$ such that $\mu \circ \tau \circ \mu^{-1}(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h})$.*

Proof. Put $\tilde{\tau} := \tau|_{\mathfrak{h}}$. Then $\tilde{\tau}$ is an involution of $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}_s$, where $\mathfrak{h}_s := [\mathfrak{h}, \mathfrak{h}]$. It is obvious that $\tilde{\tau}(\mathfrak{z}) = \mathfrak{z}$ and $\tilde{\tau}(\mathfrak{h}_s) = \mathfrak{h}_s$. Decompose $\mathfrak{h}_s$ into $\mathfrak{h}_s = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_m$ where $\mathfrak{h}_1, \ldots, \mathfrak{h}_m$ are simple ideals. From the classification of compact 4-symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ ([7]), it follows that (i) $\mathfrak{h}_i \not\cong \mathfrak{h}_j$ for any $i, j \in \{1, \ldots, m\}$ $(i \neq j)$, or (ii) there exists only one pair (p, q) such that $\mathfrak{h}_p \cong \mathfrak{h}_q$.

CASE (i). Since $\tau(\mathfrak{h}_i)$ $(1 \leq i \leq m)$ is a simple ideal of $\mathfrak{h}_s$ and $\mathfrak{h}_i \not\cong \mathfrak{h}_j$ $(i \neq j)$, it follows that $\tilde{\tau}(\mathfrak{h}_i) = \mathfrak{h}_i$ $(1 \leq i \leq m)$. Therefore we have a direct sum decomposition $\mathfrak{h}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. Let $\mathfrak{a}_i$ be a maximal abelian subspace of $\mathfrak{p}_i$ and $\mathfrak{t}_i$ be a maximal abelian subalgebra of $\mathfrak{h}_i$ containing $\mathfrak{a}_i$. We take a fundamental root system $\Pi_i = \{\lambda_1, \ldots, \lambda_{n_i}\}$ for the set of nonzero roots with respect to $(\mathfrak{h}_{i\mathbb{C}}, \mathfrak{t}_{i\mathbb{C}})$. From Theorem 5.15 of [6], there exists $\mu_i \in \mathrm{Aut}(\mathfrak{h}_i)$ such that $\mu_i \circ \tau|_{\mathfrak{h}_i} \circ \mu_i^{-1}$ is an automorphism of Π_i of order 1 or 2. Hence we have

$$\tau(\mu_i^{-1}(\mathfrak{t}_i)) = \mu_i^{-1}(\mathfrak{t}_i), \quad \tau(\mu_i^{-1}(\Pi_i)) = \mu_i^{-1}(\Pi_i). \tag{3.4}$$

Set $\tilde{\mathfrak{t}} := \mu_1^{-1}(\mathfrak{t}_1) \oplus \cdots \oplus \mu_m^{-1}(\mathfrak{t}_m) \oplus \mathfrak{z}$ and $\tilde{\Pi} := \mu_1^{-1}(\Pi_1) \cup \cdots \cup \mu_m^{-1}(\Pi_m)$. Then by (3.4), we have $\tau(\tilde{\mathfrak{t}}) = \tilde{\mathfrak{t}}$ and $\tau(\tilde{\Pi}) = \tilde{\Pi}$. Since there exist $\mu \in \mathrm{Int}(\mathfrak{h})$ and $w \in W(\mathfrak{h}, \mathfrak{t})$ such that $\mu(\tilde{\mathfrak{t}}) = \mathfrak{t}$ and $w(\mu(\tilde{\Pi})) = \Pi(\mathfrak{h})$, we obtain

$$(w\mu) \circ \tau \circ (w\mu)^{-1}(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h}),$$

which completes the proof of the lemma for the case (i).

CASE (ii). If $\tau(\mathfrak{h}_i) = \mathfrak{h}_i$ for $i = 1, \ldots, m$, then by the same argument as in the case (i), we can prove the claim. Hence we assume that and $\tau(\mathfrak{h}_p) = \mathfrak{h}_q$ and $\tau(\mathfrak{h}_i) = \mathfrak{h}_i$ for $i \neq p, q$. Define isomorphisms $\tau_1 \colon \mathfrak{h}_q \to \mathfrak{h}_p$ and $\tau_2 \colon \mathfrak{h}_p \to \mathfrak{h}_q$ by

$$\tau(X, Y) = (\tau_1(Y), \tau_2(X)), \quad X \in \mathfrak{h}_p, \ Y \in \mathfrak{h}_q.$$

Since τ is an involution, it follows that $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \mathrm{Id}$. Hence we have $\tau(X, Y) = (\tau_1(Y), \tau_1^{-1}(X))$.

Put $\mathfrak{b} := \mathfrak{h}_p$ and define an isomorphism $\nu \colon \mathfrak{h}_p \oplus \mathfrak{h}_q \to \mathfrak{b} \oplus \mathfrak{b}$ by

$$\nu(X, Y) := (X, \tau_1(Y)).$$

Then it is easy to see that $\nu\circ\tau\circ\nu^{-1}(X,Y)=(Y,X)$. Therefore, considering a symmetric pair $(\mathfrak{b}\oplus\mathfrak{b},\,\Delta\mathfrak{b})$ $(\Delta\mathfrak{b}:=\{(X,\,X);\ X\in\mathfrak{b}\})$, we can see that there exist a fundamental root system of $\mathfrak{h}_p\oplus\mathfrak{h}_q$ preserved by $\tau|_{\mathfrak{h}_p\oplus\mathfrak{h}_q}$. Hence, by an argument similar to (i), there exists $\nu\in\mathrm{Int}(\mathfrak{h})$ such that $\nu\circ\tau\circ\nu^{-1}(\Pi(\mathfrak{h}))=\Pi(\mathfrak{h})$. This completes the proof of the lemma for the case (ii). □

In the following sections, we shall classify the equivalence classes of involutive automorphisms τ within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ of $\mathfrak{g}$ such that $\tau(\mathfrak{h})=\mathfrak{h}$. From Remark 2.2 and Lemma 3.1 we have the following four type:

$$\dim\mathfrak{z}=0,\quad \tau\circ\sigma=\sigma^{\pm1}\circ\tau,$$
$$\dim\mathfrak{z}=1,\quad \tau\circ\sigma=\sigma^{\pm1}\circ\tau.$$

4. The case where $\dim\mathfrak{z}=0$

In the remaining part of this paper we use the same notation as in Section 2 and Section 3. Let $(G/H,\langle\ ,\ \rangle,\sigma)$ be a Riemannian 4-symmetric space such that σ is inner and $\dim\mathfrak{z}=0$. From Lemma 2.4 together with Remark 2.2 we may suppose that

$$\sigma=\mathrm{Ad}\left(\exp\frac{\pi}{2}\sqrt{-1}K_i\right)\quad\text{for some } i \text{ with the property } m_i=4.$$

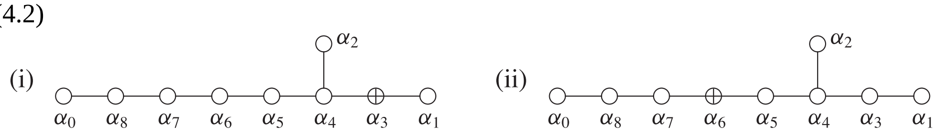
According to Section 3 and Jiménez [7], 4-symmetric pairs $(\mathfrak{g},\,\mathfrak{h})$ satisfying the condition $\dim\mathfrak{z}=0$ are given by

$$(4.1)\qquad \begin{aligned}&(\mathfrak{e}_7,\,\mathfrak{so}(6)\oplus\mathfrak{so}(6)\oplus\mathfrak{su}(2)),\quad(\mathfrak{e}_8,\,\mathfrak{su}(8)\oplus\mathfrak{su}(2)),\\&(\mathfrak{e}_8,\,\mathfrak{so}(10)\oplus\mathfrak{so}(6)),\quad(\mathfrak{f}_4,\,\mathfrak{so}(6)\oplus\mathfrak{so}(3)).\end{aligned}$$

Let τ be an involution of $\mathfrak{g}$ preserving $\mathfrak{h}$. By Lemma 3.4, we may assume $\tau(\mathfrak{t})=\mathfrak{t}$ and $\tau(\Pi(\mathfrak{h}))=\Pi(\mathfrak{h})$. If $\tau|_{\mathfrak{t}}=\mathrm{Id}_{\mathfrak{t}}$, then there exists $\sqrt{-1}H\in\mathfrak{t}$ such that $\tau=\mathrm{Ad}(\exp\pi\sqrt{-1}H)$ and $\tau\circ\sigma=\sigma\circ\tau$.

Now, we assume $\tau|_{\mathfrak{t}}\neq\mathrm{Id}_{\mathfrak{t}}$. Suppose that $\mathfrak{g}$ is of type $\mathfrak{e}_8$. From Section 3, the Dynkin diagram of $\mathfrak{h}$ coincides with the extended Dynkin diagram of $\mathfrak{e}_8$ except $\oplus$ as follows:

(4.2)



We denote $\sum_{i=1}^{8}k_i\alpha_i$ by

$$\begin{pmatrix} & & & & k_2 & & \\ k_8 & k_7 & k_6 & k_5 & k_4 & k_3 & k_1\end{pmatrix}.$$

In the above case (i), since $\tau|_{\mathfrak{t}} \neq \mathrm{Id}_{\mathfrak{t}}$ and $\tau(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h})$, the possibility of $\tau|_{\mathfrak{t}}$ is as follows:

$$\tau(\alpha_1) = \alpha_1, \quad \tau(\alpha_2) = \alpha_0, \quad \tau(\alpha_4) = \alpha_8, \quad \tau(\alpha_5) = \alpha_7, \quad \tau(\alpha_6) = \alpha_6.$$

Then we get

$$\alpha_2 = \tau(\alpha_0) = -4\tau(\alpha_3) - 3\tau(\alpha_2) - \begin{pmatrix} & & & & 0 & & \\ 6 & 5 & 4 & 3 & 2 & 0 & 2 \end{pmatrix},$$

and hence

$$\tau(\alpha_3) = \begin{pmatrix} & & & & 2 & & \\ 0 & 1 & 2 & 3 & 4 & 3 & 1 \end{pmatrix} \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}).$$

By a similar argument as above, we obtain the following proposition.

Proposition 4.1. *Suppose that* $\dim \mathfrak{z} = 0$. *Let* τ *be an involution of* $\mathfrak{g}$ *such that* $\tau(\mathfrak{h}) = \mathfrak{h}$ *and* $\tau(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h})$. *Then all the possibilities of* $\tau|_{\mathfrak{t}}$ *such that* $\tau|_{\mathfrak{t}} \neq \mathrm{Id}_{\mathfrak{t}}$ *are given by* Table 1.

For Type IV in Table 1, it is easy to see $\tau(K_3) = -4K_2 + 3K_3 \equiv -K_3 \pmod 4$. Hence we have $\tau \circ \sigma = \sigma^{-1} \circ \tau$. Similarly, for Type I we get $\tau \circ \sigma = \sigma \circ \tau$ and for the other types, we have $\tau \circ \sigma = \sigma^{-1} \circ \tau$.

Finally, in order to compute the dimension of $\mathfrak{g}^{\tau}$, we prove the following Lemma.

Lemma 4.1. *Let* $\mathfrak{t}_+$ *be the* (+1)*-eigenspace of* $\tau|_{\mathfrak{t}}$. *Then*

$$\begin{aligned} \dim \mathfrak{g}^{\tau} = \dim \mathfrak{t}_+ + \#\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) + 2\#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(E_\alpha) = E_\alpha\} \\ - \#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(\alpha) = \alpha\}. \end{aligned}$$

Table 1. The possibilities of $\tau|_{\mathfrak{t}}$ such that $\tau|_{\mathfrak{t}} \neq \mathrm{Id}$ ($\sigma = \tau_{(1/2)H}$)).

Type	$\mathfrak{g}$	H	$\tau\vert_{\mathfrak{t}}$
I	$\mathfrak{e}_7$	K_4	$\alpha_1 \mapsto \alpha_6,\ \alpha_2 \mapsto \alpha_2,\ \alpha_3 \mapsto \alpha_5,\ \alpha_4 \mapsto \alpha_4,\ \alpha_7 \mapsto \alpha_0$
II	$\mathfrak{e}_7$	K_4	$\alpha_1 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_2,\ \alpha_3 \mapsto \alpha_0,\ \alpha_5 \mapsto \alpha_7,\ \alpha_6 \mapsto \alpha_6$ $\alpha_4 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
III	$\mathfrak{e}_7$	K_4	$\alpha_1 \mapsto \alpha_6,\ \alpha_2 \mapsto \alpha_2,\ \alpha_3 \mapsto \alpha_7,\ \alpha_5 \mapsto \alpha_0$ $\alpha_4 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
IV	$\mathfrak{e}_8$	K_3	$\alpha_1 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_0,\ \alpha_4 \mapsto \alpha_8,\ \alpha_5 \mapsto \alpha_7,\ \alpha_6 \mapsto \alpha_6$ $\alpha_3 \mapsto \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
V	$\mathfrak{e}_8$	K_6	$\alpha_1 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_5,\ \alpha_3 \mapsto \alpha_3,\ \alpha_4 \mapsto \alpha_4,\ \alpha_7 \mapsto \alpha_0,\ \alpha_8 \mapsto \alpha_8$ $\alpha_6 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$
VI	$\mathfrak{f}_4$	K_3	$\alpha_1 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_0,\ \alpha_4 \mapsto \alpha_4$ $\alpha_3 \mapsto \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$

Proof. If $\tau(\alpha) = \beta$, $(\beta \neq \pm\alpha)$, we can put $\tau(E_\alpha) = cE_\beta$ for some c. Since τ is involutive and $\tau(H_\alpha) = H_\beta$, it is easy to see that $E_\alpha + cE_\beta$ and $E_{-\alpha} + c^{-1}E_{-\beta}$ are (+1)-eigenvectors of τ. If $\tau(\alpha) = \alpha$, we get $\tau(E_\alpha) = E_\alpha$ or $\tau(E_\alpha) = -E_\alpha$. Furthermore, if $\tau(\alpha) = -\alpha$, we can put $\tau(E_\alpha) = cE_{-\alpha}$ for some c. Then we have $\tau(E_\alpha \pm cE_{-\alpha}) = \pm(E_\alpha \pm cE_{-\alpha})$. Therefore we obtain

$$\begin{aligned}\dim \mathfrak{g}^\tau &= \dim \mathfrak{t}_+ + \#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(\alpha) \neq \pm\alpha\} \\ &\quad + 2\#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(E_\alpha) = E_\alpha\} + \#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(\alpha) = -\alpha\} \\ &= \dim \mathfrak{t}_+ + \#\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) + 2\#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(E_\alpha) = E_\alpha\} \\ &\quad - \#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(\alpha) = \alpha\}. \qquad \square\end{aligned}$$

5. The case where $\dim \mathfrak{z} = 0$ and $\tau \circ \sigma = \sigma^{-1} \circ \tau$

We consider the cases of Type II, III, IV, V and VI in Table 1. First we construct τ by using graded Lie algebras. Let $\mathfrak{g}^*$ be a normal real form of a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{t}^*$ be a Cartan subalgebra of $\mathfrak{g}^*$. Then we have a Cartan decomposition $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}^*$ with

$$\mathfrak{k} := \sum_{\alpha \in \Delta^+(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}})} \mathbb{R}A_\alpha, \quad \mathfrak{p}^* := \mathfrak{t}^* + \sum_{\alpha \in \Delta^+(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}})} \mathbb{R}\sqrt{-1}B_\alpha. \tag{5.1}$$

We take a gradation $\mathfrak{g}^* = \sum_{p=-4}^{4} \mathfrak{g}^*_p$ of the fourth kind on $\mathfrak{g}^*$ corresponding to a partition

$$\Pi = \Pi_0 \cup \Pi_1, \quad \Pi_1 = \{\alpha_i\},\ m_i = 4.$$

Then the characteristic element of the gradation coincides with K_i.

Let τ^* be the Cartan involution defined by (2.5). Put $\sigma := \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$. Then σ is an automorphism of order 4 on the compact dual $\mathfrak{g} := \mathfrak{k} + \sqrt{-1}\mathfrak{p}^*$ of $\mathfrak{g}^*$. Since $\tau^*(K_i) = -K_i$, it is obvious that

$$\tau^* \circ \sigma \circ (\tau^*)^{-1} = \sigma^{-1}. \tag{5.2}$$

By Lemma 3.4 and Proposition 4.1, τ^* is conjugate within $\mathrm{Int}(\mathfrak{h})$ to an involutive automorphism τ^Π of Type II, III, IV, V or VI in Table 1, that is, there exists $\mu \in \mathrm{Int}(\mathfrak{h})$ such that $\tau^\Pi|_{\mathfrak{t}} = (\mu \circ \tau^* \circ \mu^{-1})|_{\mathfrak{t}}$. Note that $\dim \mathfrak{z} = 0$ by Theorem 5.15 of Chapter X of [6], and it follows from (5.1) that

$$\mathfrak{h} \cap \mathfrak{k} = \sum_{\substack{\alpha \in \Delta^+(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}}) \\ \alpha(K_i) \equiv 0 \pmod 4}} \mathbb{R}A_\alpha. \tag{5.3}$$

Now we prove the following Lemma.

Lemma 5.1. *Let* μ *be in* $\mathrm{Int}(\mathfrak{h})$. *Then* $\mu \circ \tau^* \circ \mu^{-1}$ *is conjugate within* $\mathrm{Int}(\mathfrak{h})$ *to* $\mu \circ \tau^* \circ \mu^{-1} \circ \sigma$.

Proof. Put $\nu := \tau_{(-1/4)K_i}$. Then we have $\nu \circ \tau^* \circ \nu^{-1} = \tau^* \circ \nu^{-1} \circ \nu^{-1} = \tau^* \circ \sigma$. Since $\nu \in \mathrm{Int}(\mathfrak{h})$, it follows that

$$\mu \circ \tau^* \circ \mu^{-1} \circ \sigma = \mu \circ \tau^* \circ \sigma \circ \mu^{-1} = \mu \circ \nu \circ \tau^* \circ \nu^{-1} \circ \mu^{-1} = \nu \circ \mu \circ \tau^* \circ \mu^{-1} \circ \nu^{-1},$$

and hence $\mu \circ \tau^* \circ \mu^{-1}$ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to $\mu \circ \tau^* \circ \mu^{-1} \circ \sigma$. □

In the remaining part of this section, we shall determine all involutions for each type. Furthermore for each involution τ, we shall determine $\mathfrak{h} \cap \mathfrak{g}^\tau$ and $\mathfrak{g}^\tau$.

Let $\tau_1^\Pi, \tau_2^\Pi, \tau_3^\Pi, \tau_4^\Pi$ be the involutive automorphisms which conjugate within $\mathrm{Int}(\mathfrak{h})$ to the Cartan involutions τ^* with respect to Type IV, V, III, VI, respectively. We denote by τ any involution of each types.

Type IV: Now, we investigate involutions of Type IV in Table 1. Since $\mathfrak{g}^*$ is a normal real form and of type $\mathfrak{e}_8$, the pair $(\mathfrak{g}^*, \mathfrak{k})$ is given by $(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$. Note that $\dim \mathfrak{k} = 120$. Set $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_3)$. From (5.1) and (5.3), considering the number of roots $\alpha \in \Delta^+(\mathfrak{g}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}})$ such that $\alpha(K_3) = 0$ or 4 (for example see Freudenthal and Vries [3]), we get $\dim(\mathfrak{h} \cap \mathfrak{k}) = 29$. Then it follows from (5.2) and Proposition 4.1 that τ_1^Π is of Type IV in Table 1.

Let $\mathfrak{t}_\pm$ be the (± 1)-eigenspaces of $\tau_1^\Pi|_\mathfrak{t}$, respectively. Since $\alpha_i(\tau_1^\Pi(K_j)) = \tau_1^\Pi(\alpha_i)(K_j)$, we have

$$\mathfrak{t}_+ = \mathrm{span}\{2K_1 - K_2,\ 2K_1 - K_3,\ 2K_1 + K_6,\ 4K_1 - K_5 - K_7,\ 4K_1 + K_4 + K_8\},$$
$$\mathfrak{t}_- = \mathrm{span}\{2K_2 - K_3,\ 2K_2 - K_4 + K_8,\ K_2 - K_5 + K_7\}.$$

For τ_{h_-} $(h_- \in \mathfrak{t}_-)$, we have $\tau_1^\Pi \circ \tau_{h_-} \circ \tau_1^\Pi = \tau_{\tau_1^\Pi(h_-)} = \tau_{-h_-}$. Thus we get

$$(\tau_{h_-})^{-1} \circ \tau_1^\Pi \circ \tau_{h_-} = \tau_1^\Pi \circ \tau_{2h_-}. \tag{5.4}$$

Then using $h_- := t(K_3 - K_4 + K_8) \in \mathfrak{t}_-$, we may assume $\tau_1^\Pi(E_{\alpha_4}) = E_{\alpha_8}$. Indeed, if $\tau_1^\Pi(E_{\alpha_4}) = b_4 E_{\alpha_8}$ $(b_4 \in \mathbb{C},\ |b_4| = 1)$, then it follows from (5.4) that

$$(\tau_{h_-})^{-1} \circ \tau_1^\Pi \circ \tau_{h_-}(E_{\alpha_4}) = b_4 e^{-2t\pi\sqrt{-1}} E_{\alpha_8},$$

Taking t so that $b_4 = e^{2t\pi\sqrt{-1}}$, we may assume $\tau_1^\Pi(E_{\alpha_4}) = E_{\alpha_8}$. Similarly, using $h_- = t(2K_2 - K_3)$ or $h_- = 2(K_2 - K_5 + K_7) - (2K_2 - K_3)$, we may assume $\tau_1^\Pi(E_{\alpha_2}) = E_{\alpha_0}$ and $\tau_1^\Pi(E_{\alpha_5}) = E_{\alpha_7}$.

