

# ON ZARISKI'S PROBLEM CONCERNING THE 14TH PROBLEM OF HILBERT

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## 0. Introduction

Zariski proposed the following problem while he was trying to solve the 14th Problem of Hilbert ([1]):

Let  $A$  be a normal affine ring over a field  $k$  and let  $L$  be a function field over  $k$  such that  $L$  is a subfield of the field of fractions of  $A$ . Is then  $A \cap L$  an affine ring over  $k$ ?

The writer discussed the problem introducing a new method to construct a ring defined by an ideal  $I$  of an integral domain  $R$  ([2]). Namely, letting  $K$  be the field of fractions of  $R$ , we defined the  $I$ -transform of  $R$  to be the ring  $\{x \in K \mid xI^n \subseteq R \text{ for some } n \in \mathbf{N}\}$ . He discussed the  $I$ -transform of  $R$  also in [3].

These articles [2], [3] were written, dreaming an affirmative answer of the 14th Problem of Hilbert. But, we know already that the problem has a negative answer, and the writer wishes to write down the main results of articles [2] and [3], without such a dream and in a generalized form.

We begin with some preliminaries on Krull rings and on discrete valuation rings. Then, we give some characterization of rings which are obtained as the intersection of some normal affine ring with some function field in a generalized form (Theorems 2.1, 2.2).

In this article, by a *ring*, we mean a commutative ring with identity. By a *normal ring*, we mean an integral domain which is integrally closed in its field of fractions. The *derived normal ring* of an integral domain  $A$  means the integral closure of  $A$  in the field of fractions of  $A$ . When we say that  $A$  is an *affine ring* over a ring  $B$ , we assume always that  $B$  is a noetherian integral domain, and  $A$  is a finitely generated integral domain over  $B$ . By a *function field* over  $B$ , we mean the field of fractions of some affine ring over  $B$ .

## 1. Preliminaries

By a *discrete valuation*, we mean an additive valuation whose value group is isomorphic to  $\mathbf{Z}$ . Hence, a discrete valuation ring is a rank one discrete valuation ring.

Let  $R$  be an integral domain with field of fractions  $K$ .  $R$  is a *Krull ring* if there is a set  $V$  of discrete valuations of  $K$  such that (1) if  $0 \neq a \in K$ , then  $\{v \in V \mid v(a) \neq 0\}$  consists only of a finite number of valuations, and (2)  $R = \{a \in K \mid v(a) \geq 0 \text{ for every } v \in V\}$ . ( $K$  is regarded as a Krull ring with empty  $V$ ).

This notion was introduced by Krull (endlich discrete Hauptordnung) and the following Lemma 1.1 and Lemma 1.2 are well known:

**Lemma 1.1** ([4, §37]). *Under the circumstances, (1) if  $S$  is a multiplicatively closed subset of  $R$  not containing 0, then the ring of fractions  $R_S$  is a Krull ring with the set of valuations  $V' = \{v \in V \mid v(s) = 0 \text{ for every } s \in S\}$  and therefore, (2) with the set  $P$  of all prime ideals  $p$  of height one in  $R$ , each  $R_p$  is a member of  $V$  and  $R$  is the intersection of  $R_p$  ( $p \in P$ ).*

**Lemma 1.2** ([5, Thm.2]). *The derived normal ring of a noetherian integral domain is a Krull ring.*

**Lemma 1.3** ([3, Lemma 2.4]). *If  $I$  is an ideal of a Krull ring  $R$ . Then, the  $I$ -transform  $R^*$  of  $R$  is a Krull ring which coincides with the intersection of all  $R_p$  where  $p$  runs through height one prime ideals not containing  $I$ .*

*Proof.* Let  $P$  be as in Lemma 1.1, (2) and for each  $p \in P$ , we denote by  $v_p$  the discrete valuation defined by  $R_p$  and such that the value group is the additive group  $\mathbf{Z}$ . Set  $P' = \{p \in P \mid I \not\subseteq p\}$  and  $P'' = \{p \in P \mid I \subseteq p\}$ . Note that, for a  $p$  in  $P$ ,  $p$  is in  $P'$  if and only if  $I$  contains some element whose value by  $v_p$  is 0. In other words,  $p$  in  $P$  is in  $P''$  if and only if every element of  $I$  has a positive value by  $v_p$ . An element  $a$  of the field  $K$  of fractions of  $R$  is in  $R^*$  if and only if  $aI^n \subseteq R$  for some  $n \in \mathbf{N}$ . Since  $b$  in  $K$  is in  $R$  if and only if  $v_p(b) \geq 0$  for every  $p \in P$ ,  $aI^n \subseteq R$  implies  $v_p(a) \geq 0$  for every  $p \in P'$ . Conversely, if  $v_p(a) \geq 0$  for every  $p \in P'$ , then taking a natural number  $N$  such that  $v_p(a) \geq -N$  for all  $p \in P''$ , we see that  $aI^N \subseteq R$ . Thus,  $R^*$  is the intersection of all  $R_p$  where  $p$  runs through members of  $P'$ . Since  $\{v_p \mid p \in P\}$  satisfies the condition for a Krull ring, the subset  $\{v_p \mid p \in P'\}$  satisfies the condition for a Krull ring. Q.E.D.