On the other hand, for any involution τ of Type IV, the number of the subsets $\{\alpha, \beta\}$ such that $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\tau(\alpha) = \beta$, $\alpha \neq \pm\beta$ and $\alpha(K_3) \equiv 0 \pmod 4$ is 12. Since $\dim \mathfrak{t}_+ = 5$, by an argument similar to the proof of Lemma 4.1 we obtain

$$\dim(\mathfrak{h} \cap \mathfrak{g}^\tau) \geq 5 + (12 \times 2) = 29.$$

Because $\dim(\mathfrak{h} \cap \mathfrak{g}^{\tau_1^{\Pi}}) = \dim(\mathfrak{h} \cap \mathfrak{k}) = 29$, we obtain

$$\begin{aligned} &\tau_1^{\Pi}(E_{\alpha_1}) = -E_{\alpha_1}, \quad \tau_1^{\Pi}(E_{\alpha_2}) = E_{\alpha_0}, \quad \tau_1^{\Pi}(E_{\alpha_3}) = c_1 E_{\beta_1}, \\ &\tau_1^{\Pi}(E_{\alpha_4}) = E_{\alpha_8}, \quad \tau_1^{\Pi}(E_{\alpha_5}) = E_{\alpha_7}, \quad \tau_1^{\Pi}(E_{\alpha_6}) = -E_{\alpha_6}, \end{aligned} \tag{5.5}$$

where $\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ (see Table 1) and $c_1 \in \mathbb{C}$ with $|c_1| = 1$ (cf. Corollary 5.2 of Chapter IX of [6]).

REMARK 5.1. Except for conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$, we can determine the constant c_1 uniquely. Indeed, from the proof of Theorem 5.1 of Chapter IX of [6], there exists $\mu \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\begin{aligned} &\mu(E_{\alpha_1}) = E_{\alpha_1}, \quad \mu(E_{\alpha_2}) = E_{\alpha_0}, \quad \mu(E_{\alpha_3}) = E_{\beta_1}, \quad \mu(E_{\alpha_4}) = E_{\alpha_8}, \\ &\mu(E_{\alpha_5}) = E_{\alpha_7}, \quad \mu(E_{\alpha_6}) = E_{\alpha_6}, \quad \mu(E_{\alpha_7}) = E_{\alpha_5}, \quad \mu(E_{\alpha_8}) = E_{\alpha_4}, \\ &\mu(E_{\alpha_0}) = \epsilon_0 E_{\alpha_2}, \quad \mu(E_{\beta_1}) = \epsilon_{\beta_1} E_{\alpha_3}, \end{aligned} \tag{5.6}$$

where $\epsilon_0 = \pm 1$ and $\epsilon_{\beta_1} = \pm 1$. Note that ϵ_0 and ϵ_{β_1} are uniquely determined since $E_{\pm\alpha_i}$ $(1 \le i \le 8)$ generate $\mathfrak{g}_{\mathbb{C}}$. Since $((\tau_1^{\Pi})^{-1} \circ \mu)|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$, it follows from Proposition 5.3 of Chapter IX of [6] that there exists $\sqrt{-1}H \in \mathfrak{t}$ such that $(\tau_1^{\Pi})^{-1} \circ \mu = \tau_H$, and therefore $\mu = \tau_1^{\Pi} \circ \tau_H$. Put $H = \sum_{i=1}^{8} a_i K_i$, $a_i \in \mathbb{R}$. Then from (2.4), we have

$$E_{\alpha_1} = \mu(E_{\alpha_1}) = \tau_1^{\Pi} \circ \tau_H(E_{\alpha_1}) = e^{\pi\sqrt{-1}\alpha_1(H)} \tau_1^{\Pi}(E_{\alpha_1}) = -e^{\pi\sqrt{-1}\alpha_1(H)} E_{\alpha_1}.$$

Thus we get $a_1 = \alpha_1(H) \equiv 1 \pmod 2$. Similarly as above, we obtain $a_2 \equiv 0$, $a_4 \equiv 0$, $a_5 \equiv 0$, $a_6 \equiv 1$, $a_7 \equiv 0$, $a_8 \equiv 0 \pmod 2$. Moreover, since

$$E_{\beta_1} = \mu(E_{\alpha_3}) = e^{\pi\sqrt{-1}\alpha_3(H)} \tau_1^{\Pi}(E_{\alpha_3}) = c_1 e^{\pi\sqrt{-1}a_3} E_{\beta_1},$$

we have

$$c_1 = e^{-\pi\sqrt{-1}a_3}. \tag{5.7}$$

Then by (5.5) and (5.6) we have

$$\epsilon_0 E_{\alpha_2} = \mu(E_{\alpha_0}) = \tau_1^{\Pi} \circ \tau_H(E_{\alpha_0}) = e^{-4\pi\sqrt{-1}a_3} E_{\alpha_2},$$

and it follows from (5.7) that $c_1^4 = \epsilon_0$.

If $\epsilon_0 = 1$, then $c_1 = \pm 1$ or $\pm\sqrt{-1}$. Considering (5.4) for $h_- = 2K_2 - K_3 \in \mathfrak{t}_-$, we may assume that $c_1 = 1$ or $\sqrt{-1}$. Moreover, by Lemma 5.1 we may assume that $c_1 = 1$. If $\epsilon_0 = -1$, then by the same argument as above, we may assume $c_1 = e^{(\pi/4)\sqrt{-1}}$. Consequently, c_1 is uniquely determined except for conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$.

By an argument similar to (5.5), we may assume

$$\begin{aligned} &\tau(E_{\alpha_1}) = \pm E_{\alpha_1}, \quad \tau(E_{\alpha_2}) = E_{\alpha_0}, \quad \tau(E_{\alpha_3}) = \tilde{c}_1 E_{\beta_1}, \\ &\tau(E_{\alpha_4}) = E_{\alpha_8}, \quad \tau(E_{\alpha_5}) = E_{\alpha_7}, \quad \tau(E_{\alpha_6}) = \pm E_{\alpha_6}, \end{aligned} \tag{5.8}$$

where $\tilde{c}_1 \in \mathbb{C}$ and $|\tilde{c}| = 1$. Then, by Proposition 5.3 of Chapter IX of [6], there exists $\sqrt{-1}h \in \mathfrak{t}$ such that $\tau = \tau_1^{\Pi} \circ \tau_h$. Put

$$\begin{aligned} h &:= h_+ + h_-, \\ h_+ &:= k_1(2K_1 - K_2) + k_2(2K_1 - K_3) + k_3(2K_1 + K_6) \\ &\qquad + k_4(4K_1 - K_5 - K_7) + k_5(4K_1 + K_4 + K_8) \in \sqrt{-1}\mathfrak{t}_+, \\ h_- &:= k_6(2K_2 - K_3) + k_7(2K_2 - K_4 + K_8) + k_8(K_2 - K_5 + K_7) \in \sqrt{-1}\mathfrak{t}_-, \end{aligned}$$

where $k_1, \ldots, k_8 \in \mathbb{R}$. Then since $\tau^2 = \mathrm{Id}$ and $\tau_1^{\Pi}(h) = h_+ - h_-$, we have $\tau_{2h_+} = \mathrm{Id}$ and hence $2h_+ \equiv 0 \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$. Therefore we get $k_1, \ldots, k_5 \in \mathbb{Z}$. Then we have

$$\begin{aligned} h &\equiv k_1K_2 + k_2K_3 + k_5K_4 + k_4K_5 + k_3K_6 + k_4K_7 + k_5K_8 \\ &\quad + k_6(2K_2 - K_3) + k_7(2K_2 - K_4 + K_8) + k_8(K_2 - K_5 + K_7) \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}. \end{aligned}$$

Considering (5.5) and (5.8) together with (2.4), we obtain

$$\alpha_2(h) \equiv \alpha_4(h) \equiv \alpha_8(h) \equiv \alpha_5(h) \equiv \alpha_7(h) \equiv 0 \pmod 2,$$

and therefore

$$h \equiv (k_2 - k_6)K_3 + k_3K_6 \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}.$$

Furthermore, since $\tau(E_{\alpha_0}) = \tau_1^{\Pi}(E_{\alpha_0}) = E_{\alpha_2}$, it follows that $\alpha_0(h) \equiv 0 \pmod 2$, and therefore $2k_6 \in \mathbb{Z}$. Hence we may assume that τ is one of the following:

$$\tau_1^{\Pi},\ \tau_1^{\Pi} \circ \tau_{K_j},\ \tau_1^{\Pi} \circ \tau_{K_j} \circ \sigma,\ \tau_1^{\Pi} \circ \tau_{K_3+K_6},\ \tau_1^{\Pi} \circ \tau_{K_3+K_6} \circ \sigma, \quad j = 3, 6.$$

Indeed, $\tau_1^{\Pi} \circ \tau_{-k_6K_3}$ is conjugate within $\mathrm{Int}(\mathfrak{t})$ to one of τ_1^{Π} and $\tau_1^{\Pi} \circ \sigma$ since

$$\tau_1^{\Pi} \circ \tau_{-(1/2)K_3} = \tau_1^{\Pi} \circ \sigma^{-1} = \sigma \circ (\tau_1^{\Pi} \circ \sigma) \circ \sigma^{-1}.$$

Moreover, since $\tau_{K_3} = \sigma^2$ and $\tau_1^{\Pi} \circ \sigma = \sigma^{-1} \circ \tau_1^{\Pi}$, it follows that $\tau_1^{\Pi} \circ \tau_{K_3}$ and $\tau_1^{\Pi} \circ \tau_{K_3+K_6}$ are conjugate within $\mathrm{Int}(\mathfrak{t})$ to τ_1^{Π} and $\tau_1^{\Pi} \circ \tau_{K_6}$, respectively. Consequently, τ is conjugate within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ to one of following:

$$\tau_1^{\Pi}, \quad \tau_1^{\Pi} \circ \tau_{K_6}, \quad \tau_1^{\Pi} \circ \tau_{K_6} \circ \sigma.$$

Now we shall compute the dimension of $\mathfrak{h} \cap \mathfrak{g}^{\theta}$ and $\mathfrak{g}^{\theta}$, where θ is one of τ_1^{Π}, $\tau_1^{\Pi} \circ \tau_{K_6}$ and $\tau_1^{\Pi} \circ \tau_{K_6} \circ \sigma$. Since $\tau_1^{\Pi} \circ \tau_{K_6}(E_{\alpha_6}) = E_{\alpha_6}$ and $\dim(\mathfrak{h} \cap \mathfrak{g}^{\tau_1^{\Pi}}) = 29$, we have $\dim(\mathfrak{h} \cap \mathfrak{g}^{\tau_1^{\Pi} \circ \tau_{K_6}}) = 36$. Therefore we get $\mathfrak{h} \cap \mathfrak{g}^{\tau_1^{\Pi}} \cong D_4 \oplus D_1$ and $\mathfrak{h} \cap \mathfrak{g}^{\tau_1^{\Pi} \circ \tau_{K_6}} \cong C_4 \oplus D_1$. Put $\nu := \tau|_{\mathfrak{t}}$. It is easy to see that positive roots α such that $\nu(\alpha) = \alpha$ are

$$\Delta_{\nu}^{+} := \left\{ \begin{array}{l} \alpha_1,\ \alpha_6,\ \alpha_5 + \alpha_6 + \alpha_7,\ \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8,\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6,\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \\ \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\ 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \end{array} \right\}.$$

We consider the case of $\tau_1^{\Pi} \circ \tau_{K_6}$. Put $\gamma := \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$. Take a Weyl basis so that $\tau_1^{\Pi}(E_{\gamma}) = E_{\gamma}$ (cf. see Gilkey and Seitz [4]). Then it is easy to see that $\tau_1^{\Pi}(E_{\alpha}) = E_{\alpha}$ for any $\alpha \in \Delta_{\nu}^{+} \setminus \{\alpha_1, \alpha_6, \alpha_5 + \alpha_6 + \alpha_7, \alpha_4 + \alpha_5 + \alpha_7, 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8\}$ and therefore $\tau_1^{\Pi} \circ \tau_{K_6}(E_{\alpha}) = E_{\alpha}$ for any $\alpha \in \Delta_{\nu}^{+} \setminus \{\alpha_1\}$. It follows from Lemma 4.1 that $\dim \mathfrak{g}^{\tau_1^{\Pi} \circ \tau_{K_6}} = 136$. By using the classification of symmetric spaces, we get $\mathfrak{g}^{\tau_1^{\Pi} \circ \tau_{K_6}} \cong E_7 \oplus A_1$.

Similarly as above we can obtain $\mathfrak{h} \cap \mathfrak{g}^{\theta}$ and $\mathfrak{g}^{\theta}$ for $\theta = \tau_1^{\Pi} \circ \tau_{K_6} \circ \sigma$.

By an argument similar to above, we can obtain all involutions τ of Type V and VI, and determine $\mathfrak{h} \cap \mathfrak{g}^{\tau}$ and $\mathfrak{g}^{\tau}$, which are listed in Table 2.

Now we investigate involutions of Type II and III in Table 1. Since $\mathfrak{g}^*$ is a normal real form and of type $\mathfrak{e}_7$, the pair $(\mathfrak{g}^*, \mathfrak{k})$ is given by $(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$. It is easy to see that $\dim(\mathfrak{h} \cap \mathfrak{k}) = 13$. On the other hand, for an involution τ of $\mathfrak{g}$, we can see that

$$\dim \mathfrak{t}_{+} = \begin{cases} 4 & \text{if} \quad \tau \text{ is of Type II,} \\ 5 & \text{if} \quad \tau \text{ is of Type III,} \end{cases}$$

and if τ is of Type II (resp. Type III), the number of the subsets $\{\alpha, \beta\}$ such that $\alpha \in \Delta^{+}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\tau(\alpha) = \beta$, $\alpha \neq \pm\beta$ and $\alpha(K_4) \equiv 0 \pmod 4$ is 6 (resp. 4). Hence we obtain

$$\begin{cases} \dim(\mathfrak{h} \cap \mathfrak{g}^{\tau}) \geq 16 & \text{if} \quad \tau \text{ is of Type II,} \\ \dim(\mathfrak{h} \cap \mathfrak{g}^{\tau}) \geq 13 & \text{if} \quad \tau \text{ is of Type III.} \end{cases}$$

Therefore the Cartan involution τ^* of $\mathfrak{g}^* = \mathfrak{e}_{7(7)}$ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to an involution τ_3^{Π} of Type III. By an argument similar to Type IV, we can obtain all involutions τ of Type III, which are listed in Table 2.

Table 2. $\dim \mathfrak{z} = 0$, $\tau \circ \sigma = \sigma^{-1} \circ \tau$, $\sigma = \tau_{(1/2)H}$, $\mathfrak{k} = \mathfrak{g}^{\tau}$.

$(\mathfrak{g}, \mathfrak{h}, H)$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathfrak{su}(2), K_3)$	τ_1^{Π}	D_8	$D_4 \oplus D_1$
	$\tau_1^{\Pi} \circ \tau_{K_6}$	$E_7 \oplus A_1$	$C_4 \oplus D_1$
	$\tau_1^{\Pi} \circ \tau_{K_6} \circ \tau_{(1/2)K_3}$	D_8	$C_4 \oplus D_1$
$(\mathfrak{e}_8, \mathfrak{so}(10) \oplus \mathfrak{so}(6), K_6)$	τ_2^{Π}	D_8	$B_2 \oplus B_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_4}$	D_8	$B_2 \oplus B_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_4} \circ \tau_{(1/2)K_6}$	D_8	$B_2 \oplus B_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_8}$	$E_7 \oplus A_1$	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_8} \circ \tau_{(1/2)K_6}$	D_8	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_4+K_8}$	$E_7 \oplus A_1$	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_4+K_8} \circ \tau_{(1/2)K_6}$	D_8	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_3}$	D_8	$B_2 \oplus B_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_3} \circ \tau_{(1/2)K_6}$	D_8	$B_2 \oplus B_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_3+K_4}$	$E_7 \oplus A_1$	$D_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_3+K_4} \circ \tau_{(1/2)K_6}$	$E_7 \oplus A_1$	$D_2 \oplus B_1 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_3+K_8}$	D_8	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_3+K_8} \circ \tau_{(1/2)K_6}$	$E_7 \oplus A_1$	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_3+K_4+K_8}$	$E_7 \oplus A_1$	$B_3 \oplus B_2 \oplus B_1$
	$\tau_2^{\Pi} \circ \tau_{K_3+K_4+K_8} \circ \tau_{(1/2)K_6}$	D_8	$B_3 \oplus B_2 \oplus B_1$
$(\mathfrak{e}_7, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	τ_3^{Π}	A_7	$B_1 \oplus B_1 \oplus B_1 \oplus B_1 \oplus B_1 \oplus \mathbb{R}$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_2}$	$D_6 \oplus A_1$	$B_2 \oplus B_1 \oplus B_1 \oplus A_1$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_2} \circ \tau_{(1/2)K_4}$	$D_6 \oplus A_1$	$B_2 \oplus B_1 \oplus B_1 \oplus A_1$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_6}$	$E_6 \oplus \mathbb{R}$	$B_2 \oplus B_2 \oplus \mathbb{R}$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_6} \circ \tau_{(1/2)K_4}$	A_7	$B_2 \oplus B_2 \oplus \mathbb{R}$
	$\tau_3^{\Pi} \circ \tau_{K_2+K_6}$	$D_6 \oplus A_1$	$B_2 \oplus B_1 \oplus B_1 \oplus A_1$
	$\tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ \tau_{(1/2)K_4}$	$D_6 \oplus A_1$	$B_2 \oplus B_1 \oplus B_1 \oplus A_1$
	$\tau_3^{\Pi} \circ \varphi$	$D_6 \oplus A_1$	$D_3 \oplus D_1$
	$\tau_3^{\Pi} \circ \varphi \circ \tau_{(1/2)K_4}$	A_7	$D_3 \oplus D_1$
$(\mathfrak{f}_4, \mathfrak{so}(6) \oplus \mathfrak{so}(3), K_3)$	τ_4^{Π}	$C_3 \oplus A_1$	$B_1 \oplus B_1 \oplus D_1$
	$\tau_4^{\Pi} \circ \tau_{K_1+K_4}$	B_4	$B_2 \oplus B_1$
	$\tau_4^{\Pi} \circ \tau_{K_1+K_4} \circ \tau_{(1/2)K_3}$	$C_3 \oplus A_1$	$B_2 \oplus B_1$

τ_1^{Π}: $E_{\alpha_1} \mapsto -E_{\alpha_1}$, $E_{\alpha_2} \mapsto E_{\alpha_0}$, $E_{\alpha_3} \mapsto c_1 E_{\beta_1}$, $E_{\alpha_4} \mapsto E_{\alpha_8}$, $E_{\alpha_5} \mapsto E_{\alpha_7}$, $E_{\alpha_6} \mapsto -E_{\alpha_6}$, ($\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$)
τ_2^{Π}: $E_{\alpha_1} \mapsto -E_{\alpha_1}$, $E_{\alpha_2} \mapsto E_{\alpha_5}$, $E_{\alpha_3} \mapsto -E_{\alpha_3}$, $E_{\alpha_4} \mapsto -E_{\alpha_4}$, $E_{\alpha_6} \mapsto c_2 E_{\beta_2}$, $E_{\alpha_7} \mapsto E_{\alpha_0}$, $E_{\alpha_8} \mapsto -E_{\alpha_8}$, ($\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$)
τ_3^{Π}: $E_{\alpha_1} \mapsto -E_{\alpha_1}$, $E_{\alpha_2} \mapsto -E_{\alpha_2}$, $E_{\alpha_3} \mapsto E_{\alpha_0}$, $E_{\alpha_4} \mapsto c_3 E_{\beta_3}$, $E_{\alpha_5} \mapsto E_{\alpha_7}$, $E_{\alpha_6} \mapsto -E_{\alpha_6}$, ($\beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$)
τ_4^{Π}: $E_{\alpha_1} \mapsto -E_{\alpha_1}$, $E_{\alpha_2} \mapsto E_{\alpha_0}$, $E_{\alpha_3} \mapsto c_4 E_{\beta_4}$, $E_{\alpha_4} \mapsto -E_{\alpha_4}$, ($\beta_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$)
where c_i ($i = 1, 2, 3, 4$) is some complex number with $|c_i| = 1$.

Finally we consider the Type II. Put $\beta_3 := \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. Let τ_3^Π be as above. Then, since $\dim(\mathfrak{h} \cap \mathfrak{g}^{\tau_3^\Pi}) = \dim(\mathfrak{h} \cap \mathfrak{k}) = 13$, it follows that

$$\begin{aligned} &\tau_3^\Pi(E_{\alpha_1}) = -E_{\alpha_1}, \quad \tau_3^\Pi(E_{\alpha_2}) = -E_{\alpha_2}, \quad \tau_3^\Pi(E_{\alpha_3}) = E_{\alpha_0}, \\ &\tau_3^\Pi(E_{\alpha_4}) = c_3 E_{\beta_3}, \quad \tau_3^\Pi(E_{\alpha_5}) = E_{\alpha_7}, \quad \tau_3^\Pi(E_{\alpha_6}) = -E_{\alpha_6}, \end{aligned} \tag{5.9}$$

for some $c_3 \in \mathbb{C}$ with $|c_3| = 1$. On the other hand, from Theorem 5.1 of Chapter IX of [6], There exists an automorphism φ on $\mathfrak{g}$ such that

$$\begin{aligned} &\varphi(E_{\alpha_1}) = E_{\alpha_6}, \quad \varphi(E_{\alpha_2}) = E_{\alpha_2}, \quad \varphi(E_{\alpha_3}) = E_{\alpha_5}, \quad \varphi(E_{\alpha_4}) = E_{\alpha_4}, \\ &\varphi(E_{\alpha_5}) = E_{\alpha_3}, \quad \varphi(E_{\alpha_6}) = E_{\alpha_1}, \quad \varphi(E_{\alpha_7}) = E_{\alpha_0}, \quad \varphi(E_{\alpha_0}) = \epsilon E_{\alpha_7}, \end{aligned}$$

where $\epsilon = \pm 1$. If $\epsilon = -1$, then we have

$$\begin{cases} \varphi^2(E_{\alpha_i}) = E_{\alpha_i} \quad (1 \le i \le 6), \\ \varphi^2(E_{\alpha_7}) = -E_{\alpha_7}. \end{cases}$$

Thus the inner automorphism φ^2 has the form τ_{K_7}. Hence we have

$$\mathfrak{g}^{\varphi^2} = \mathfrak{t} + \sum_{\substack{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \\ \alpha(K_7) = 0}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{5.10}$$

Put $\gamma := \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$. Then we get $\varphi(\gamma) = -\gamma$ and from the proof of Theorem 5.1 of Chapter IX of [6], we get

$$\varphi(E_\gamma) = \epsilon_\gamma E_{-\gamma}, \ \varphi(E_{-\gamma}) = \epsilon_{-\gamma} E_\gamma \quad (\epsilon_\gamma = \epsilon_{-\gamma} = \pm 1).$$

Therefore we obtain

$$\begin{aligned} \varphi(A_\gamma) &= \epsilon_\gamma (E_{-\gamma} - E_\gamma) = -\epsilon_\gamma A_\gamma, \\ \varphi(B_\gamma) &= \sqrt{-1}\epsilon_\gamma (E_\gamma + E_{-\gamma}) = \epsilon_\gamma B_\gamma. \end{aligned}$$

This implies that

$$A_\gamma \quad \text{or} \quad B_\gamma \in \mathfrak{g}^\varphi \subset \mathfrak{g}^{\varphi^2}.$$

This contradicts (5.10). Thus $\varphi(E_{\alpha_0}) = E_{\alpha_7}$, that is,

$$\begin{aligned} &\varphi(E_{\alpha_1}) = E_{\alpha_6}, \quad \varphi(E_{\alpha_2}) = E_{\alpha_2}, \quad \varphi(E_{\alpha_3}) = E_{\alpha_5}, \quad \varphi(E_{\alpha_4}) = E_{\alpha_4}, \\ &\varphi(E_{\alpha_5}) = E_{\alpha_3}, \quad \varphi(E_{\alpha_6}) = E_{\alpha_1}, \quad \varphi(E_{\alpha_7}) = E_{\alpha_0}, \quad \varphi(E_{\alpha_0}) = E_{\alpha_7}. \end{aligned} \tag{5.11}$$

Then $\tau_3^\Pi \circ \varphi$ maps

$$\alpha_1 \mapsto \alpha_6, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \alpha_7, \quad \alpha_5 \mapsto \alpha_0, \quad \alpha_4 \mapsto \beta_3,$$

and it is easy to see

$$\begin{aligned}(5.12)\qquad &\tau_3^{\Pi}\circ\varphi(E_{\alpha_1})=\varphi\circ\tau_3^{\Pi}(E_{\alpha_1})=-E_{\alpha_6},\quad \tau_3^{\Pi}\circ\varphi(E_{\alpha_2})=\varphi\circ\tau_3^{\Pi}(E_{\alpha_2})=-E_{\alpha_2},\\ &\tau_3^{\Pi}\circ\varphi(E_{\alpha_3})=\varphi\circ\tau_3^{\Pi}(E_{\alpha_3})=E_{\alpha_7},\quad \tau_3^{\Pi}\circ\varphi(E_{\alpha_5})=\varphi\circ\tau_3^{\Pi}(E_{\alpha_5})=E_{\alpha_0},\\ &\tau_3^{\Pi}\circ\varphi(E_{\alpha_4})=c_3E_{\beta_3},\quad \varphi\circ\tau_3^{\Pi}(E_{\alpha_4})=c_3\varphi(E_{\beta_3}).\end{aligned}$$

Therefore $(\tau_3^{\Pi}\circ\varphi)^2=\mathrm{Id}$ if and only if $\varphi(E_{\beta_3})=E_{\beta_3}$. Since $\dim\mathfrak{g}_{\beta_3}=1$, we have $\mathfrak{g}_{\beta_3}=\mathbb{C}X_{\beta_3}$, where

$$X_{\beta_3}:=[[[[[[[[E_{\alpha_2},E_{\alpha_4}],E_{\alpha_3}],E_{\alpha_5}],E_{\alpha_4}],E_{\alpha_1}],E_{\alpha_6}],E_{\alpha_3}],E_{\alpha_5}],E_{\alpha_4}].$$

Because $[E_{\alpha_3},E_{\alpha_5}]=[E_{\alpha_1},E_{\alpha_6}]=0$, we get $\varphi(X_{\beta_3})=X_{\beta_3}$ and therefore $\varphi|_{\mathfrak{g}_{\beta_3}}=\mathrm{Id}$. Thus $\tau_3^{\Pi}\circ\varphi$ is an involutive automorphism of $\mathfrak{e}_7$. From (5.12), we obtain

$$\begin{aligned}(5.13)\qquad &\tau_3^{\Pi}\circ\varphi(E_{\alpha_1})=-E_{\alpha_6},\quad \tau_3^{\Pi}\circ\varphi(E_{\alpha_2})=-E_{\alpha_2},\quad \tau_3^{\Pi}\circ\varphi(E_{\alpha_3})=E_{\alpha_7},\\ &\tau_3^{\Pi}\circ\varphi(E_{\alpha_5})=E_{\alpha_0},\quad \tau_3^{\Pi}\circ\varphi(E_{\alpha_4})=c_3E_{\beta_3}.\end{aligned}$$

Hence we can construct an involution of Type II. By an argument similar to Type IV, we can give all involutions of Type II.

Consequently we obtain the following proposition.

Proposition 5.1. *Suppose that* $\dim\mathfrak{z}=0$. *Let* τ *be an involution of* $\mathfrak{g}$ *such that* $\tau\circ\sigma=\sigma^{-1}\circ\tau$. *Then* τ *is conjugate within* $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *to one of automorphisms listed in* Table 2.

6. The case where $\dim\mathfrak{z}=1$, $\tau\circ\sigma=\sigma^{-1}\circ\tau$

In this section we investigate involutions τ of $\mathfrak{g}$ such that $\dim\mathfrak{z}=1$ and $\tau\circ\sigma=\sigma^{-1}\circ\tau$. First, we construct such involutions by using graded Lie algebras. Let $\mathfrak{g}^*$ be a noncompact simple Lie algebra over $\mathbb{R}$ such that $\mathfrak{g}_{\mathbb{C}}$ is simple. Let $\mathfrak{g}^*=\mathfrak{k}+\mathfrak{p}^*$ and τ be the Cartan involution as in (2.5), and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}^*$. Let $\mathfrak{t}^*$ be a Cartan subalgebra of $\mathfrak{g}^*$ such that $\mathfrak{a}\subset\mathfrak{t}^*$. We take compatible orderings on $\mathfrak{a}$ and $\mathfrak{t}^*$.

Take a gradation $\mathfrak{g}^*=\sum_{p=-3}^{3}\mathfrak{g}_p^*$ of the third kind on $\mathfrak{g}^*$ corresponding to a partition

$$\Pi=\Pi_0\cup\Pi_1,\ \Pi_1=\{\lambda_i\},\quad n_i=3.$$

Put $\sigma:=\mathrm{Ad}(\exp(\pi/2)\sqrt{-1}h_i)$. It is obvious that $\sigma\in\mathrm{Int}(\mathfrak{g})$ ($\mathfrak{g}=\mathfrak{k}+\sqrt{-1}\mathfrak{p}^*$), $\sigma^4=\mathrm{Id}$ and $\tau\circ\sigma=\sigma^{-1}\circ\tau$. Considering the classification of the Satake diagram, there exists a unique $\alpha_j\in\Delta(\mathfrak{g}_{\mathbb{C}}^*,\mathfrak{t}_{\mathbb{C}}^*)$ such that $\alpha_j|_{\mathfrak{a}}=\lambda_i$ with $m_j=3$, and it follows from Lemma 2.5 that $h_i=K_j$. Therefore, by Remark 2.2 we have $\dim\mathfrak{z}=1$.

Generally, let $(G/H, \langle\ ,\ \rangle, \sigma)$ be a compact Riemannian 4-symmetric space such that G is simple and σ is inner. As before, let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{h}$. We suppose that $\sigma = \tau_{(1/2)K_i}$ for some $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_i = 3$. Then $\mathfrak{z} = \mathbb{R}\sqrt{-1}K_i$. From [7], a pair $(\mathfrak{g}, \mathfrak{h})$ is one of the following:

$$(\mathfrak{e}_6, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{e}_7, \mathfrak{su}(2) \oplus \mathfrak{su}(6) \oplus \mathbb{R}), \quad (\mathfrak{e}_7, \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}),$$
$$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathbb{R}), \quad (\mathfrak{e}_8, \mathfrak{su}(2) \oplus \mathfrak{e}_6 \oplus \mathbb{R}), \quad (\mathfrak{f}_4, \mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathbb{R}), \quad (\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathbb{R}).$$

REMARK 6.1. Each 4-symmetric pair described in the above is neither symmetric nor 3-symmetric. Indeed, except for $(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathbb{R})$, it follows from the classifications of compact k-symmetric spaces ($k = 2, 3$) that each 4-symmetric pair described in the above is not k-symmetric ($k = 2, 3$). Now, for $(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathbb{R})$, we prove that it is not isomorphic to a k-symmetric pair ($k = 2, 3$). First, we note that $\sigma = \tau_{(1/2)K_1}$ with $m_1 = 3$. From the classification of compact symmetric spaces, it is obvious that the pair $(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathbb{R})$ is not symmetric. Let $(\mathfrak{g}_2, \theta)$ be a 3-symmetric pair. Then θ is conjugate to $\tau_{(2/3)K_2}$ and

$$\mathfrak{g}_2{}^{\tau_{(2/3)K_2}} = \mathfrak{su}_{\alpha_1}(2) \oplus \mathbb{R}\sqrt{-1}K_2 \cong \mathfrak{su}(2) \oplus \mathbb{R}.$$

If there exists $\mu \in \mathrm{Aut}(\mathfrak{g})$ such that $\mu(\mathfrak{g}_2{}^{\sigma}) = \mu(\mathfrak{g}_2{}^{\tau_{(1/2)K_1}}) = \mathfrak{g}_2{}^{\tau_{(2/3)K_2}}$, then we have $\mu(\mathfrak{su}_{\alpha_2}(2)) = \mathfrak{su}_{\alpha_1}(2)$. Therefore it follows that there exists $k \in \mathbb{C}$ with $|c| = 1$ such that

$$\mu(E_{\alpha_2}) = cE_{\pm\alpha_1}, \quad \mu(E_{-\alpha_2}) = c^{-1}E_{\mp\alpha_1},$$

which implies that $\mu(H_{\alpha_2}) = \pm H_{\alpha_1}$. However, this is a contradiction because $|\alpha_1| \neq |\alpha_2|$. Consequently, the 4-symmetric pair $(\mathfrak{g}_2, \sigma)$ is not k-symmetric ($k = 2, 3$).

Now we assume that $\sigma = \tau_{(1/2)K_i}$ for some $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_i = 3$. Let τ be an involution of $\mathfrak{g}$ such that $\tau \circ \sigma = \sigma^{-1} \circ \tau$. Then it is easy to see that $\tau(\mathfrak{h}) = \mathfrak{h}$ and $\tau(\mathfrak{z}) = \mathfrak{z}$. Thus we have $\tau(\sqrt{-1}K_i) = \pm\sqrt{-1}K_i$. If $\tau(\sqrt{-1}K_i) = \sqrt{-1}K_i$, then $\tau \circ \sigma = \sigma \circ \tau$. Hence we get $\tau(\sqrt{-1}K_i) = -\sqrt{-1}K_i$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ corresponding to τ. Then we have $\sqrt{-1}K_i \in \mathfrak{p}$. Put

$$\Delta^+_{\mathfrak{m}} := \{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ A_\alpha, B_\alpha \in \mathfrak{m}\}, \quad \Delta^+_{\mathfrak{h}} := \{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ A_\alpha, B_\alpha \in \mathfrak{h}\}.$$

Lemma 6.1.

$$\Delta^+_{\mathfrak{h}} = \left\{\alpha = \sum_{j=1}^{n} k_j\alpha_j \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ k_i = 0\right\}.$$

Proof. Since $m_i = 3$ and $E_\alpha = \sigma(E_\alpha) = e^{(\pi\sqrt{-1}/2)\alpha(K_i)}E_\alpha$ for any $\alpha \in \Delta^+_{\mathfrak{h}}$, we have $\alpha(K_i) = \sum_{j=1}^{n} k_j\alpha_j(K_i) = k_i = 0$. □

We define a subset Δ_s^+ ($s = 1, 2, 3$) of $\Delta_{\mathfrak{m}}^+$ as follows:

$$\Delta_s^+ := \left\{ \alpha = \sum_{j=1}^m \in \Delta^+;\ k_i = s \right\}.$$

Then we have an orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, where

$$\mathfrak{m}_s = \sum_{\alpha \in \Delta_s^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).$$

Lemma 6.2.

$$\tau(\mathfrak{m}_s) = \mathfrak{m}_s, \quad s = 1, 2, 3.$$

Proof. Since

$$\sigma(E_\alpha) = e^{(\pi\sqrt{-1}/2)\alpha(K_i)} E_\alpha = \begin{cases} \sqrt{-1}E_\alpha, & \alpha \in \Delta_1^+, \\ -E_\alpha, & \alpha \in \Delta_2^+, \\ -\sqrt{-1}E_\alpha, & \alpha \in \Delta_3^+, \end{cases}$$

it follows that

$$\sigma(X) = -X \iff X \in \mathfrak{m}_2.$$

Hence if $X \in \mathfrak{m}_2$, then

$$\sigma(\tau(X)) = \tau \circ \sigma^{-1}(X) = -\tau(X).$$

Thus we obtain $\tau(\mathfrak{m}_2) = \mathfrak{m}_2$.

Next for $\alpha \in \Delta_1^+$ (resp. Δ_3^+), we get $\tau(\alpha) \in -\Delta_1^+$ (resp. $-\Delta_3^+$). Indeed, since

$$[K_i, \tau(E_\alpha)] = \tau[\tau(K_i), E_\alpha] = -\tau[K_i, E_\alpha] = -\alpha(K_i)\tau(E_\alpha),$$

and $\tau(E_\alpha) \in \mathfrak{g}_{\tau(\alpha)}$, we get $\tau(\alpha)(K_i) = -\alpha(K_i) = -1$ (resp. -3). This completes the proof of the lemma. $\square$

Put

$$\mathfrak{h}^\pm := \{X \in \mathfrak{h};\ \tau(X) = \pm X\}, \quad \mathfrak{m}_s^\pm := \{X \in \mathfrak{m}_s;\ \tau(X) = \pm X\}.$$

Since $\tau(\mathfrak{h}) = \mathfrak{h}$ and $\tau(\mathfrak{m}) = \mathfrak{m}$, we can write

$$\mathfrak{g} = (\mathfrak{h}^+ + \mathfrak{h}^-) \oplus \sum_{s=1}^3 (\mathfrak{m}_s^+ + \mathfrak{m}_s^-),$$

$$\mathfrak{k} = \mathfrak{h}^+ \oplus \mathfrak{m}_1^+ \oplus \mathfrak{m}_2^+ \oplus \mathfrak{m}_3^+, \quad \mathfrak{p} = \mathfrak{h}^- \oplus \mathfrak{m}_1^- \oplus \mathfrak{m}_2^- \oplus \mathfrak{m}_3^-.$$

Put $\mathfrak{g}^* := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then we have $Z := K_i \in \sqrt{-1}\mathfrak{p}$. We shall prove the following lemma.

Lemma 6.3. *The eigenvalues of* $\mathrm{ad}(Z)\colon \mathfrak{g}^* \to \mathfrak{g}^*$ *are* $0, \pm 1, \pm 2$ *and* ± 3.

Proof. First we note that $\mathfrak{h}^+ + \sqrt{-1}\mathfrak{h}^-$ is the 0-eigenspace of $\mathrm{ad}(Z)$.
It is easy to see that

$$\begin{cases} \alpha \in \Delta_1^+ \Longrightarrow \sigma(A_\alpha) = B_\alpha,\ \sigma(B_\alpha) = -A_\alpha, \\ \alpha \in \Delta_2^+ \Longrightarrow \sigma(A_\alpha) = -A_\alpha,\ \sigma(B_\alpha) = -B_\alpha, \\ \alpha \in \Delta_3^+ \Longrightarrow \sigma(A_\alpha) = -B_\alpha,\ \sigma(B_\alpha) = A_\alpha, \end{cases} \tag{6.1}$$

and

$$\sigma^2|_{\mathfrak{m}_1} = -\mathrm{Id}_{\mathfrak{m}_1}, \quad \sigma^2|_{\mathfrak{m}_3} = -\mathrm{Id}_{\mathfrak{m}_3}. \tag{6.2}$$

If $X \in \mathfrak{m}_1^+$, then by (6.2), we have

$$\tau(\sigma(X)) = \sigma^{-1}(\tau(X)) = \sigma^3(X) = -\sigma(X).$$

Thus we have $\sigma(\mathfrak{m}_1^+) \subset \mathfrak{m}_1^-$. Similarly, we get $\sigma(\mathfrak{m}_1^-) \subset \mathfrak{m}_1^+$. Therefore it follows that

$$\sigma(\mathfrak{m}_1^+) = \mathfrak{m}_1^-, \quad \sigma(\mathfrak{m}_1^-) = \mathfrak{m}_1^+. \tag{6.3}$$

Similarly, we obtain

$$\sigma(\mathfrak{m}_3^+) = \mathfrak{m}_3^-, \quad \sigma(\mathfrak{m}_3^-) = \mathfrak{m}_3^+. \tag{6.4}$$

By a straightforward computation we have

$$[\sqrt{-1}H, A_\alpha] = \alpha(H)B_\alpha, \quad [\sqrt{-1}H, B_\alpha] = -\alpha(H)A_\alpha. \tag{6.5}$$

Put $X_1 = \sum_{\alpha\in\Delta_1^+}(a_\alpha A_\alpha + b_\alpha B_\alpha) \in \mathfrak{m}_1$. Then by (6.1), we have $\sigma(X_1) = \sum_{\alpha\in\Delta_1^+}(a_\alpha B_\alpha - b_\alpha A_\alpha)$. Using (6.5), it is easy to see that

$$[\sqrt{-1}Z, X_1] = \sigma(X_1).$$

Similarly, we get

$$[\sqrt{-1}Z, X_3] = -3\sigma(X_3), \quad [\sqrt{-1}Z, [\sqrt{-1}Z, X_2]] = -4X_2,$$

for $X_j \in \mathfrak{m}_j$ $(j = 2, 3)$. Therefore it follows from (6.2) that

$$\begin{aligned} &[Z, X_1 \pm \sqrt{-1}\sigma(X_1)] = \mp(X_1 \pm \sqrt{-1}\sigma(X_1)), \\ &[Z, X_3 \pm \sqrt{-1}\sigma(X_3)] = \pm 3(X_3 \pm \sqrt{-1}\sigma(X_3)). \end{aligned} \tag{6.6}$$

Note that $X_s \pm \sqrt{-1}\sigma(X_s) \in \mathfrak{g}^*$ for $X_s \in \mathfrak{m}_s^+$ $(s = 1, 3)$ from (6.3) and (6.4). Moreover, $Y_2 := [\sqrt{-1}Z, X_2] \neq 0$ and $Y_2 \in \mathfrak{m}_2^-$ for $X_2 \in \mathfrak{m}_2^-$, and

$$\left[Z, X_2 \pm \frac{1}{2}\sqrt{-1}Y_2\right] = \mp 2\left(X_2 \pm \frac{1}{2}\sqrt{-1}Y_2\right). \tag{6.7}$$

Consequently, from (6.6) and (6.7) the lemma is proved. □

Now, we are in a position to prove the following proposition which classifies involutions preserving $\mathfrak{h}$ for this case.