By a *divisorial valuation ring*  $D$  over  $B$ , we understand that  $B$  is a noetherian integral domain and there is a pair of an affine ring  $A$  over  $B$  and a prime ideal  $p$  of height one in the derived normal ring  $A^*$  of  $A$  such that  $D$  coincides with the discrete valuation ring  $A_p^*$ .

**Lemma 1.4** ([6, Lemma 2.1]). *If  $A$  is an affine ring over a discrete valuation ring  $B$  with maximal ideal  $m$ , and if  $p$  is a minimal prime ideal of  $A$  containing  $mA$ , then it holds that*

$$\text{trans.deg}_{B/m} A/p = \text{trans.deg}_B A$$

For the proof, see [6].

**Lemma 1.5.** *Let  $D$  be a discrete valuation ring of a function field  $K$  over  $B$ . Let  $k$  be the field of fractions of  $B$  and  $m$  the maximal ideal of  $D$ .*

(1) *If  $D$  contains  $k$ , then  $D$  is divisorial over  $B$  if and only if*

$$\textcircled{1} \quad 1 + \text{trans.deg}_k D/m = \text{trans.deg}_k K$$

(2) *When  $D$  does not contain  $k$ , then, setting  $B' = D \cap k$ ,  $m' = m \cap k$ , we see that  $D$  is divisorial over  $B'$  if and only if*

$$\textcircled{2} \quad \text{trans.deg}_{B'/m'} D/m = \text{trans.deg}_k K$$

Proof. (1) is obvious, and we consider (2) only. Assume first that  $D$  is divisorial over  $B'$ . Then the equality  $\textcircled{2}$  follows from Lemma 1.4.

Conversely, assume that  $\textcircled{2}$  holds. Let  $r = \text{trans.deg}_k K$ . Then, there are elements  $z_1, \dots, z_r$  of  $D$  such that  $z_1, \dots, z_r$  modulo  $m'$  are algebraically independent over  $B'/m'$ . Since  $B'$  is a discrete valuation ring,  $z_1, \dots, z_r$  are algebraically independent over  $k$  and therefore, denoting by  $(m')$  the ideal of  $B[z_1, \dots, z_r]$  generated by  $m'$ , the ring of fractions  $D' = B'[z_1, \dots, z_r]_{(m')}$  is a discrete valuation ring dominated by  $D$ . By the equality  $\textcircled{2}$ ,  $D$  is algebraic over  $D'$  and therefore,  $D$  is of the form  $D_{m^*}^*$  with integral closure  $D^*$  of  $D'$  in  $K$  and a maximal ideal  $m^*$ . Therefore,  $D$  is divisorial over  $B'$ . Q.E.D.

Noether's normalization theorem is easily generalized as follows (see [7, Chapt.1, Cor.1 to Prop.1]):

**Lemma 1.6.** *If  $A$  is an affine ring over  $B$ , then there are algebraically independent elements  $z_1, \dots, z_t$  ( $\in A$ ) over  $B$  and a non-zero element  $c$  of  $B$  such that  $A[c^{-1}]$  is integral over  $B[z_1, \dots, z_t, c^{-1}]$ .*

## 2. Derived normal rings of affine rings

Let  $A$  be an affine ring over  $B$ .  $B$  is noetherian by our assumption, and therefore  $A$  is noetherian. It follows that the derived normal ring of  $A$  is a Krull ring.

Our main aim is to prove the following two theorems, which are generalizations of similar results in [2].