Proposition 6.1. (1) *Let* $\mathfrak{g}^* = \sum_{p=-3}^{3} \mathfrak{g}_p^*$ *be a graded simple Lie algebra of the third kind with a grade-reversing Cartan involution* τ, *which is corresponding to a partition* $\{\Pi_0, \Pi_1\}$ *of* $\Pi = \{\lambda_1, \ldots, \lambda_l\}$ *such that* $\Pi_1 = \{\lambda_i\}$ *with* $n_i = 3$. *Put* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}h_i)$. *Then* σ *is an automorphism of order* 4 *on the compact dual* $\mathfrak{g}$ *of* $\mathfrak{g}^*$ *such that* $\dim \mathfrak{z} = 1$ *and* $\tau \circ \sigma = \sigma^{-1} \circ \tau$.
(2) *Let* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ *for some* $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *with* $m_i = 3$. *Then for each involution* τ *of* $\mathfrak{g}$ *satisfying* $\tau \circ \sigma = \sigma^{-1} \circ \tau$, *there exists* $\theta \in \mathrm{Aut}(\mathfrak{g})$ *such that* $\theta \circ \sigma \circ \theta^{-1}$ *and* $\theta \circ \tau \circ \theta^{-1}$ *are obtained from a graded Lie algebra by the method described in* (1).

Proof. We have proved (1) in the above.

Now we prove (2). For each $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ and τ with $\tau \circ \sigma = \sigma^{-1} \circ \tau$, it follows from Lemma 6.3 that there exists a graded Lie algebra $\mathfrak{g}^* = \sum_{p=-3}^{3} \mathfrak{g}_p^*$ with the characteristic element $Z := K_i$ such that τ is the Cartan involution. As above, let $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}^*$ be the Cartan decomposition of $\mathfrak{g}^*$ corresponding to τ and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}^*$ such that $Z \in \mathfrak{a}$. Moreover, let $\mathfrak{t}^*$ be a Cartan subalgebra of $\mathfrak{g}^*$ containing $\mathfrak{a}$ equipped with a compatible ordering. By Lemma 6.3, we have $\lambda(Z) = 0, \pm 1, \pm 2$ or ± 3 for any $\lambda \in \Delta := \Delta(\mathfrak{g}^*, \mathfrak{a})$. If Δ is a reduced root system, then from Lemma 2.4 together with Lemma 2.4 of [15] there exists $w \in W(\mathfrak{g}^*, \mathfrak{a})$ such that

$$\frac{1}{4}w(Z) = \frac{1}{4}h + T.$$

Here T is an element in $\mathfrak{a}$ satisfying $\lambda(T) \in \mathbb{Z}$ for any $\lambda \in \Delta$, and h is one of the following:

$$h_p, \quad h_{q_1} + h_{q_2}, \quad 2h_{r_1} + h_{r_2}, \quad h_{s_1} + h_{s_2} + h_{s_3},$$

with $n_p = 1, 2, 3$ or 4, $(n_{q_1}, n_{q_2}) = (1, 1)$, $(1, 2)$ or $(2, 2)$, $n_{r_1} = n_{r_2} = 1$ and $n_{s_1} = n_{s_2} = n_{s_3} = 1$. If Δ is a nonreduced root system, then $\Delta' := \{\lambda \in \Delta; 2\lambda \notin \Delta\}$ is a reduced root system of type B_l with the fundamental root system Π. Applying Lemma 2.4 together with Lemma 2.4 of [15] to Δ', we can see that there exists $w \in W(\mathfrak{g}^*, \mathfrak{a})$ such that $(1/4)w(Z) = (1/4)h + T$ with $\lambda(T) \in \mathbb{Z}$ for any $\lambda \in \Delta'$ and h is one of

$$h_a, \quad h_b + h_c, \quad n_a = n_b = n_c = 2.$$

Hence we may assume that there exists $\theta \in \mathrm{Int}(\mathfrak{k})$ such that

$$\theta \circ \sigma \circ \theta^{-1} = \mathrm{Ad}\left(\exp \frac{\pi}{2}\sqrt{-1}h\right).$$

Note that $\theta \circ \tau \circ \theta^{-1} = \tau$ because $\theta \in \mathrm{Int}(\mathfrak{k})$.

Next, we shall prove that $h = h_p$ for some $\lambda_p \in \Pi$ with $n_p = 3$. In the case where $h = h_p$ with $n_p = 1$, there exists a unique $\alpha_{i_p} \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $m_{i_p} = 1$ and $\alpha_{i_p}|_{\mathfrak{a}} = \lambda_p$. Therefore by Lemma 2.5 together with Remark 2.2 we have $h_p = K_{i_p}$ and $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair, which also contradicts Remark 6.1. Similarly, if $h = h_p$, $h_{q_1} + h_{q_2}$, $2h_{r_1} + h_{r_2}$ or h_a with $n_p = n_a = 2$ and $n_{q_1} = n_{q_2} = n_{r_1} = n_{r_2} = 1$, then a pair $(\mathfrak{g}, \mathfrak{h})$ is 3-symmetric, which contradicts Remark 6.1.

In the case where $h = h_{s_1} + h_{s_2} + h_{s_3}$ with $n_{s_1} = n_{s_2} = n_{s_3} = 1$, there exist unique $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3} \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\alpha_{i_k}|_{\mathfrak{a}} = \lambda_{s_k}$ $(k = 1, 2, 3)$. Then we have $h = K_{i_1} + K_{i_2} + K_{i_3}$ and hence $\dim \mathfrak{z} = 3$, which is a contradiction.

In the case where $h = h_{q_1} + h_{q_2}$, then we obtain

$$h = K_{i_1} + K_{i_2} \quad \text{or} \quad K_{i_1} + K_{j_1} + K_{j_2}.$$

Here $\alpha_{i_k}|_{\mathfrak{a}} = \lambda_{q_k}$, $m_{i_1} = 1$, $m_{i_2} = 2$, or $\alpha_{i_1}|_{\mathfrak{a}} = \lambda_{q_1}$, $\alpha_{j_k}|_{\mathfrak{a}} = \lambda_{q_k}$, $m_{i_k} = 1$ $(k = 1, 2)$. Therefore by Remark 2.2 we have $\dim \mathfrak{z} \neq 1$.

In the case where $h = h_p$ with $n_p = 4$, then we have

(i) $$h = K_{i_p} \quad \text{with} \quad m_{i_p} = 4,$$

or

(ii) $$h = K_{i_1} + K_{i_2} \quad \text{with} \quad \alpha_{i_k}|_{\mathfrak{a}} = \lambda_p,\ m_{i_k} = 2\ (k = 1, 2).$$

For the case (i), it follow from Remark 2.2 that $\dim \mathfrak{z} = 0$. For the case (ii), it is easy to see that the center $\mathfrak{z}(\mathfrak{g}^{\tau_{(1/2)h}})$ of $\mathfrak{g}^{\tau_{(1/2)h}}$ coincides with

$$\mathfrak{z}(\mathfrak{g}^{\tau_{(1/2)h}}) = \mathbb{R}\sqrt{-1}(K_{i_1} - K_{i_2}). \tag{6.8}$$

Note that if $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ with $m_i = 3$, then the center $\mathfrak{z}$ of $\mathfrak{h}$ coincides with $\mathbb{R}\sqrt{-1}K_i$ as mentioned before. It is easy to see that $\mathfrak{h}$ is the centralizer of $\mathfrak{z}$ in $\mathfrak{g}$. However, $\mathfrak{g}^{\tau_{(1/2)h}}$ is not the centralizer of $\mathfrak{z}(\mathfrak{g}^{\tau_{(1/2)h}})$. Indeed, let $\alpha = \sum_j k_j\alpha_j$ be a root satisfying $k_{i_1} = k_{i_2} = 1$. Since $\alpha(h) = 2$ and $\alpha(K_{i_1} - K_{i_2}) = 0$, we obtain

$$[\sqrt{-1}(K_{i_1} - K_{i_2}),\ A_\alpha] = 0, \quad \tau_{(1/2)h}(A_\alpha) = -A_\alpha,$$

which implies that A_α belongs to the centralizer of $\mathfrak{z}(\mathfrak{g}^{\tau_{(1/2)h}})$ and $A_\alpha \notin \mathfrak{g}^{\tau_{(1/2)h}}$. Hence σ is not conjugate to $\tau_{(1/2)h}$.

Finally, consider the case where $h = h_j + h_k$ with $n_j = n_k = 2$. In this case, we have $h = K_{i_j} + K_{i_k}$ with $m_{i_j} = m_{i_k} = 2$, or $h = K_{j_1} + K_{j_2} + K_{i_k}$ with $m_{j_1} = m_{j_2} = 1$, $m_{i_k} = 2$. By the same argument as (i) above, the first case is impossible. Moreover, if $h = K_{j_1} + K_{j_2} + K_{i_k}$, then the center of $\mathfrak{g}^{\tau(1/2)h}$ coincides with

$$\mathbb{R}\sqrt{-1}(2K_{j_1} - K_{i_k}) + \mathbb{R}\sqrt{-1}(K_{j_2} - K_{i_k}),$$

since $\mathfrak{g}^{\tau(1/2)h}$ is generated by $\mathfrak{t}$ and $\{A_\alpha, B_\alpha;\ \alpha(h) \equiv 0 \pmod 4\}$. However, this is a contradiction.

Consequently we obtain $h = h_p$ with $n_p = 3$ which completes the proof of (2) of the proposition. $\square$

7. The case where $\dim \mathfrak{z} = 0$ and $\tau \circ \sigma = \sigma \circ \tau$

In this section we consider the case where $\dim \mathfrak{z} = 0$ and $\tau \circ \sigma = \sigma \circ \tau$. In this case, it follows from Proposition 4.1 that $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$ or τ is of Type I in Table 1.

First we consider the Type I in Table 1. From Section 5 there exists an automorphism φ satisfying (5.11). We note that φ is an involution of Type I in Table 1. Let $\mathfrak{t}_\pm$ be the (± 1)-eigenspaces of φ. Then we have

$$\begin{aligned}
\mathfrak{t}_+ &= \mathrm{span}\{K_1 + K_6 - 2K_7,\ K_2 - K_7,\ K_3 + K_5 - 3K_7,\ K_4 - 2K_7\},\\
\mathfrak{t}_- &= \mathrm{span}\{-K_1 + K_6,\ -K_3 + K_5,\ K_7\}.
\end{aligned}$$

For any involution τ of Type I, it follows from Proposition 5.3 of Chapter IX of [6] that there exists $\sqrt{-1}h \in \mathfrak{t}$ such that $\tau = \varphi \circ \tau_h$. We put $h = h_+ + h_-$, $h_\pm \in \sqrt{-1}\mathfrak{t}_\pm$. Then since $\tau^2 = \mathrm{Id}$, we can write

$$\begin{aligned}
h_+ = k_1(K_1 + K_6 - 2K_7) + k_2(K_2 + K_7) + k_3(K_3 + K_5 - 3K_7) + k_4(K_4 - 2K_7),&\\
k_i \in \mathbb{Z}.&
\end{aligned} \tag{7.1}$$

As in the case of Type IV in Section 5, we may assume $\tau(E_{\alpha_1}) = E_{\alpha_6}$, $\tau(E_{\alpha_3}) = E_{\alpha_5}$ and $\tau(E_{\alpha_7}) = E_{\alpha_0}$. Indeed, for example, if $\tau(E_{\alpha_1}) = b_1 E_{\alpha_6}$ for some $b_1 \in \mathbb{C}$ with $|b_1| = 1$, then using $h_- = k(-K_1 + K_6)$ with $e^{2\pi\sqrt{-1}k} = b_1$, we have $(\tau_{h_-})^{-1} \circ \tau \circ \tau_{h_-}(E_{\alpha_1}) = E_{\alpha_6}$.

Using (7.1), by an argument similar to the case of Type IV in Section 5 we can prove that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following involutions:

$$\varphi, \quad \varphi \circ \tau_{K_2}, \quad \varphi \circ \tau_{K_4}, \quad \varphi \circ \tau_{K_2} \circ \tau_{K_4}.$$

Note that $\mathfrak{su}_{\alpha_2}(2) \subset \mathfrak{h} \cap \mathfrak{g}^\varphi$, and hence $t_{\alpha_2} \in \mathrm{Int}(\mathfrak{h} \cap \mathfrak{g}^\varphi)$. Therefore we have $\varphi \circ t_{\alpha_2} = t_{\alpha_2} \circ \varphi$. Moreover, since $t_{\alpha_2}(K_2) = -K_2 + K_4$, it follows that $\varphi \circ \tau_{K_2}$ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to $\varphi \circ \tau_{K_2} \circ \tau_{K_4}$.

Put $\nu := \tau|_{\mathfrak{t}}$. It is easy to see that the set Δ_ν^+ of positive roots α satisfying $\nu(\alpha) = \alpha$ coincides with

$$\Delta_\nu^+ = \left\{ \begin{array}{l} \alpha_2,\ \alpha_4,\ \alpha_2+\alpha_4,\ \alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5, \\ \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \\ \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ \delta \end{array} \right\}.$$

Using (5.11), we can check that $\varphi(E_\alpha) = E_\alpha$ for any $\alpha \in \Delta_\nu^+$. For example

$$\varphi([E_{\alpha_5}, [E_{\alpha_3}, E_{\alpha_4}]]) = [E_{\alpha_3}, [E_{\alpha_5}, E_{\alpha_4}]] = [E_{\alpha_5}, [E_{\alpha_3}, E_{\alpha_4}]],$$

and thus $\varphi(E_{\alpha_3+\alpha_4+\alpha_5}) = E_{\alpha_3+\alpha_4+\alpha_5}$. Hence it follows from Lemma 4.1 that $\dim \mathfrak{g}^\varphi = 79$. By using the classification of symmetric spaces, we get $\mathfrak{g}^\varphi \cong E_6 \oplus \mathbb{R}$.

The number of subsets $\{\alpha, \beta\}$ such that $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\tau(\alpha) = \beta$, $\alpha \neq \pm\beta$ and $\alpha(K_4) \equiv 0 \pmod 4$ is 6. Furthermore $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\tau(\alpha) = \alpha$ and $\alpha(K_4) \equiv 0 \pmod 4$ is only α_2. Since $\dim \mathfrak{t}_+ = 4$, we get

$$\dim(\mathfrak{h} \cap \mathfrak{g}^\varphi) = 4 + ((6+1) \times 2) = 18.$$

Therefore we get $\mathfrak{h} \cap \mathfrak{g}^\varphi \cong D_8 \oplus C_1$.

Similarly as above we can compute $\dim(\mathfrak{h} \cap \mathfrak{g}^\tau)$ and $\dim \mathfrak{g}^\tau$ for the other types.

Next we consider the case $\tau|_{\mathfrak{t}} = \mathrm{Id}$. First we suppose that $\mathfrak{g}$ is of type $\mathfrak{e}_8$ and $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_3)$. Then by (4.2) (i), we have

$$\mathfrak{h} \cong A_7 \oplus A_1.$$

Furthermore a maximal abelian subalgebra $\mathfrak{t}$ is decomposed into $\mathfrak{t} = (A_7 \cap \mathfrak{t}) \oplus (A_1 \cap \mathfrak{t})$. Hence we can write

$$\tau = \tau_{T_1} \circ \tau_{T_2}, \quad \sqrt{-1}T_1 \in A_7 \cap \mathfrak{t},\ \sqrt{-1}T_2 \in A_1 \cap \mathfrak{t}.$$

We define $v_i \in \sqrt{-1}(A_7 \cap \mathfrak{t})$, $i \in \Lambda := \{0, 2, 4, 5, 6, 7, 8\}$ and $v_1 \in \sqrt{-1}(A_1 \cap \mathfrak{t})$ by $\alpha_i(v_j) = \delta_{ij}$. Since $(\tau_{T_1}|_{A_7})^2 = \mathrm{Id}_{A_7}$ and $(\tau_{T_2}|_{A_1})^2 = \mathrm{Id}_{A_1}$, it follows from Lemma 2.2 and Remark 2.1 that there exist $\mu_1 \in \mathrm{Int}(A_7)$ and $\mu_2 \in \mathrm{Int}(A_1)$ such that

$$\mu_1(T_1) \equiv \begin{cases} 0 & \mathrm{mod}\ 2\Pi_{A_7}, \\ v_i & \mathrm{mod}\ 2\Pi_{A_7}\ (i \in \Lambda), \end{cases} \qquad \mu_2(T_2) \equiv \begin{cases} 0 & \mathrm{mod}\ 2\Pi_{A_1}, \\ v_1 & \mathrm{mod}\ 2\Pi_{A_1}, \end{cases} \tag{7.2}$$

where Π_{A_l} denotes the fundamental root system of Type A_l. Therefore considering Lemma 2.3 we may assume

$$T_1 = \begin{cases} 2m_0v_0 + 2m_2v_2 + 2m_3v_3 + \cdots + 2m_8v_8, \\ v_i + 2m_0v_0 + 2m_2v_2 + 2m_3v_3 + \cdots + 2m_8v_8, \end{cases} \qquad T_2 = \begin{cases} 2m_1v_1, \\ v_1 + 2m_1v_1, \end{cases} \tag{7.3}$$

where $i = 2, 4, 5, 6$ and $m_0, m_1, m_2, m_4, \dots, m_8 \in \mathbb{Z}$. Consequently τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following automorphisms:

$$(7.4)\qquad \begin{cases} \mathrm{Ad}(\exp \pi\sqrt{-1}(2m_0v_0 + 2m_1v_1 + 2m_2v_2 + 2m_4v_4 + \cdots + 2m_8v_8)), \\ \mathrm{Ad}(\exp \pi\sqrt{-1}(v_i + 2m_0v_0 + 2m_1v_1 + 2m_2v_2 + 2m_4v_4 + \cdots + 2m_8v_8)), \\ \mathrm{Ad}(\exp \pi\sqrt{-1}(v_1 + v_j + 2m_0v_0 + 2m_1v_1 + 2m_2v_2 + 2m_4v_4 + \cdots + 2m_8v_8)), \end{cases}$$

where $i = 1, 2, 4, 5, 6$, $j = 2, 4, 5, 6$ and $m_0, m_1, m_2, m_4, \dots, m_8 \in \mathbb{Z}$.

Now we compute v_j. Put $v_1 = \sum_{i=1}^{8} a_i K_i$, $a_i \in \mathbb{R}$. Since $A_1 \cap \mathfrak{t} = \mathbb{R}\sqrt{-1}H_{\alpha_1}$ and

$$A_1 \cap \mathfrak{t} = \{\sqrt{-1}H \in \mathfrak{t};\ \alpha_j(H) = 0,\ j = 0, 2, 4, 5, 6, 7, 8\},$$

we have $a_1 = 1$, $a_2 = a_4 = \cdots = a_8 = 0$ and $a_1 + 2a_3 = 0$. Hence we obtain $v_1 = K_1 - (1/2)K_3$.

Moreover, since $A_7 \cap \mathfrak{t} = \{\sqrt{-1}H \in \mathfrak{t};\ \alpha_1(H) = 0\}$, we can put $v_i = \sum_{k=2}^{8} b_k^i K_k$, $b_k^i \in \mathbb{R}$, $i \in \Lambda$. Then computing simultaneous equations $\alpha_i(v_j) = \delta_{ij}$, $i, j \in \Lambda$, we obtain

$$(7.5)\qquad \begin{aligned} &v_0 = -\frac{1}{4}K_3, \quad v_1 = K_1 - \frac{1}{2}K_3, \quad v_2 = K_2 - \frac{3}{4}K_3, \quad v_4 = K_4 - \frac{3}{2}K_3, \\ &v_5 = K_5 - \frac{5}{4}K_3, \quad v_6 = K_6 - K_3, \quad v_7 = K_7 - \frac{3}{4}K_3, \quad v_8 = K_8 - \frac{1}{2}K_3. \end{aligned}$$

Thus (7.4) implies that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following:

$$(7.6)\qquad \tau_{mK_3}, \quad \tau_{v_i+mK_3}, \quad \tau_{v_1+v_j+mK_3},$$

where $m = -((1/2)m_0 + m_1 + (3/2)m_2 + 3m_4 + (5/2)m_5 + (3/2)m_7 + m_8)$. From (7.5) if $i, j = 2, 5$, then $\tau^2 \neq \mathrm{Id}$. Therefore $i = 1, 4, 6$ and $j = 4, 6$. Hence τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_3 + K_j, \quad K_1 + K_k, \quad K_1 + K_3 + K_k,$$

where $i = 1, 3, 4, 6$, $j = 1, 4, 6$ and $k = 4, 6$. If $h = K_1$, then $\mathfrak{g}^{\tau_{K_1}} \cong D_8$ (cf. Theorem 5.15 of Chapter X of [6]). Furthermore

$$\mathfrak{h} \cap \mathfrak{g}^{\tau_{K_1}} = \mathfrak{t} \oplus \sum_{\substack{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \\ \alpha(K_3) \equiv 0 \bmod 4 \\ \alpha(K_1) \equiv 0 \bmod 2}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \subset \mathfrak{h} \quad (\cong \mathfrak{su}(8) \oplus \mathfrak{su}(2)).$$

In this case, $\tau_{K_1}|_{A_7} = \mathrm{Id}$ and $A_1^{\tau_{K_1}} \cong \mathbb{R}$, and hence

$$\mathfrak{h} \cap \mathfrak{k} \cong A_7 \oplus \mathbb{R}.$$

Similarly as above, we can get $(\mathfrak{g}^\tau, \mathfrak{h} \cap \mathfrak{g}^\tau)$ for each $\tau = \tau_h$.

Now we consider the reflection $t_{\alpha_1} \in \mathrm{Int}(\mathfrak{su}_{\alpha_1}(2)) \subset \mathrm{Int}(\mathfrak{h})$. It is easy to check that t_{α_1} maps $K_1 \mapsto -K_1 + K_3$, $K_1 + K_4 \mapsto -K_1 + K_3 + K_4$, $K_1 + K_6 \mapsto -K_1 + K_3 + K_6$ and $K_3 \mapsto K_3$. Therefore we have $\tau_{K_1} \approx \tau_{K_1+K_3}$, $\tau_{K_1+K_4} \approx \tau_{K_1+K_3+K_4}$ and $\tau_{K_1+K_6} \approx \tau_{K_1+K_3+K_6}$, where we write $\tau_H \approx \tau_{H'}$ if τ_H is conjugate to $\tau_{H'}$ within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$.

Next, we consider the case where $\mathfrak{g}$ is of type $\mathfrak{e}_8$ and $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_6)$. Then by (4.2) (ii), we have $\mathfrak{h} \cong A_3 \oplus D_5$. By a computation similar to the above case, we obtain

$$\begin{aligned}
&v_0 = -\frac{1}{4}K_6, \quad v_1 = K_1 - \frac{1}{2}K_6, \quad v_2 = K_2 - \frac{3}{4}K_6, \quad v_3 = K_3 - K_6,\\
&v_4 = K_4 - \frac{3}{2}K_6, \quad v_5 = K_5 - \frac{5}{4}K_3, \quad v_7 = K_7 - \frac{3}{4}K_6, \quad v_8 = K_8 - \frac{1}{2}K_6.
\end{aligned} \tag{7.7}$$

Then, considering Lemma 2.3, τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following automorphisms:

$$\tau_{mK_6}, \quad \tau_{v_a+mK_6}, \quad \tau_{v_b+v_c+mK_6}, \tag{7.8}$$

where $a = 1, 2, 3, 7, 8$, $b = 1, 2, 3$, $c = 7, 8$, and m is equal to that of the above case. Since $\tau^2 = \mathrm{Id}$, it follows from (7.7) that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_6, \quad K_2 + K_7, \quad K_k + K_8, \quad K_k + K_6 + K_8, \quad K_2 + K_6 + K_7,$$

where $i = 1, 3, 6, 8$, $j = 1, 3, 8$ and $k = 1, 3$. By a computation similar to the above case, we obtain $\mathfrak{g}^\tau$ and $\mathfrak{h} \cap \mathfrak{g}^\tau$. We put

$$\begin{aligned}
&\beta_1 := \begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 4 & 5 & 6 & 4 & 2 \end{pmatrix}, \quad \beta_2 := \begin{pmatrix} & & & & 1 & & \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix},\\
&\beta_3 := \begin{pmatrix} & & & & 1 & & \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{7.9}$$

It is easy to check that

$$t_{\beta_1} \circ t_{\alpha_8}(K_6) = -3K_6 + 4K_7, \quad t_{\beta_2} \circ t_{\alpha_1}(K_6) = K_6, \quad t_{\beta_3} \circ t_{\alpha_3}(K_6) = K_6,$$

and so $t_{\beta_1} \circ t_{\alpha_8}$, $t_{\beta_2} \circ t_{\alpha_1}$, $t_{\beta_3} \circ t_{\alpha_3} \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$. Moreover we have

$$\begin{aligned}
&t_{\beta_1} \circ t_{\alpha_8}(K_8) = -K_6 + 2K_7 - K_8, \quad t_{\beta_2} \circ t_{\alpha_1}(K_1) = -K_1 + K_6,\\
&t_{\beta_2} \circ t_{\alpha_1}(K_1 + K_8) = -K_1 + K_6 + K_8, \quad t_{\beta_3} \circ t_{\alpha_3}(K_3) = 2K_1 - K_3 + K_6,\\
&t_{\beta_3} \circ t_{\alpha_3}(K_3 + K_8) = 2K_1 - K_3 + K_6 + K_8.
\end{aligned}$$

Therefore we have

$$\tau_{K_6+K_8} \approx \tau_{K_8}, \quad \tau_{K_1+K_6} \approx \tau_{K_1}, \quad \tau_{K_3+K_6} \approx \tau_{K_3}, \quad \tau_{K_1+K_6+K_8} \approx \tau_{K_1+K_8}, \quad \tau_{K_3+K_6+K_8} \approx \tau_{K_3+K_8}.$$

For the case where $\mathfrak{g} = \mathfrak{e}_7$ and $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_4)$, we can check that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_k, \quad K_3 + K_7, \quad K_l + K_m + K_n,$$
$$K_p + K_3 + K_7, \quad K_1 + K_2 + K_4 + K_6, \quad K_2 + K_3 + K_4 + K_7,$$

where $i = 1, 2, 4, 6$, $j, k = 1, 2, 4, 6$ $(j < k)$, $l, m, n = 1, 2, 4, 6$ $(l < m < n)$ and $p = 2, 4$. Using the reflection $t_{\alpha_2} \in \mathrm{Int}(\mathfrak{h})$ we have

$$\tau_{K_2+K_4} \approx \tau_{K_2}, \quad \tau_{K_1+K_2+K_4} \approx \tau_{K_1+K_2}, \quad \tau_{K_2+K_4+K_6} \approx \tau_{K_2+K_6},$$
$$\tau_{K_1+K_2+K_4+K_6} \approx \tau_{K_1+K_2+K_6}, \quad \tau_{K_2+K_3+K_4+K_7} \approx \tau_{K_2+K_3+K_7},$$

and since $t_{\alpha_5+\alpha_6+\alpha_7} \circ t_{\alpha_6} \in \mathrm{Int}(\mathfrak{h})$, we have

$$\tau_{K_6} \approx \tau_{K_4+K_6}.$$

Furthermore put $\gamma_1 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Then $t_{\gamma_1} \circ t_{\alpha_1} \in \mathrm{Int}(\mathfrak{h})$ gives the following conjugations:

$$\tau_{K_1+K_4} \approx \tau_{K_1}, \quad \tau_{K_1+K_4+K_6} \approx \tau_{K_1+K_6}.$$

Finally we consider an involution $\varphi \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ (see (5.11)). Then it is easy to see that

$$\varphi(K_1) = K_6 - 2K_7, \quad \varphi(K_2) = K_2 - 2K_7, \quad \varphi(K_3) = K_5 - 3K_7, \quad \varphi(K_4) = K_4 + 4K_7,$$
$$\varphi(K_5) = K_3 - 3K_7, \quad \varphi(K_6) = K_1 - 2K_7, \quad \varphi(K_7) = -K_7,$$

and therefore φ gives the following conjugations:

$$\tau_{K_1} \approx \tau_{K_6}, \quad \tau_{K_1+K_2} \approx \tau_{K_2+K_6}.$$

For the case where $\mathfrak{g} = \mathfrak{f}_4$ and $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_3)$, we can check that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_k, \quad K_1 + K_3 + K_4.$$

Here $i = 1, 3, 4$ and $j, k = 1, 3, 4$ $(j < k)$. Using the reflection $t_{\alpha_4} \in \mathrm{Int}(\mathfrak{h})$ and $t_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}$ we have

$$\tau_{K_3+K_4} \approx \tau_{K_4}, \quad \tau_{K_1+K_3+K_4} \approx \tau_{K_1+K_4}, \quad \tau_{K_1+K_3} \approx \tau_{K_1}.$$

Consequently we obtain the following proposition.

Proposition 7.1. *Suppose that* $\dim \mathfrak{z} = 0$. *Let* τ *be an involution of* $\mathfrak{g}$ *such that* $\tau \circ \sigma = \sigma \circ \tau$. *Then* τ *is conjugate within* $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *to one of automorphisms listed in* Table 3.

Table 3. $\dim \mathfrak{z} = 0$, $\tau \circ \sigma = \sigma \circ \tau$, $\sigma = \tau_{(1/2)H}$ and $\mathfrak{k} = \mathfrak{g}^{\tau}$.

$(\mathfrak{g}, \mathfrak{h}, H)$	h $(\tau = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathfrak{su}(2), K_3)$	K_1	D_8	$A_7 \oplus \mathbb{R}$
	K_3	$E_7 \oplus A_1$	$A_7 \oplus A_1$
	K_4	$E_7 \oplus A_1$	$A_5 \oplus A_1 \oplus A_1 \oplus \mathbb{R}$
	K_6	D_8	$A_3 \oplus A_3 \oplus A_1 \oplus \mathbb{R}$
	$K_3 + K_4$	D_8	$A_5 \oplus A_1 \oplus A_1 \oplus \mathbb{R}$
	$K_3 + K_6$	$E_7 \oplus A_1$	$A_3 \oplus A_3 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_4$	$E_7 \oplus A_1$	$A_5 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_6$	D_8	$A_5 \oplus A_1 \oplus \mathbb{R}$
$(\mathfrak{e}_8, \mathfrak{so}(10) \oplus \mathfrak{so}(6), K_6)$	K_1	D_8	$D_4 \oplus D_3 \oplus \mathbb{R}$
	K_3	$E_7 \oplus A_1$	$D_3 \oplus D_3 \oplus D_2$
	K_6	D_8	$D_5 \oplus D_3$
	K_8	$E_7 \oplus A_1$	$D_5 \oplus D_2 \oplus \mathbb{R}$
	$K_1 + K_8$	$E_7 \oplus A_1$	$D_4 \oplus D_2 \oplus \mathbb{R}^2$
	$K_2 + K_7$	$E_7 \oplus A_1$	$A_4 \oplus A_2 \oplus \mathbb{R}^2$
	$K_3 + K_8$	D_8	$D_3 \oplus D_2 \oplus D_2 \oplus \mathbb{R}$
	$K_2 + K_6 + K_7$	D_8	$A_4 \oplus A_2 \oplus \mathbb{R}^2$
$(\mathfrak{e}_7, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	K_1	$D_6 \oplus A_1$	$D_3 \oplus D_2 \oplus A_1 \oplus \mathbb{R}$
	K_2	A_7	$D_3 \oplus D_3 \oplus \mathbb{R}$
	K_4	$D_6 \oplus A_1$	$D_3 \oplus D_3 \oplus A_1$
	$K_1 + K_2$	$E_6 \oplus \mathbb{R}$	$D_3 \oplus D_2 \oplus \mathbb{R}^2$
	$K_1 + K_3$	$D_6 \oplus A_1$	$D_3 \oplus D_2 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_6$	$D_6 \oplus A_1$	$D_2 \oplus D_2 \oplus A_1 \oplus \mathbb{R}^2$
	$K_3 + K_4$	$D_6 \oplus A_1$	$D_3 \oplus D_3 \oplus A_1$
	$K_3 + K_7$	A_7	$A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_2 + K_6$	A_7	$D_2 \oplus D_2 \oplus \mathbb{R}^3$
	$K_1 + K_3 + K_4$	$D_6 \oplus A_1$	$D_3 \oplus D_2 \oplus A_1 \oplus \mathbb{R}$
	$K_2 + K_3 + K_7$	$D_6 \oplus A_1$	$A_2 \oplus A_2 \oplus \mathbb{R}^3$
	$K_3 + K_4 + K_7$	$E_6 \oplus \mathbb{R}$	$A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R}^2$
$(\mathfrak{f}_4, \mathfrak{so}(6) \oplus \mathfrak{so}(3), K_3)$	K_1	$B_3 \oplus A_1$	$D_2 \oplus C_2 \oplus \mathbb{R}$
	K_3	C_4	$D_3 \oplus C_2$
	K_4	C_4	$D_3 \oplus \mathbb{R}$
	$K_1 + K_4$	$B_3 \oplus A_1$	$D_3 \oplus \mathbb{R}$
$(\mathfrak{g}, \mathfrak{h}, H)$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_7, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	φ	$E_6 \oplus \mathbb{R}$	$D_8 \oplus C_1$
	$\varphi \circ \tau_{K_2}$	A_7	$D_8 \oplus D_1$
	$\varphi \circ \tau_{K_4}$	A_7	$D_8 \oplus C_1$
	$\varphi \circ \tau_{K_2} \circ \tau_{K_4}$	A_7	$D_8 \oplus D_1$

$\varphi : E_{\alpha_1} \mapsto E_{\alpha_6},\ E_{\alpha_2} \mapsto E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_5},\ E_{\alpha_4} \mapsto E_{\alpha_4},\ E_{\alpha_7} \mapsto E_{\alpha_0}$

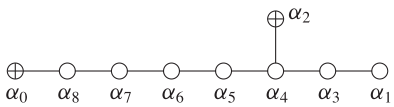
8. The case where $\dim \mathfrak{z} = 1$, $\tau \circ \sigma = \sigma \circ \tau$

Let $(G/H, \langle\ ,\ \rangle, \sigma)$ be a compact Riemannian 4-symmetric space such that G is simple and $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ for some $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_i = 3$. By Remark 2.2, we have $\dim \mathfrak{z} = 1$. We shall classify the equivalence classes of involutions τ such that $\tau \circ \sigma = \sigma \circ \tau$. According to Section 3 and Jiménez [7], 4-symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ satisfying the condition $\dim \mathfrak{z} = 1$ are given by

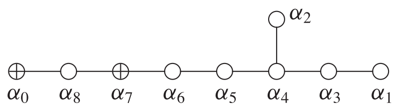
$$\begin{aligned}
&(\mathfrak{e}_6, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{e}_7, \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}), \\
&(\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathbb{R}), \quad (\mathfrak{e}_8, \mathfrak{su}(2) \oplus \mathfrak{e}_6 \oplus \mathbb{R}), \\
&(\mathfrak{f}_4, \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathbb{R}).
\end{aligned} \tag{8.1}$$

Suppose that $\mathfrak{g}$ is of type $\mathfrak{e}_8$. From Section 3, the Dynkin diagram of $\mathfrak{h}$ is one of the following:

(i)



(ii)



CASE (i): In this case, $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_2)$. From Lemma 3.2, the possibilities of positive roots whose coefficients of α_2 are 3 are as follows:

(8.2)

$$\begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 3 & 4 & 5 & 3 & 1 \end{pmatrix}, \begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 3 & 4 & 5 & 3 & 2 \end{pmatrix}, \begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 3 & 4 & 5 & 4 & 2 \end{pmatrix},$$
$$\begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 3 & 4 & 6 & 4 & 2 \end{pmatrix}, \begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 3 & 5 & 6 & 4 & 2 \end{pmatrix}, \begin{pmatrix} & & & & 3 & & \\ 1 & 2 & 4 & 5 & 6 & 4 & 2 \end{pmatrix},$$
$$\begin{pmatrix} & & & & 3 & & \\ 1 & 3 & 4 & 5 & 6 & 4 & 2 \end{pmatrix}, \begin{pmatrix} & & & & 3 & & \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{pmatrix}.$$

Since $\tau(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h})$ and $\delta + \alpha_j \notin \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ $(j \neq 2)$, we have $\tau(\delta) + \alpha_k \notin \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ $(k \neq 2)$. Thus it follows from (8.2) that $\tau(\delta) = \delta$. If τ satisfies

$$\tau(\alpha_1) = \alpha_8, \quad \tau(\alpha_3) = \alpha_7, \quad \tau(\alpha_4) = \alpha_6, \quad \tau(\alpha_5) = \alpha_5,$$

we get

$$\begin{pmatrix} & & & & 3 & & \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{pmatrix} = \tau(\delta) = 3\tau(\alpha_2) + \begin{pmatrix} & & & & 0 & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

Hence we have

$$3\tau(\alpha_2) = \begin{pmatrix} & & & & 3 & & \\ 0 & -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix},$$

which is a contradiction. Therefore τ satisfies $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$. Hence from Proposition 5.3 of Chapter IX of [6], τ has a form τ_H for a suitable element $H \in \sqrt{-1}\mathfrak{t}$. From (8.1), we have

$$\mathfrak{h} \cong A_7 \oplus \mathbb{R}\sqrt{-1}K_2 \cong \mathfrak{su}(8) \oplus \mathbb{R} \quad \text{and} \quad \mathfrak{t} = (A_7 \cap \mathfrak{t}) \oplus \mathbb{R}\sqrt{-1}K_2, \tag{8.3}$$

and we can write

$$\tau = \tau_H = \tau_{T+kK_2} = \tau_T \circ \tau_{kK_2}, \quad \sqrt{-1}T \in A_7 \cap \mathfrak{t},\, k \in \mathbb{R}.$$

Note that $\tau_T = \tau|_{A_7} \colon A_7 \to A_7$ and $(\tau_T)^2 = \mathrm{Id}$ on A_7.

We define $v_i \in \sqrt{-1}(A_7 \cap \mathfrak{t})$, $i \in \Lambda := \{1, 3, 4, 5, 6, 7, 8\}$ by $\alpha_i(v_j) = \delta_{ij}$. From Lemma 2.3, we may suppose that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following automorphisms:

$$\begin{cases} \mathrm{Ad}(\exp \pi\sqrt{-1}(2m_1 v_1 + 2m_3 v_3 + \cdots + 2m_8 v_8 + kK_2)), \\ \mathrm{Ad}(\exp \pi\sqrt{-1}(v_i + 2m_1 v_1 + 2m_3 v_3 + \cdots + 2m_8 v_8 + kK_2)), \end{cases} \tag{8.4}$$

where $i = 1, 3, 4, 5$ and $m_1, m_3, m_4, \ldots, m_8 \in \mathbb{Z}$. Put $K_2 = \sum_{i=1}^{8} b_i H_{\alpha_i}$. Then we have

$$\delta_{j2} = \alpha_j(K_2) = \sum_{i=1}^{8} b_i \alpha_j(H_{\alpha_i}), \quad \text{for} \quad j = 1, 2, \ldots, 8,$$

and therefore

$$b_1 - \frac{b_3}{2} = 0, \quad b_2 - \frac{b_4}{2}\alpha_2(H_{\alpha_2}) = 1, \quad -\frac{b_1}{2} + b_3 - \frac{b_4}{2} = 0, \quad -\frac{b_2}{2} - \frac{b_3}{2} + b_4 - \frac{b_5}{2} = 0,$$
$$-\frac{b_4}{2} + \frac{b_5}{2} - \frac{b_6}{2} = 0, \quad -\frac{b_5}{2} + b_6 - \frac{b_7}{2} = 0, \quad -\frac{b_6}{2} + b_7 - \frac{b_8}{2} = 0, \quad -\frac{b_7}{2} + b_8 = 0.$$

Indeed if $j = 1$, considering the α_1 series containing α_i, we have $\alpha_i(H_{\alpha_1}) = 0$ for $i \neq 1, 3$ and $2\alpha_3(H_{\alpha_1})/\alpha_1(H_{\alpha_1}) = -1$. Thus we get

$$\begin{aligned} 0 = \alpha_1(K_2) &= \sum_{i=1}^{8} b_i \alpha_1(H_{\alpha_i}) = b_1\alpha_1(H_{\alpha_1}) + b_3\alpha_1(H_{\alpha_3}) \\ &= b_1\alpha_1(H_{\alpha_1}) + b_3\left(-\frac{1}{2}\alpha_1(H_{\alpha_1})\right) = \left(b_1 - \frac{b_3}{2}\right)\alpha_1(H_{\alpha_1}). \end{aligned}$$

We can obtain the other equations by a similar computation as above.