**Theorem 2.1.** *Assume that  $B$  is a noetherian integral domain, satisfying the following condition (\*).*

(\*)  $\left\{ \begin{array}{l} \text{Let } k \text{ be the field of fractions of } B. \text{ Then, for every divisorial valuation ring } D \\ \text{over } B, D \cap k \text{ is a divisorial valuation ring over } B, \text{ or } D \cap k = k. \end{array} \right.$

Let  $A^*$  be the derived normal ring of an affine ring  $A$  over  $B$  and let  $L$  be a function field over  $B$  which is a subfield of the field  $K$  of fractions of  $A$ . Then, there is a pair of an affine ring  $A'$  and an ideal  $I$  of the derived normal ring  $A'^*$  of  $A'$  such that  $A^* \cap L$  coincides with the  $I$ -transform of  $A'^*$ . Here, unless  $A'^* = A^* \cap L$ ,  $I$  can be chosen so that  $I$  is the intersection of a finite number of height one prime ideals of  $A'^*$ .

**Theorem 2.2.** Assume that  $A'$  is an affine ring over a noetherian integral domain  $B$  and let  $L$  be the field of fractions of  $A'$ . Furthermore, let  $I$  be an ideal of the derived normal ring  $A'^*$  of  $A'$ . Then there is an affine ring  $A$  over  $B$  such that the  $I$ -transform  $T$  of  $A'^*$  coincides with  $A^* \cap L$ , where  $A^*$  is the derived normal ring  $A^*$  of  $A$ .

Proof of Theorem 2.2. We may assume that  $I = \cap_{i=1}^s p_i^*$  with height one prime ideals  $p_i^*$  of  $A'^*$ . Set  $p_i = p_i^* \cap A'$ . Since there are only a finite number of prime ideals in  $A'^*$  which lie over  $p_i$ , enlarging  $A'$  if necessary, we may assume that  $p_i^*$  is the unique prime ideal lying over  $p_i$ , for each  $i$ . Then, there are two elements  $a, b$  of  $A'$  such that height one prime ideals of  $A'$  containing  $J = aA' + bA'$  are exactly  $p_1, \dots, p_s$ . Let  $x, y$  be indeterminates and set  $A = A'[x, y, (ax + by)^{-1}]$ . The derived normal ring  $A^*$  of  $A$  coincides with  $A'^*[x, y, (ax + by)^{-1}]$ .

Obviously  $A^* \cap L$  contains  $A'^*$ . An element  $c$  of  $L$  is in  $A^*$  if and only if  $c$  is expressed as  $f(x, y)/(ax + by)^n$  with  $f(x, y) \in A'^*[x, y]$  and  $n \in \mathbb{N}$ , or equivalently,  $c(ax + by)^n \in A'^*[x, y]$ . This means  $ca^i b^{n-i} \in A'^*$  for  $i = 0, 1, 2, \dots, n$ . This condition is equivalent to  $c \in T$ . Thus,  $A^* \cap L$  coincides with  $T$ . Q.E.D.

Proof of Theorem 2.1. Since  $A^*$  is a Krull ring (Lemma 1.2),  $A^* \cap L$  is a Krull ring. Replacing  $L$  by the field of fractions of  $A^* \cap L$ , we may assume that  $L$  is the field of fractions of  $A^* \cap L$ . There are elements  $c_1, \dots, c_m$  of  $A^* \cap L$  such that  $L$  is the field of fractions of  $B' = B[c_1, \dots, c_m]$ . We may replace  $A$  with  $A[c_1, \dots, c_m]$  and we apply Lemma 1.6 to  $A$  over  $B'$ . Then, with  $0 \neq c \in B'$ , elements of  $A$  are integral over the polynomial ring  $B'[c^{-1}][z_1, \dots, z_t]$  over  $B'[c^{-1}]$ . This means that if  $p'$  is a height one prime ideal of  $B'$  such that  $c$  is not in  $p'$ , then there is a height one prime ideal of  $A$  which lies over  $p'$ . Thus, if  $p''$  is a height one prime ideal of  $A'^*$  containing such  $p'$ , there is a height one prime ideal  $p^*$  of  $A^*$  lying over  $p''$ . So, let  $p_1^*, \dots, p_w^*$  be the prime ideals of height one in  $A^*$  containing  $c$ . Then, if  $p^*$  is a prime ideal of height one in  $A^*$  other than them,  $p^* \cap B'$  is either  $\{0\}$  or a prime ideal of height one. Consider such  $p_i^*$ .  $D_i = A^*_{p_i^*}$  is a divisorial valuation ring over  $B$ . Let  $k$  be the field of fractions of  $B$ , and set  $D_{0i} = D_i \cap k$  and  $D'_i = D_i \cap L$ . Let  $P_{0i}, P_i, P'_i$  be the maximal ideals of  $D_{0i}, D_i, D'_i$ , respectively. By our condition (\*),  $D_{0i}$  is either  $k$  or a divisorial valuation