Computing these simultaneous equations we have

$$K_2 = \frac{c_8}{3}(5H_{\alpha_1} + 8H_{\alpha_2} + 10H_{\alpha_3} + 15H_{\alpha_4} + 12H_{\alpha_5} + 9H_{\alpha_6} + 6H_{\alpha_7} + 3H_{\alpha_8}). \tag{8.5}$$

Now put $v_1 = a_1K_1 + \cdots + a_8K_8$, $a_1, \ldots, a_8 \in \mathbb{R}$. Then we get $a_3 = \cdots = a_8 = 0$ and $a_1 = 1$, since $\alpha_i(v_1) = \delta_{i1}$ for $i \in \Lambda$. Thus we have $v_1 = K_1 + a_2K_2$. Since $v_1 \perp K_2$, it follows from (8.5) that $0 = (5/3)c_8 + (8/3)a_2c_8$ and therefore $a_2 = -5/8$. Hence we have $v_1 = K_1 - (5/8)K_2$. By a similar computation, we obtain

$$\begin{aligned} &v_1 = K_1 - \frac{5}{8}K_2, \quad v_3 = -\frac{5}{4}K_2 + K_3, \quad v_4 = -\frac{15}{8}K_2 + K_4, \quad v_5 = -\frac{3}{2}K_2 + K_5, \\ &v_6 = -\frac{9}{8}K_2 + K_6, \quad v_7 = -\frac{3}{4}K_2 + K_7, \quad v_8 = -\frac{3}{8}K_2 + K_8. \end{aligned} \tag{8.6}$$

Therefore by (8.4) and (8.6) it follows that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following:

$$\tau_{mK_2}, \quad \tau_{v_i+mK_2}, \tag{8.7}$$

where $m = -(1/4)(5m_1 + 10m_3 + 15m_4 + 12m_5 + 9m_6 + 6m_7 + 3m_8 - 2k)$. Moreover, since $\tau^2 = \mathrm{Id}$, it follows from (8.6) that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i,\ K_j + K_2, \quad i = 1, 2, 3, 4, 5,\ j = 1, 3, 4, 5.$$

By a computation similar to Section 7 we can obtain $(\mathfrak{g}^{\tau_h}, \mathfrak{h} \cap \mathfrak{g}^{\tau_h})$ for each h.

REMARK 8.1. From Lemma 2.3, we can see that $\tau_{v_8}|_{A_7}$ is conjugate within $\mathrm{Int}(A_7)$ ($\subset \mathrm{Int}(\mathfrak{h})$) to $\tau_{v_1}|_{A_7}$. Therefore by the above argument, τ_{K_8} is conjugate within $\mathrm{Int}(\mathfrak{h})$ to τ_{K_1} or $\tau_{K_1+K_2}$. However, $\mathfrak{g}^{\tau_{K_8}} \ncong \mathfrak{g}^{\tau_{K_1}}$, and hence $\tau_{K_8} \approx \tau_{K_1+K_2}$.

CASE (ii): In this case, $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_7)$ and

$$\begin{aligned} \mathfrak{h} &\cong A_1 \oplus E_6 \oplus \mathbb{R}\sqrt{-1}K_7 \cong \mathfrak{su}(2) \oplus \mathfrak{e}_6 \oplus \mathbb{R}, \\ \mathfrak{t} &= (A_1 \cap \mathfrak{t}) \oplus (E_6 \cap \mathfrak{t}) \oplus \mathbb{R}\sqrt{-1}K_7. \end{aligned}$$

By a computation similar to the case (i), we have $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$. Hence we can write

$$\tau = \tau_{T_1} \circ \tau_{T_2} \circ \tau_{kK_7},$$

where $\sqrt{-1}T_1 \in A_1 \cap \mathfrak{t}$, $\sqrt{-1}T_2 \in E_6 \cap \mathfrak{t}$, $k \in \mathbb{R}$. We define $v_8 \in \sqrt{-1}(A_1 \cap \mathfrak{t})$ and $v_a \in \sqrt{-1}(E_6 \cap \mathfrak{t})$, $a \in \Lambda := \{1, 2, 3, 4, 5, 6\}$ by $\alpha_i(v_j) = \delta_{ij}$. Then from Lemma 2.3, we may assume τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of following automorphisms:

$$\begin{cases} \mathrm{Ad}(\exp \pi\sqrt{-1}(2m_1v_1 + \cdots + 2m_6v_6 + 2m_8v_8 + kK_7)), \\ \mathrm{Ad}(\exp \pi\sqrt{-1}(v_a + 2m_1v_1 + \cdots + 2m_6v_6 + 2m_8v_8 + kK_7)), \\ \mathrm{Ad}(\exp \pi\sqrt{-1}(v_8 + v_b + 2m_1v_1 + \cdots + 2m_6v_6 + 2m_8v_8 + kK_7)), \end{cases} \tag{8.8}$$

where $a = 1, 2, 8$, $b = 1, 2$ and $m_1, \ldots, m_6, m_8 \in \mathbb{Z}$. Furthermore we obtain

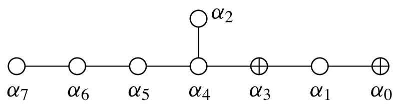
$$v_1 = K_1 - \frac{2}{3}K_7, \quad v_2 = K_2 - K_7, \quad v_3 = K_3 - \frac{4}{3}K_7, \quad v_4 = K_4 - 2K_7,$$
$$v_5 = K_6 - \frac{5}{3}K_7, \quad v_6 = K_6 - \frac{4}{3}K_7, \quad v_8 = K_8 - \frac{1}{2}K_7.$$

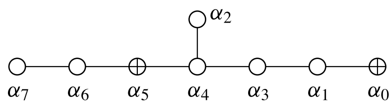
Similarly as in Case (i), we can see that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_7, \quad K_k + K_8, \quad K_k + K_7 + K_8,$$

where $i = 1, 2, 7, 8$, $j = 1, 2, 8$ and $k = 1, 2$. It is easy to check that the reflection $t_{\alpha_8} \in \mathrm{Int}(\mathfrak{h})$ maps $K_7 + K_8 \mapsto 2K_7 - K_8$, $K_1 + K_7 + K_8 \mapsto K_1 + 2K_7 - K_8$, $K_2 + K_7 + K_8 \mapsto K_2 + 2K_7 - K_8$ and $K_7 \mapsto K_7$. Therefore we have $\tau_{K_7+K_8} \approx \tau_{K_8}$, $\tau_{K_1+K_7+K_8} \approx \tau_{K_1+K_8}$ and $\tau_{K_2+K_7+K_8} \approx \tau_{K_2+K_8}$.

In the case where $\mathfrak{g} = \mathfrak{e}_7$, the Dynkin diagram of $\mathfrak{h}$ is one of the following:

(i) 

(ii) 

CASE (i): In this case, $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_3)$ and $\mathfrak{h} \cong A_1 \oplus A_5 \oplus \mathbb{R}\sqrt{-1}K_3$. By an argument similar to the above, we can see that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_3, \quad K_k + K_1, \quad K_k + K_1 + K_3,$$

where $i = 1, 2, 3, 4, 5$, $j = 1, 2, 4, 5$ and $k = 2, 4, 5$. Using the reflection $t_{\alpha_1} \in \mathrm{Int}(\mathfrak{h})$ we obtain

$$\tau_{K_1+K_3} \approx \tau_{K_1}, \quad \tau_{K_1+K_2+K_3} \approx \tau_{K_1+K_2}, \quad \tau_{K_1+K_3+K_4} \approx \tau_{K_2+K_4}, \quad \tau_{K_1+K_3+K_5} \approx \tau_{K_1+K_5}.$$

Furthermore, similarly as in Remark 8.1 we get $\tau_{K_2+K_3} \approx \tau_{K_7}$.

CASE (ii): In this case, $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_5)$ and $\mathfrak{h} \cong A_2 \oplus A_4 \oplus \mathbb{R}\sqrt{-1}K_5$. Moreover, τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_5, \quad K_k + K_6, \quad K_k + K_5 + K_6,$$

where $i = 1, 3, 5, 6$, $j = 1, 3, 6$ and $k = 1, 3$. Similarly as in Remark 8.1 we get $\tau_{K_5+K_6} \approx \tau_{K_7}$, $\tau_{K_1+K_5+K_6} \approx \tau_{K_1+K_7}$ and $\tau_{K_3+K_5+K_6} \approx \tau_{K_3+K_7}$.

If $\mathfrak{g} = \mathfrak{f}_4$, then $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_2)$ and $\mathfrak{h} \cong A_1 \oplus A_2 \oplus \mathbb{R}\sqrt{-1}K_2$. In this case, τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_k, \quad K_1 + K_2 + K_3,$$

where $i = 1, 2, 3$ and $j, k = 1, 2, 3$ $(j \neq k)$. Using the reflection $t_{\alpha_1} \in \mathrm{Int}(\mathfrak{h})$ we have $\tau_{K_1+K_2} \approx \tau_{K_1}$ and $\tau_{K_1+K_2+K_3} \approx \tau_{K_1+K_3}$.

If $\mathfrak{g} = \mathfrak{g}_2$, then $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_1)$ and $\mathfrak{h} \cong A_1 \oplus \mathbb{R}\sqrt{-1}K_1$. In this case, τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of

$$K_i,\ K_1 + K_2, \quad i = 1, 2.$$

Using the reflection $t_{\alpha_2} \in \mathrm{Int}(\mathfrak{h})$ we have $\tau_{K_1+K_2} \approx \tau_{K_2}$.

If $\mathfrak{g} = \mathfrak{e}_6$, then $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_4)$ and $\mathfrak{h} \cong A_1 \oplus A_2 \oplus A_2 \oplus \mathbb{R}\sqrt{-1}K_4$. By an argument similar to the case where $\mathfrak{g} = \mathfrak{e}_8$, we obtain $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$ or

$$\tau(\alpha_1) = \alpha_6, \quad \tau(\alpha_3) = \alpha_5, \quad \tau(\alpha_4) = \alpha_4, \quad \tau(\alpha_2) = \alpha_2.$$

If $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$, then τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_j + K_k, \quad K_l + K_m + K_n, \quad K_1 + K_2 + K_4 + K_5,$$

where $i = 1, 2, 4, 5$, $j, k = 1, 2, 4, 5$ $(j < k)$ and $l, m, n = 1, 2, 4, 5$ $(l < m < n)$. Using the reflection $t_{\alpha_2} \in \mathrm{Int}(\mathfrak{h})$ we have

$$\tau_{K_2+K_4} \approx \tau_{K_2}, \quad \tau_{K_1+K_2+K_4} \approx \tau_{K_1+K_2}, \quad \tau_{K_2+K_4+K_5} \approx \tau_{K_2+K_5}, \quad \tau_{K_1+K_2+K_4+K_5} \approx \tau_{K_1+K_2+K_5}.$$

Next suppose that $\mathfrak{g} = \mathfrak{e}_6$ and τ satisfies

$$\tau(\alpha_1) = \alpha_6, \quad \tau(\alpha_3) = \alpha_5, \quad \tau(\alpha_4) = \alpha_4, \quad \tau(\alpha_2) = \alpha_2.$$

Let $\mathfrak{t}_\pm$ be the (± 1)-eigenspaces of $\tau|_{\mathfrak{t}}$, respectively. Then we have

$$\mathfrak{t}_+ = \mathrm{span}\{K_1 + K_6, K_2, K_3 + K_5, K_4\}, \quad \mathfrak{t}_- = \mathrm{span}\{K_1 - K_6, K_3 - K_5\}$$

It is known that there exists an involutive automorphism ψ of outer type satisfying

$$\psi(E_{\alpha_1}) = E_{\alpha_6}, \quad \psi(E_{\alpha_2}) = E_{\alpha_2}, \quad \psi(E_{\alpha_3}) = E_{\alpha_5}, \quad \psi(E_{\alpha_4}) = E_{\alpha_4}. \tag{8.9}$$

Therefore there exists $\sqrt{-1}h_+ \in \mathfrak{t}_+$ such that $\tau_{h_+}^2 = \mathrm{Id}$ and $\tau \approx \psi \circ \tau_{h_+}$. Then by an argument similar to that in Section 7, we can see that τ is conjugate within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ to one of the following involutions:

$$\psi, \quad \psi \circ \tau_{K_2}, \quad \psi \circ \tau_{K_4}, \quad \psi \circ \tau_{K_2+K_4}.$$

Since $\mathfrak{su}_{\alpha_2}(2) \subset \mathfrak{g}^\psi$ and $t_{\alpha_2}(K_2) = -K_2 + K_4$, we obtain $\psi \circ t_{\alpha_2} = t_{\alpha_2} \circ \psi$ and

$$\psi \circ \tau_{K_2+K_4} = \psi \circ \tau_{t_{\alpha_2}(K_2)} = \psi \circ t_{\alpha_2} \circ \tau_{K_2} \circ t_{\alpha_2}^{-1} = t_{\alpha_2}(\psi \circ \tau_{K_2}) t_{\alpha_2}^{-1}.$$

Thus we obtain $\psi \circ \tau_{K_2} \approx \psi \circ \tau_{K_2+K_4}$. Furthermore by an argument similar to that in Section 7, we can compute $\dim(\mathfrak{h} \cap \mathfrak{g}^{\tau_h})$ and $\dim \mathfrak{g}^{\tau_h}$ for each h. Consequently we have the following proposition.

Proposition 8.1. *Suppose that* $\dim \mathfrak{z} = 1$ *and* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ *for some* $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *with* $m_i = 3$. *Let* τ *be an involution of* $\mathfrak{g}$ *such that* $\tau \circ \sigma = \sigma \circ \tau$. *Then* τ *is conjugate within* $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *to one of automorphisms listed in* Table 4.

Table 4. $\dim \mathfrak{z} = 1$, $\tau \circ \sigma = \sigma \circ \tau$, $\sigma = \tau_{(1/2)H}$ and $\mathfrak{k} = \mathfrak{g}^{\tau}$.

$(\mathfrak{g}, \mathfrak{h}, H)$	h $(\tau = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathbb{R}, K_2)$	K_1	D_8	$A_6 \oplus \mathbb{R}^2$
	K_2	D_8	$A_7 \oplus \mathbb{R}$
	K_3	$E_7 \oplus A_1$	$A_5 \oplus A_1 \oplus \mathbb{R}^2$
	K_4	$E_7 \oplus A_1$	$A_4 \oplus A_2 \oplus \mathbb{R}^2$
	K_5	D_8	$A_3 \oplus A_3 \oplus \mathbb{R}^2$
	K_8	$E_7 \oplus A_1$	$A_6 \oplus \mathbb{R}^2$
	$K_2 + K_3$	$E_7 \oplus A_1$	$A_5 \oplus A_1 \oplus \mathbb{R}^2$
	$K_2 + K_4$	D_8	$A_4 \oplus A_2 \oplus \mathbb{R}^2$
	$K_2 + K_5$	D_8	$A_3 \oplus A_3 \oplus \mathbb{R}^2$
$(\mathfrak{e}_8, \mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	K_1	D_8	$D_5 \oplus A_1 \oplus \mathbb{R}^2$
	K_2	D_8	$A_5 \oplus A_1 \oplus A_1 \oplus \mathbb{R}$
	K_7	$E_7 \oplus A_1$	$E_6 \oplus A_1 \oplus \mathbb{R}$
	K_8	$E_7 \oplus A_1$	$E_6 \oplus \mathbb{R}^2$
	$K_1 + K_7$	$E_7 \oplus A_1$	$D_5 \oplus A_1 \oplus \mathbb{R}^2$
	$K_2 + K_7$	$E_7 \oplus A_1$	$A_5 \oplus A_1 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_8$	$E_7 \oplus A_1$	$D_5 \oplus \mathbb{R}^3$
	$K_2 + K_8$	D_8	$A_5 \oplus A_1 \oplus \mathbb{R}^2$
$(\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_1	$D_6 \oplus A_1$	$A_5 \oplus \mathbb{R}^2$
	K_2	A_7	$A_4 \oplus A_1 \oplus \mathbb{R}^2$
	K_3	$D_6 \oplus A_1$	$A_5 \oplus A_1 \oplus \mathbb{R}$
	K_4	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	K_5	A_7	$A_2 \oplus A_2 \oplus \mathbb{R}^2$
	K_7	$E_6 \oplus \mathbb{R}$	$A_4 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_2$	$E_6 \oplus \mathbb{R}$	$A_4 \oplus \mathbb{R}^3$
	$K_1 + K_4$	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus \mathbb{R}^3$
	$K_1 + K_5$	A_7	$A_2 \oplus A_2 \oplus \mathbb{R}^3$
	$K_3 + K_4$	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_3 + K_5$	$E_6 \oplus \mathbb{R}$	$A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R}^2$
$(\mathfrak{e}_7, \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	K_1	$D_6 \oplus A_1$	$A_3 \oplus A_2 \oplus \mathbb{R}^2$
	K_3	$D_6 \oplus A_1$	$A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R}^2$
	K_5	A_7	$A_4 \oplus A_2 \oplus \mathbb{R}$
	K_6	$D_6 \oplus A_1$	$A_4 \oplus A_1 \oplus \mathbb{R}^2$
	K_7	$E_6 \oplus \mathbb{R}$	$A_4 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_5$	A_7	$A_3 \oplus A_2 \oplus \mathbb{R}^2$
	$K_1 + K_6$	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus \mathbb{R}^3$
	$K_3 + K_5$	$E_6 \oplus \mathbb{R}$	$A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R}^2$
	$K_3 + K_6$	$D_6 \oplus A_1$	$A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^3$
	$K_1 + K_7$	$E_6 \oplus \mathbb{R}$	$A_3 \oplus A_1 \oplus \mathbb{R}^3$
	$K_3 + K_7$	A_7	$A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^3$

$(\mathfrak{e}_6,\ \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R},\ K_4)$	K_1	$D_5 \oplus \mathbb{R}$	$A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	K_4	$A_5 \oplus A_1$	$A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R}$
	K_5	$A_5 \oplus A_1$	$A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_2$	$D_5 \oplus \mathbb{R}$	$A_2 \oplus A_1 \oplus \mathbb{R}^3$
	$K_1 + K_4$	$A_5 \oplus A_1$	$A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_5$	$A_5 \oplus A_1$	$A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^3$
	$K_2 + K_4$	$A_5 \oplus A_1$	$A_2 \oplus A_2 \oplus \mathbb{R}^2$
	$K_2 + K_5$	$D_5 \oplus \mathbb{R}$	$A_2 \oplus A_1 \oplus \mathbb{R}^3$
	$K_4 + K_5$	$D_5 \oplus \mathbb{R}$	$A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_2 + K_5$	$A_5 \oplus A_1$	$A_1 \oplus A_1 \oplus \mathbb{R}^4$
	$K_1 + K_4 + K_5$	$D_5 \oplus \mathbb{R}$	$A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^3$
$(\mathfrak{f}_4,\ \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R},\ K_2)$	K_1	$C_3 \oplus A_1$	$A_2 \oplus \mathbb{R}^2$
	K_2	$C_3 \oplus A_1$	$A_2 \oplus A_1 \oplus \mathbb{R}$
	K_4	B_4	$A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_3$	$C_3 \oplus A_1$	$A_1 \oplus \mathbb{R}^3$
	$K_2 + K_4$	$C_3 \oplus A_1$	$A_1 \oplus A_1 \oplus \mathbb{R}^2$
$(\mathfrak{g}_2,\ \mathfrak{su}(2) \oplus \mathbb{R},\ K_1)$	K_1	$A_1 \oplus A_1$	$A_1 \oplus \mathbb{R}$
	K_2	$A_1 \oplus A_1$	$\mathbb{R}^2$

$(\mathfrak{g},\ \mathfrak{h},\ H)$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_6,\ \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R},\ K_4)$	ψ	F_4	$A_2 \oplus A_1 \oplus C_1 \oplus \mathbb{R}$
	$\psi \circ \tau_{K_2}$	C_4	$A_2 \oplus A_1 \oplus D_1 \oplus \mathbb{R}$
	$\psi \circ \tau_{K_4}$	C_4	$A_2 \oplus A_1 \oplus C_1 \oplus \mathbb{R}$

$\psi : E_{\alpha_1} \mapsto E_{\alpha_6},\ E_{\alpha_2} \mapsto E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_5},\ E_{\alpha_4} \mapsto E_{\alpha_4}$

9. Remarks on conjugations

9.1. $\dim \mathfrak{z} = 1$, $\tau \circ \sigma = \sigma \circ \tau$. *The case where* $\mathfrak{g} = \mathfrak{e}_6$ *and* $\sigma = \tau_{(1/2)K_4}$. We shall show that $\tau_{K_2+K_5} \approx \tau_{K_1+K_2}$, $\tau_{K_1+K_4} \approx \tau_{K_5}$ and $\tau_{K_4+K_5} \approx \tau_{K_1}$. We consider $\psi \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ (see (8.9)). For $\mu_1 := t_{\alpha_5+\alpha_6} \circ t_{\alpha_2} \circ \psi \in \mathrm{Aut}(\mathfrak{g})$, we have

$$\mu_1(\alpha_1) = -\alpha_5, \quad \mu_1(\alpha_2) = -\alpha_2, \quad \mu_1(\alpha_3) = -\alpha_6,$$
$$\mu_1(\alpha_4) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \quad \mu_1(\alpha_5) = \alpha_3, \quad \mu_1(\alpha_6) = \alpha_1.$$

Hence we get $\mu_1^{-1}(K_4) = K_4$ and $\mu_1^{-1}(K_2 + K_5) = K_1 - K_2 + 2K_4$. Thus μ_1^{-1} is in $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ and gives a conjugation between $\tau_{K_2+K_5}$ and $\tau_{K_1+K_2}$.

Similarly as above, by using $\mu_2 := t_{\alpha_3} \circ t_{\alpha_1+\alpha_3} \circ \psi$, $\mu_3 := t_{\alpha_1} \circ t_{\alpha_3} \circ \psi \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$, we obtain $\tau_{K_1+K_4} \approx \tau_{K_5}$ and $\tau_{K_4+K_5} \approx \tau_{K_1}$.

The case where $\mathfrak{g} = \mathfrak{e}_8$ *and* $\sigma = \tau_{(1/2)K_2}$. From Proposition 8.1, we see that $\mathfrak{g}^{\tau_{K_3}} \cong \mathfrak{g}^{\tau_{K_2+K_3}}$ and $\mathfrak{h} \cap \mathfrak{g}^{\tau_{K_3}} \cong \mathfrak{h} \cap \mathfrak{g}^{\tau_{K_2+K_3}}$. Now, we shall show that τ_{K_3} is not conjugate within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ to $\tau_{K_2+K_3}$. Put $\mathfrak{k}_1 := \mathfrak{g}^{\tau_{K_3}}$ and $\mathfrak{k}_2 := \mathfrak{g}^{\tau_{K_2+K_3}}$, then we have $\mathfrak{k}_1 \cong \mathfrak{k}_2 \cong A_1 \oplus E_7$ and $\mathfrak{k}_1 \cap \mathfrak{h} \cong \mathfrak{k}_2 \cap \mathfrak{h}$. We denote $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ by $\alpha = \sum_{i=1}^{8} n_i\alpha_i$ and put

$$\Delta_{\mathfrak{k}_1} := \{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \alpha(K_3) = 0, 2, 4\},$$
$$\Delta_{\mathfrak{k}_2} := \{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \alpha(K_3) = 0, 2, 4, 6\}.$$

Then

$$\mathfrak{k}_i = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_{\mathfrak{k}_i}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha), \quad i = 1, 2.$$

Put $\gamma := 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$. Then for any $\alpha \in \Delta_{\mathfrak{k}_1}$ and $\alpha' \in \Delta_{\mathfrak{k}_2}$, we can see that $\alpha_1 \pm \alpha \notin \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $\gamma \pm \alpha' \notin \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (cf. [3]). Therefore, we get

$$\mathfrak{k}_1 = \mathfrak{su}_{\alpha_1}(2) \oplus \mathfrak{su}_{\alpha_1}(2)^{\perp}, \quad \mathfrak{k}_2 = \mathfrak{su}_{\gamma}(2) \oplus \mathfrak{su}_{\gamma}(2)^{\perp}, \tag{9.1}$$

where $\mathfrak{su}_{\alpha_1}(2)^{\perp} \cong \mathfrak{su}_{\beta}(2)^{\perp} \cong \mathfrak{e}_7$. For any $\nu \in \mathrm{Aut}(\mathfrak{g})$ satisfying $\nu(\mathfrak{k}_1) = \mathfrak{k}_2$, it follows from (9.1) that $\nu(\mathfrak{su}_{\alpha_1}(2)) = \mathfrak{su}_{\beta}(2)$. Hence there is no automorphism in $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ such that it maps $\mathfrak{k}_1$ to $\mathfrak{k}_2$ because $\mathfrak{su}_{\alpha_1}(2) \subset \mathfrak{h}$ and $\mathfrak{su}_{\beta}(2) \not\subset \mathfrak{h}$.

Next we shall show that τ_{K_5} is conjugate within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ to $\tau_{K_2+K_5}$. Set

$$\gamma_1 := \alpha_8, \quad \gamma_2 := -\alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8,$$
$$\gamma_3 := \alpha_7, \quad \gamma_4 := \alpha_8, \quad \gamma_5 := \alpha_5, \quad \gamma_6 := \alpha_4, \quad \gamma_7 := \alpha_3, \quad \gamma_8 := \alpha_1.$$

It is easy to see that $\Pi' := \{\gamma_1, \ldots, \gamma_8\}$ is a fundamental root system of $\mathfrak{e}_8$ (cf. [3]). Therefore there exists a unique ν in $W(\mathfrak{g}, \mathfrak{h})$ such that $\nu(\Pi) = \Pi'$. Hence we have $\nu(\alpha_i) = \gamma_i$ $(i = 1, \ldots, 8)$. Then it is easy to see that $\nu^{-1}(K_2) = -K_2$ and $\nu^{-1}(K_5) = -3K_2 + K_5$. Hence $\nu^{-1} \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ and $\tau_{K_5} \approx \tau_{K_2+K_5}$.

The case where $\mathfrak{g} = \mathfrak{e}_7$ *and* $\sigma = \tau_{(1/2)K_3}$. From Proposition 8.1, we can see that $\mathfrak{g}^{\tau_{K_4}} \cong \mathfrak{g}^{\tau_{K_3+K_4}}$ and $\mathfrak{h} \cap \mathfrak{g}^{\tau_{K_4}} \cong \mathfrak{h} \cap \mathfrak{g}^{\tau_{K_3+K_4}}$. However τ_{K_4} is not conjugate within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$

to $\tau_{K_3+K_4}$. Indeed, note that $\mathfrak{g}^{\tau_{K_4}} \cong \mathfrak{g}^{\tau_{K_3+K_4}} \cong A_1 \oplus D_6$, where $A_1 \subset \mathfrak{g}^{\tau_{K_4}}$ and $A_1 \subset \mathfrak{g}^{\tau_{K_3+K_4}}$ coincide with $\mathfrak{su}_{\alpha_2}(2)$ ($\subset \mathfrak{h}$) and $\mathfrak{su}_{\beta}(2)$ ($\not\subset \mathfrak{h}$), respectively. Here $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Therefore, there is no automorphism in $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ which maps $\mathfrak{g}^{\tau_{K_4}}$ to $\mathfrak{g}^{\tau_{K_3+K_4}}$, since $\nu(\mathfrak{su}_{\alpha_2}(2)) = \mathfrak{su}_{\beta}(2)$ for any $\nu \in \mathrm{Aut}(\mathfrak{g})$ satisfying $\nu(\mathfrak{g}^{\tau_{K_4}}) = \mathfrak{g}^{\tau_{K_3+K_4}}$.

9.2. $\dim \mathfrak{z} = 0$, $\tau \circ \sigma = \sigma^{-1} \circ \tau$. *The case where $\mathfrak{g} = \mathfrak{e}_8$ and $\sigma = \tau_{(1/2)K_6}$.* For the reflection $t_{\alpha_1} \in \mathrm{Int}(\mathfrak{h})$, it follows from Proposition 5.1 that $t_{\alpha_1}^{-1} \circ \tau_2^{\Pi} \circ t_{\alpha_1}(E_{\alpha_3}) = -\tau_2^{\Pi}(E_{\alpha_3})$ and $t_{\alpha_1}^{-1} \circ \tau_2^{\Pi} \circ t_{\alpha_1}(E_{\alpha_i}) = \tau_2^{\Pi}(E_{\alpha_i})$ ($i \neq 3$). Hence we get

$$t_{\alpha_1}^{-1} \circ \tau_2^{\Pi} \circ t_{\alpha_1} = \tau_2^{\Pi} \circ \tau_{K_3}, \tag{9.2}$$

and therefore $\tau_2^{\Pi} \circ \tau_{K_3} \approx \tau_2^{\Pi} \approx \tau_2^{\Pi} \circ \sigma \approx \tau_2^{\Pi} \circ \tau_{K_3} \circ \sigma$. By an argument similar to the case $\mathfrak{g} = \mathfrak{e}_7$, for the reflection $t_{\alpha_3} \in \mathrm{Int}(\mathfrak{su}_{\alpha_3}(2)) \subset \mathrm{Int}(\mathfrak{h})$, it follows that

$$\tau_2^{\Pi} \circ t_{\alpha_3}(E_{\alpha_i}) = -t_{\alpha_3} \circ \tau_2^{\Pi}(E_{\alpha_i}), \quad \tau_2^{\Pi} \circ t_{\alpha_3}(E_{\alpha_j}) = t_{\alpha_3} \circ \tau_2^{\Pi}(E_{\alpha_j}),$$

where $i = 1, 4$ and $j \neq 1, 4$. Therefore

$$t_{\alpha_3}^{-1} \circ \tau_2^{\Pi} \circ t_{\alpha_3} = \tau_2^{\Pi} \circ \tau_{K_1+K_4}, \tag{9.3}$$

which implies $\tau_2^{\Pi} \circ \tau_{K_1+K_4} \approx \tau_2^{\Pi} \approx \tau_2^{\Pi} \circ \sigma \approx \tau_2^{\Pi} \circ \tau_{K_1+K_4} \circ \sigma$.

Next, we shall prove

$$\begin{aligned}
&\tau_2^{\Pi} \circ \tau_{K_1+K_8} \approx \tau_2^{\Pi} \circ \tau_{K_4+K_8} \approx \tau_2^{\Pi} \circ \tau_{K_3+K_4+K_8} \approx \tau_2^{\Pi} \circ \tau_{K_1+K_3+K_8} \circ \sigma, \\
&\tau_2^{\Pi} \circ \tau_{K_1+K_8} \circ \sigma \approx \tau_2^{\Pi} \circ \tau_{K_4+K_8} \circ \sigma \approx \tau_2^{\Pi} \circ \tau_{K_3+K_4+K_8} \circ \sigma \approx \tau_2^{\Pi} \circ \tau_{K_1+K_3+K_8}.
\end{aligned} \tag{9.4}$$

It is easy to see that $t_{\alpha_i}(K_4 + K_8) = K_4 + K_8$ ($i = 1, 3$), and it follows from (9.2) and (9.3) that

$$\begin{aligned}
\tau_2^{\Pi} \circ \tau_{K_1+K_8} &= \tau_2^{\Pi} \circ \tau_{K_1+K_4} \circ \tau_{K_4+K_8} = t_{\alpha_3}^{-1} \circ \tau_2^{\Pi} \circ t_{\alpha_3} \circ \tau_{K_4+K_8} \\
&= t_{\alpha_3}^{-1} \circ \tau_2^{\Pi} \circ \tau_{K_4+K_8} \circ t_{\alpha_3}, \\
\tau_2^{\Pi} \circ \tau_{K_3+K_4+K_8} &= t_{\alpha_1}^{-1} \circ \tau_2^{\Pi} \circ t_{\alpha_1} \circ \tau_{K_4+K_8} = t_{\alpha_1}^{-1} \circ \tau_2^{\Pi} \circ \tau_{K_4+K_8} \circ t_{\alpha_1}.
\end{aligned}$$

For $t_{\alpha_4} \in \mathrm{Int}(\mathfrak{h})$, we get

$$\tau_2^{\Pi} \circ t_{\alpha_4}(E_{\alpha_3}) = -t_{\alpha_4} \circ \tau_2^{\Pi}(E_{\alpha_3}), \quad \tau_2^{\Pi} \circ t_{\alpha_4}(E_{\alpha_i}) = t_{\alpha_4} \circ \tau_2^{\Pi}(E_{\alpha_i}),$$

where $i = 1, 4, 6, 7, 8$. Moreover, we obtain

$$\begin{aligned}
&\tau_2^{\Pi} \circ t_{\alpha_4}(E_{\alpha_2}) = b_2 \tau_2^{\Pi}(E_{\alpha_2+\alpha_4}) = b_2 k E_{\alpha_4+\alpha_5}, \\
&t_{\alpha_4} \circ \tau_2^{\Pi}(E_{\alpha_2}) = t_{\alpha_4}(E_{\alpha_5}) = b_5 E_{\alpha_4+\alpha_5}, \\
&\tau_2^{\Pi} \circ t_{\alpha_4}(E_{\alpha_5}) = b_5 \tau_2^{\Pi}(E_{\alpha_4+\alpha_5}) = b_5 k^{-1} E_{\alpha_2+\alpha_4}, \\
&t_{\alpha_4} \circ \tau_2^{\Pi}(E_{\alpha_5}) = t_{\alpha_4}(E_{\alpha_2}) = b_2 E_{\alpha_2+\alpha_4},
\end{aligned}$$

for some $b_2, b_5, k \in \mathbb{C}$ with $|b_2| = |b_5| = |k| = 1$. Put $a := b_2k/b_5$. Then we have

$$\tau_2^\Pi \circ t_{\alpha_4}(E_{\alpha_2}) = at_{\alpha_4} \circ \tau_2^\Pi(E_{\alpha_2}), \quad \tau_2^\Pi \circ t_{\alpha_4}(E_{\alpha_5}) = a^{-1}t_{\alpha_4} \circ \tau_2^\Pi(E_{\alpha_5}).$$

Hence we have $t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_4} = \tau_2^\Pi \circ \tau_{K_3} \circ \tau_{s(K_2-K_5)}$, where $a = e^{s\pi\sqrt{-1}}$. Note that $s \in \mathbb{Z}$ since $(t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_4})^2 = \mathrm{Id}$, and thus we may assume $s = 0$ or 1. If $s = 0$, then we have $t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_4} \circ \tau_{K_1+K_8} = t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_8} \circ t_{\alpha_4} = \tau_2^\Pi \circ \tau_{K_1+K_3+K_8}$, which contradicts Proposition 5.1. Thus $t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_4} = \tau_2^\Pi \circ \tau_{K_3} \circ \tau_{K_2-K_5}$ and

$$\begin{aligned} t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_8} \circ t_{\alpha_4} &= t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_4} \circ \tau_{K_1+K_8} = \tau_2^\Pi \circ \tau_{K_3+K_2-K_5+K_1+K_8} \\ &= \tau_2^\Pi \circ \tau_{K_1+K_3+K_8} \circ \tau_{K_7} \circ \tau_{K_2-K_5+K_7} \\ &= \tau_2^\Pi \circ \tau_{K_1+K_3+K_8} \circ \sigma \circ \tau_{(-(1/2)K_6+K_7)+(K_2-K_5+K_7))}. \end{aligned}$$

It is easy to see that $\sqrt{-1}h := \sqrt{-1}(-(1/2)K_6+K_7+K_2-K_5+K_7)$ is a (-1)-eigenvector of τ_2^Π. Therefore $t_{\alpha_4}^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_8} \circ \tau_{\alpha_4} = \tau_{-(1/2)h} \circ \tau_2^\Pi \circ \tau_{K_1+K_3+K_8} \circ \sigma \circ \tau_{(1/2)h}$, which implies $\tau_2^\Pi \circ \tau_{K_1+K_8} \approx \tau_2^\Pi \circ \tau_{K_1+K_3+K_8} \circ \sigma$. Moreover since

$$\tau_2^\Pi \circ \tau_{K_1+K_8} \circ \sigma \approx \tau_2^\Pi \circ \tau_{K_1+K_3+K_8} \circ \sigma^2 = \sigma^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_3+K_8} \circ \sigma,$$

we have $\tau_2^\Pi \circ \tau_{K_1+K_8} \circ \sigma \approx \tau_2^\Pi \circ \tau_{K_1+K_3+K_8}$. We have thus proved (9.4).

Finally, by using $t_{\alpha_8} \in \mathrm{Int}(\mathfrak{h})$, we shall show that $\tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \approx \tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \circ \sigma$. It is easy to see that $\tau_2^\Pi \circ t_{\alpha_8}(E_{\alpha_i}) = t_{\alpha_8} \circ \tau_2^\Pi(E_{\alpha_i})$, $i = 1, \dots, 6$, and

$$\begin{aligned} \tau_2^\Pi \circ t_{\alpha_8}(E_{\alpha_7}) &= b_7\tau_2^\Pi(E_{\alpha_7+\alpha_8}) = b_7kE_{\alpha_0+\alpha_8}, \\ t_{\alpha_8} \circ \tau_2^\Pi(E_{\alpha_7}) &= t_{\alpha_8}(E_{\alpha_0}) = b_0E_{\alpha_0+\alpha_5}, \end{aligned}$$

for some $b_0, b_7, k \in \mathbb{C}$ with $|b_0| = |b_7| = |k| = 1$. By an argument similar to the above, we obtain $t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_8} = \tau_2^\Pi$ or $\tau_2^\Pi \circ \tau_{K_7}$. If $t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_8} = \tau_2^\Pi$, then since $t_{\alpha_8}(K_1) = K_1$, $t_{\alpha_8}(K_8) = K_7 - K_8$, it follows that

$$\begin{aligned} \tau_2^\Pi \circ \tau_{K_1} \circ \tau_{K_8} &= t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_8} \circ \tau_{K_1+K_8} = t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_7+K_8} \circ t_{\alpha_8} \\ &= t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_8} \circ \sigma \circ \tau_{K_7-(1/2)K_6} \circ t_{\alpha_8}. \end{aligned}$$

Therefore $\tau_2^\Pi \circ \tau_{K_1+K_8} \approx \tau_2^\Pi \circ \tau_{K_1+K_8} \circ \sigma$ since $\sqrt{-1}(K_7-(1/2)K_6)$ is a (-1)-eigenvector of τ_2^Π. This contradicts Proposition 5.1, and hence $t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_8} = \tau_2^\Pi \circ \tau_{K_7}$. Thus

$$\begin{aligned} t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \circ t_{\alpha_8} &= t_{\alpha_8}^{-1} \circ \tau_2^\Pi \circ t_{\alpha_8} \circ \tau_{K_1+K_3+K_4} = \tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \circ \tau_{K_7} \\ &= \tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \circ \sigma \circ \tau_{K_7-(1/2)K_6} \approx \tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \circ \sigma, \end{aligned}$$

which implies $\tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \approx \tau_2^\Pi \circ \tau_{K_1+K_3+K_4} \circ \sigma$.

The case where $\mathfrak{g} = \mathfrak{e}_7$ *and* $\sigma = \tau_{(1/2)K_4}$. We consider $\tau_3^\Pi \circ \tau_{K_1+K_2}$, $\tau_3^\Pi \circ \tau_{K_2+K_6}$, $\tau_3^\Pi \circ \tau_{K_1+K_2} \circ \sigma$ and $\tau_3^\Pi \circ \tau_{K_2+K_6} \circ \sigma$ (see Proposition 5.1). Let φ be the automorphism of $\mathfrak{g}$ given by (5.11). It follows from (5.9) and (5.11) that $\varphi \circ \tau_3^{\Pi} = \tau_3^{\Pi} \circ \varphi$. Since $\varphi(K_1 + K_2) = K_2 + K_6 - 4K_7$ and $\varphi(K_4) = K_4 - 4K_7$, we have $\varphi \circ \sigma = \sigma \circ \varphi$ and

$$\varphi \circ \tau_3^{\Pi} \circ \tau_{K_1+K_2} \circ \varphi^{-1} = \tau_3^{\Pi} \circ \tau_{K_2+K_6},$$

and therefore

$$\varphi \circ \tau_3^{\Pi} \circ \tau_{K_1+K_2} \circ \sigma \circ \varphi^{-1} = \varphi \circ \tau_3^{\Pi} \circ \tau_{K_1+K_2} \circ \varphi^{-1} \circ \sigma = \tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ \sigma.$$

Thus we obtain $\tau_3^{\Pi} \circ \tau_{K_1+K_2} \approx \tau_3^{\Pi} \circ \tau_{K_2+K_6}$ and $\tau_3^{\Pi} \circ \tau_{K_1+K_2} \circ \sigma \approx \tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ \sigma$.