ring over  $B$ . Since  $D_{0i}$  contains  $B$  and contained in  $D_i$ ,  $D_i$  is a divisorial valuation ring over  $B$  and also over  $D_{0i}$ . Therefore, according to whether  $D_{0i}$  is  $k$  or not,  $\text{trans.deg}_{D_{0i}/P_{0i}} D_i/P_i = \text{trans.deg}_k K - 1$  or  $\text{trans.deg}_k K$  (Lemma 1.5). Similarly, since  $D'_i$  contains  $B$  and contained in  $D_i$ ,  $D_i$  is a divisorial valuation ring over  $D'_i$ . Therefore,  $\text{trans.deg}_{D'_i/P'_i} D_i/P_i = \text{trans.deg}_L K$ . Thus,  $D'_i$  is a divisorial valuation ring over  $B$ . This means that there are elements  $b_1, \dots, b_u$  in  $A^* \cap L$  such that  $D'_i$  is the ring of fractions  $(A^*)_{P^*_i}$  where  $A^*_i$  is the derived normal ring of  $B[b_1, \dots, b_u]$  and  $P^*_i$  is a prime ideal of height one in  $A^*_i$  is the derived normal ring of  $B[b_1, \dots, b_u]$  and  $P^*_i$  is a prime ideal of height one in  $A^*_i$ . This can be applied to all  $i$  and we see that there is an affine ring  $A'$  over  $B$  such that  $B' \subseteq A' \subseteq A^* \cap L$  and such that each  $D'_i$  is obtained as the ring of fractions of the derived normal ring  $A'^*$  of  $A'$  with respect to a height one prime ideals. Let  $p'^*_1, \dots, p'^*_v$  be all of height one prime ideals of  $A'^*$  such that there is no prime ideals of height one in  $A^*$  which lie over them. Now, if  $p^*$  is a height one prime ideal in  $A^*$ , then  $p^* \cap A'^*$  is either  $\{0\}$  or a height one prime ideal of  $A'^*$  other than  $p'^*_1, \dots, p'^*_v$ . Now,  $A^* \cap L = \bigcap (A^*_p \cap L)$ , where  $p$  runs through height one prime ideals in  $A^*$ , hence,  $A^* \cap L$  coincides with the intersection of all  $A'^*_p$  where  $P$  runs through prime ideals of height one in  $A'^*$  other than  $p'^*_1, \dots, p'^*_v$ . Therefore,  $A^* \cap L$  coincides with the  $I$ -transform of  $A'^*$  with  $I = p'^*_1 \cap \dots \cap p'^*_v$ . Q.E.D.

We add here two questions.

**Question 1.** *What is the class of noetherian integral domains for which the condition (\*) above holds?*

Particularly,

**Question 2.** *Assume that  $B$  is a noetherian integral domain such that for any affine ring  $A$  over  $B$  and for two prime ideals  $P, Q$  in  $A$  such that  $P \supset Q$ , the length of maximal chain of prime ideals which begins with  $P$  and ends with  $Q$  is uniquely determined by the pair  $P, Q$ .*

*Is this condition sufficient to hold the condition (\*)?*

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#### References

- [1] O. Zariski: *Interprétations algébrique-géométriques du quatorzième problème de Hilbert*, Bull. Sci. Math. (2), Vol. 78 (1954), 155–168.
- [2] M. Nagata: *A treatise on the 14-th problem of Hilbert*, Mem. Coll. Sci. Univ. Kyoto, Ser. A-Math., Vol. 30, No. 1 (1956), 57–70; *Addition and correction*, ibid., No. 3 (1957), 197–200.
- [3] M. Nagata: *A theorem on finite generation of a ring*, Nagoya Math. Journ., Vol. 27 (1996), 193–205.
- [4] W. Krull: *Idealtheorie*, Springer (1935), (1968)
- [5] M. Nagata: *On the derived normal rings of Noetherian integral domains*, Mem. Coll. Sci. Univ.

Kyoto, Vol. **29**, No. **3** (1955), 293–303.

- [6] M. Nagata: *Finitely generated rings over a valuation ring*, Journ. Math. Koto Univ., Vol **5**, No. **2** (1966), 163–169.
- [7] M. Nagata: *A general theory of algebraic geometry over Dedekind domains, I*, Amer. Journ. Math., Vol. **78** (1956), 78–116.

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