Next considering reflection $t_{\alpha_1} \in \mathrm{Int}(\mathfrak{su}_{\alpha_1}(2)) \subset \mathrm{Int}(\mathfrak{h})$, we get $t_{\alpha_1}(E_{\alpha_i}) = E_{\alpha_i}$ for $i = 2, 4, 5, 6, 7$ because $\alpha_1 \pm \alpha_i$ are not roots. Put $\beta := \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. Then $\tau_3^{\Pi}(\alpha_4) = \beta$ and $\beta \pm \alpha_1 \notin \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Hence we have $t_{\alpha_1}(E_\beta) = E_\beta$. Since

$$\begin{cases} \tau_3^{\Pi} \circ t_{\alpha_1}(E_{\alpha_1}) = b_1 \tau_3^{\Pi}(E_{-\alpha_1}) = -b_1 E_{-\alpha_1}, \\ t_{\alpha_1} \circ \tau_3^{\Pi}(E_{\alpha_1}) = -t_{\alpha_1}(E_{\alpha_1})) = -b_1 E_{-\alpha_1}, \end{cases}$$

$$\begin{cases} \tau_3^{\Pi} \circ t_{\alpha_1}(E_{\alpha_3}) = b_3 \tau_3^{\Pi}(E_{\alpha_1+\alpha_3}) = b_3 k E_{\alpha_0+\alpha_1}, \\ t_{\alpha_1} \circ \tau_3^{\Pi}(E_{\alpha_3}) = t_{\alpha_1}(E_{\alpha_0})) = b_0 E_{\alpha_0+\alpha_1}, \end{cases}$$

for some $b_0, b_1, b_3, k \in \mathbb{C}$ with $|b_0| = |b_1| = |b_3| = |k| = 1$, there exists $s \in \mathbb{R}$ such that $t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1} = \tau_3^{\Pi} \circ \tau_{sK_3}$. Moreover, since $(t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1})^2 = \mathrm{Id}$ and $\tau_3^{\Pi}(K_3) = -3K_3 + 2K_4$, we get $s \in \mathbb{Z}$, and thus $t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1} = \tau_3^{\Pi}$ or $\tau_3^{\Pi} \circ \tau_{K_3}$. If $t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1} = \tau_3^{\Pi}$, then

$$\begin{aligned} \tau_3^{\Pi} \circ \tau_{K_1+K_6} &= t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1} \circ \tau_{K_1+K_6} = t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ \tau_{t_{\alpha_1}(K_1+K_6)} \circ t_{\alpha_1} \\ &= t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ \tau_{K_1+K_3} \circ \tau_{K_6} \circ t_{\alpha_1} = t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ \tau_{K_1+K_6} \circ \tau_{K_3} \circ t_{\alpha_1} \\ &= t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ \tau_{K_1+K_6} \circ \sigma \circ \tau_{K_3-(1/2)K_4} \circ t_{\alpha_1}. \end{aligned}$$

Put $\mu := \tau_{(1/2)(K_3-(1/2)K_4)}$. Then since $K_3 - (1/2)K_4$ is a (-1)-eigenvector of τ_3^{Π}, it is easy to see that

$$\tau_3^{\Pi} \circ \tau_{K_1+K_6} = t_{\alpha_1}^{-1} \circ \mu^{-1} \circ \tau_3^{\Pi} \circ \tau_{K_1+K_6} \circ \sigma \circ \mu \circ t_{\alpha_1}.$$

Because $\mu \circ t_{\alpha_1} \in \mathrm{Int}(\mathfrak{h})$, it follows that $\tau_3^{\Pi} \circ \tau_{K_1+K_6} \approx \tau_3^{\Pi} \circ \tau_{K_1+K_6} \circ \sigma$, which contradicts Proposition 5.1. Hence $t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1} = \tau_3^{\Pi} \circ \tau_{K_3}$, and

$$\begin{aligned} t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ t_{\alpha_1} &= t_{\alpha_1}^{-1} \circ \tau_3^{\Pi} \circ t_{\alpha_1} \circ \tau_{K_2+K_6} = \tau_3^{\Pi} \circ \tau_{K_3} \circ \tau_{K_2+K_6} \\ &= \tau_3^{\Pi} \circ \sigma \circ \mu^2 \circ \tau_{K_2+K_6} = \tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ \sigma \circ \mu^2 \\ &= \mu^{-1} \circ \tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ \sigma \circ \mu, \end{aligned}$$

since $t_{\alpha_1}(K_2 + K_6) = K_2 + K_6$ and $\tau_3^{\Pi}(K_3 - (1/2)K_4) = -(K_3 - (1/2)K_4)$. Consequently we obtain $\tau_3^{\Pi} \approx \tau_3^{\Pi} \circ \tau_{K_3}$ and $\tau_3^{\Pi} \circ \tau_{K_2+K_6} \approx \tau_3^{\Pi} \circ \tau_{K_2+K_6} \circ \sigma$.

10. Classifications

From Propositions 5.1, 6.1, 7.1 and 8.1 together with the results in Section 9, we obtain the following theorem which gives the complete classification of involutions preserving $\mathfrak{h}$.

Theorem 10.1. *Let* $(G/H, \langle\ ,\ \rangle, \sigma)$ *be a Riemannian* 4*-symmetric space such that* G *is compact and simple. Suppose that* $\sigma = \mathrm{Ad}(\exp(\pi/2)\sqrt{-1}K_i)$ *for some* $\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *with* $m_i = 3$ *or* 4*. Then the following* Tables 5, 6, 7 *and* 8 *give the complete lists of the equivalence classes within* $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *of involutions* τ *satisfying* $\tau(\mathfrak{h}) = \mathfrak{h}$.

Table 5. $\dim \mathfrak{z} = 0$, $\tau \circ \sigma = \sigma^{-1} \circ \tau$, $\sigma = \tau_{(1/2)H}$ and $\mathfrak{k} = \mathfrak{g}^{\tau}$.

$(\mathfrak{g}, \mathfrak{h}, H)$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathfrak{su}(2), K_3)$	τ_1^{Π}	$\mathfrak{so}(16)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(2)$
	$\tau_1^{\Pi} \circ \tau_{K_6}$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(4) \oplus \mathfrak{so}(2)$
	$\tau_1^{\Pi} \circ \tau_{K_6+(1/2)K_3}$	$\mathfrak{so}(16)$	$\mathfrak{sp}(4) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_8, \mathfrak{so}(10) \oplus \mathfrak{so}(6), K_6)$	τ_2^{Π}	$\mathfrak{so}(16)$	$(\mathfrak{so}(5) + \mathfrak{so}(5)) \oplus (\mathfrak{so}(3) + \mathfrak{so}(3))$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_8}$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(7) + \mathfrak{so}(3)) \oplus \mathfrak{so}(5)$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_3+K_4}$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(9) \oplus (\mathfrak{so}(3) + \mathfrak{so}(3))$
	$\tau_2^{\Pi} \circ \tau_{K_1+K_3+K_8}$	$\mathfrak{so}(16)$	$(\mathfrak{so}(7) + \mathfrak{so}(3)) \oplus \mathfrak{so}(5)$
$(\mathfrak{e}_7, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	τ_3^{Π}	$\mathfrak{su}(8)$	$(\mathfrak{so}(3) + \mathfrak{so}(3)) \oplus (\mathfrak{so}(3) + \mathfrak{so}(3)) \oplus \mathfrak{so}(2)$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_2}$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(5) \oplus (\mathfrak{so}(3) + \mathfrak{so}(3)) \oplus \mathfrak{su}(2)$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_6}$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(5) \oplus \mathfrak{so}(5) \oplus \mathfrak{su}(2)$
	$\tau_3^{\Pi} \circ \tau_{K_1+K_6+(1/2)K_4}$	$\mathfrak{su}(8)$	$\mathfrak{so}(5) \oplus \mathfrak{so}(5) \oplus \mathfrak{su}(2)$
	$\tau_3^{\Pi} \circ \varphi$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$
	$\tau_3^{\Pi} \circ \varphi \circ \tau_{(1/2)K_4}$	$\mathfrak{su}(8)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$
$(\mathfrak{f}_4, \mathfrak{so}(6) \oplus \mathfrak{so}(3), K_3)$	τ_4^{Π}	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(3) + \mathfrak{so}(3)) \oplus \mathfrak{so}(2)$
	$\tau_4^{\Pi} \circ \tau_{K_1+K_4}$	$\mathfrak{so}(9)$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3)$
	$\tau_4^{\Pi} \circ \tau_{K_1+K_4+(1/2)K_3}$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3)$

$\varphi: E_{\alpha_1} \mapsto E_{\alpha_6},\ E_{\alpha_2} \mapsto E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_5},\ E_{\alpha_4} \mapsto E_{\alpha_4},\ E_{\alpha_7} \mapsto E_{\alpha_0}$
$\tau_1^{\Pi}: E_{\alpha_1} \mapsto -E_{\alpha_1},\ E_{\alpha_2} \mapsto E_{\alpha_0},\ E_{\alpha_3} \mapsto c_1 E_{\beta_1},\ E_{\alpha_4} \mapsto E_{\alpha_8},\ E_{\alpha_5} \mapsto E_{\alpha_7},\ E_{\alpha_6} \mapsto -E_{\alpha_6},$
$(\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$
$\tau_2^{\Pi}: E_{\alpha_1} \mapsto -E_{\alpha_1},\ E_{\alpha_2} \mapsto E_{\alpha_5},\ E_{\alpha_3} \mapsto -E_{\alpha_3},\ E_{\alpha_4} \mapsto -E_{\alpha_4},\ E_{\alpha_6} \mapsto c_2 E_{\beta_2},\ E_{\alpha_7} \mapsto E_{\alpha_0},$
$E_{\alpha_8} \mapsto -E_{\alpha_8},\ (\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)$
$\tau_3^{\Pi}: E_{\alpha_1} \mapsto -E_{\alpha_1},\ E_{\alpha_2} \mapsto -E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_0},\ E_{\alpha_4} \mapsto c_3 E_{\beta_3},\ E_{\alpha_5} \mapsto E_{\alpha_7},\ E_{\alpha_6} \mapsto -E_{\alpha_6},$
$(\beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$
$\tau_4^{\Pi}: E_{\alpha_1} \mapsto -E_{\alpha_1},\ E_{\alpha_2} \mapsto E_{\alpha_0},\ E_{\alpha_3} \mapsto c_4 E_{\beta_4},\ E_{\alpha_4} \mapsto -E_{\alpha_4},\ (\beta_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)$
where c_i $(i = 1, 2, 3, 4)$ is some complex number with $|c_i| = 1$.

Table 6. $\dim \mathfrak{z} = 1$, $\tau \circ \sigma = \sigma^{-1} \circ \tau$, $\sigma = \tau_{(1/2)H}$ and $\mathfrak{k} = \mathfrak{g}^{\tau}$.

$(\mathfrak{g}, \mathfrak{h}, H)$	Π	Π_1	$(\mathfrak{g}^*, \mathfrak{k})$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathbb{R}, K_2)$	E_8	α_2	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	$\mathfrak{so}(8)$
$(\mathfrak{e}_8, \mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	E_8	α_7	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	$\mathfrak{sp}(4)$
	F_4	α_7	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \oplus \mathfrak{su}(2))$	$\mathfrak{f}_4$
$(\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	E_7	α_3	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$
	F_4	α_3	$(\mathfrak{e}_{7(5)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2))$	$\mathfrak{sp}(3) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_7, \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	E_7	α_5	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3)$
$(\mathfrak{e}_6, \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_4)$	E_6	α_4	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)$
	F_4	α_4	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$
$(\mathfrak{f}_4, \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_2)$	F_4	α_2	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) \oplus \mathfrak{su}(2))$	$\mathfrak{so}(3) \oplus \mathfrak{so}(2)$
$(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathbb{R}, K_1)$	G_2	α_1	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$	$\mathfrak{so}(2)$

Table 7. $\dim \mathfrak{z} = 0$, $\tau \circ \sigma = \sigma \circ \tau$, $\sigma = \tau_{(1/2)H}$ and $\mathfrak{k} = \mathfrak{g}^{\tau}$.

$(\mathfrak{g}, \mathfrak{h}, H)$	h $(\tau = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathfrak{su}(2), K_3)$	K_1	$\mathfrak{so}(16)$	$\mathfrak{su}(8) \oplus \mathfrak{so}(2)$
	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{su}(8) \oplus \mathfrak{su}(2)$
	K_4	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(4) \oplus \mathfrak{su}(2)$
	K_6	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(4)) \oplus \mathfrak{su}(2)$
	$K_3 + K_4$	$\mathfrak{so}(16)$	$\mathfrak{sp}(4) \oplus \mathfrak{su}(2)$
	$K_3 + K_6$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(4)) \oplus \mathfrak{su}(2)$
	$K_1 + K_4$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2)$
	$K_1 + K_6$	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2)$
$(\mathfrak{e}_8, \mathfrak{so}(10) \oplus \mathfrak{so}(6), K_6)$	K_1	$\mathfrak{so}(16)$	$(\mathfrak{so}(8) + \mathfrak{so}(2)) \oplus \mathfrak{so}(6)$
	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(6) + \mathfrak{so}(4)) \oplus \mathfrak{so}(6)$
	K_6	$\mathfrak{so}(16)$	$\mathfrak{so}(10) \oplus \mathfrak{so}(6)$
	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(10) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2))$
	$K_1 + K_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(8) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2))$
	$K_2 + K_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(5)$
	$K_3 + K_8$	$\mathfrak{so}(16)$	$(\mathfrak{so}(6) + \mathfrak{so}(4)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2))$
	$K_2 + K_6 + K_7$	$\mathfrak{so}(16)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(5)$
$(\mathfrak{e}_7, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
	K_2	$\mathfrak{su}(8)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{so}(2)$
	K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2)$
	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(2)$
	$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{su}(2)$
	$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{su}(2)$
	$K_1 + K_2 + K_6$	$\mathfrak{su}(8)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus (\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(2)$
	$K_2 + K_3 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{so}(2)$
	$K_3 + K_4 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{u}(3) \oplus \mathfrak{u}(3) \oplus \mathfrak{su}(2)$
$(\mathfrak{f}_4, \mathfrak{so}(6) \oplus \mathfrak{so}(3), K_3)$	K_1	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(3)$
	K_3	$\mathfrak{so}(9)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(3)$
	K_4	$\mathfrak{so}(9)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$
	$K_1 + K_4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(2)$
$(\mathfrak{g}, \mathfrak{h}, H)$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_7, \mathfrak{so}(6) \oplus \mathfrak{so}(6) \oplus \mathfrak{su}(2), K_4)$	φ	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(16) \oplus \mathfrak{sp}(1)$
	$\varphi \circ \tau_{K_2}$	$\mathfrak{su}(8)$	$\mathfrak{so}(16) \oplus \mathfrak{so}(2)$
	$\varphi \circ \tau_{K_4}$	$\mathfrak{su}(8)$	$\mathfrak{so}(16) \oplus \mathfrak{sp}(1)$

φ is the same involution as in Table 5.

Table 8. $\dim \mathfrak{z} = 1$, $\tau \circ \sigma = \sigma \circ \tau$, $\sigma = \tau_{(1/2)H}$ and $\mathfrak{k} = \mathfrak{g}^\tau$.

$(\mathfrak{g}, \mathfrak{h}, H)$	h $(\tau = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_8, \mathfrak{su}(8) \oplus \mathbb{R}, K_2)$	K_1	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(7) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	K_2	$\mathfrak{so}(16)$	$\mathfrak{su}(8) \oplus \mathbb{R}$
	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathbb{R}$
	K_4	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(3)) \oplus \mathbb{R}$
	K_5	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(4)) \oplus \mathbb{R}$
	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(7) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	$K_2 + K_3$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(6) + \mathfrak{u}(2)) \oplus \mathbb{R}$
	$K_2 + K_4$	$\mathfrak{so}(16)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(3)) \oplus \mathbb{R}$
$(\mathfrak{e}_8, \mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_7)$	K_1	$\mathfrak{so}(16)$	$(\mathfrak{so}(10) + \mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_2	$\mathfrak{so}(16)$	$(\mathfrak{su}(6) + \mathfrak{su}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_7	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{e}_6 \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_1 + K_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(10) + \mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_1 + K_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(10) + \mathbb{R}) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_2 + K_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{su}(6) + \mathfrak{su}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_2 + K_8$	$\mathfrak{so}(16)$	$(\mathfrak{su}(6) + \mathfrak{su}(2)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}, K_3)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	K_2	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_5	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_1 + K_4$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_1 + K_5$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_3 + K_4$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_3 + K_5$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}, K_5)$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	K_5	$\mathfrak{su}(8)$	$\mathfrak{su}(5) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(5) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(5) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	$K_1 + K_5$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	$K_1 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	$K_3 + K_5$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3) \oplus \mathbb{R}$
	$K_3 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$
	$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(2)) \oplus \mathfrak{s}(\mathfrak{u}(2) + \mathfrak{u}(1)) \oplus \mathbb{R}$

$(\mathfrak{e}_6,\ \mathfrak{su}(3)\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R},\ K_4)$	K_1	$\mathfrak{so}(10)\oplus\mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	K_4	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	K_5	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	K_1+K_2	$\mathfrak{so}(10)\oplus\mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(3)\oplus\mathfrak{so}(2)\oplus\mathbb{R}$
	K_1+K_5	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	K_2+K_4	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)\oplus\mathfrak{so}(2)\oplus\mathbb{R}$
	$K_1+K_2+K_5$	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{so}(2)\oplus\mathbb{R}$
	$K_1+K_4+K_5$	$\mathfrak{so}(10)\oplus\mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
$(\mathfrak{f}_4,\ \mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R},\ K_2)$	K_1	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(3)\oplus\mathfrak{so}(2)\oplus\mathbb{R}$
	K_2	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	K_4	$\mathfrak{so}(9)$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
	K_1+K_3	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{so}(2)\oplus\mathbb{R}$
	K_2+K_4	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(1))\oplus\mathfrak{su}(2)\oplus\mathbb{R}$
$(\mathfrak{g}_2,\ \mathfrak{su}(2)\oplus\mathbb{R},\ K_1)$	K_1	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(2)\oplus\mathbb{R}$
	K_2	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	$\mathfrak{so}(2)\oplus\mathbb{R}$

$(\mathfrak{g},\ \mathfrak{h},\ H)$	τ	$\mathfrak{k}$	$\mathfrak{h}\cap\mathfrak{k}$
$(\mathfrak{e}_6,\ \mathfrak{su}(3)\oplus\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathbb{R},\ K_4)$	ψ	$\mathfrak{f}_4$	$\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$
	$\psi\circ\tau_{K_2}$	$\mathfrak{sp}(4)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{so}(2)\oplus\mathbb{R}$
	$\psi\circ\tau_{K_4}$	$\mathfrak{sp}(4)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{sp}(1)\oplus\mathbb{R}$

$\psi\colon E_{\alpha_1}\mapsto E_{\alpha_6},\ E_{\alpha_2}\mapsto E_{\alpha_2},\ E_{\alpha_3}\mapsto E_{\alpha_5},\ E_{\alpha_4}\mapsto E_{\alpha_4}$

References

[1] M. Berger: *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.
[2] N. Bourbaki: Groupes et Algébres de Lie, Chapitre 4, 5 et 6, Masson, Paris, 1981.
[3] H. Freudenthal and H. de Vries: Linear Lie Groups, Academic Press, New York, 1969.
[4] P.B. Gilkey and G.M. Seitz: *Some representations of exceptional Lie algebras*, Geom. Dedicata **25** (1988), 407–416.
[5] A. Gray: *Riemannian manifolds with geodesic symmetries of order* 3, J. Differential Geometry **7** (1972), 343–369.
[6] S. Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.
[7] J.A. Jiménez: *Riemannian* 4*-symmetric spaces*, Trans. Amer. Math. Soc. **306** (1988), 715–734.
[8] V.G. Kac: Infinite-Dimensional Lie Algebras, third edition, Cambridge Univ. Press, Cambridge, 1990.
[9] S. Kaneyuki and H. Asano: *Graded Lie algebras and generalized Jordan triple systems*, Nagoya Math. J. **112** (1988), 81–115.
[10] S. Murakami: *Sur la classification des algèbres de Lie réelles et simples*, Osaka J. Math. **2** (1965), 291–307.
[11] H. Naitoh: *Compact simple Lie algebras with two involutions and submanifolds of compact symmetric spaces*, I, II, Osaka J. Math. **30** (1993), 653–690, 691–732.
[12] H. Naitoh: *Grassmann geometries on compact symmetric spaces of general type*, J. Math. Soc. Japan **50** (1998), 557–592.
[13] K. Tojo: *Totally real totally geodesic submanifolds of compact* 3*-symmetric spaces*, Tohoku Math. J. (2) **53** (2001), 131–143.
[14] K. Tojo: *Classification of totally real and totally geodesic submanifolds of compact* 3*-symmetric spaces*, J. Math. Soc. Japan **58** (2006), 17–53.
[15] J.A. Wolf and A. Gray: *Homogeneous spaces defined by Lie group automorphisms*, I, II, J. Differential Geometry **2** (1968), 77–114, 115–159.

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