Kobayashi, T. and Nishi, H.
Osaka J. Math.
31 (1994), 109–136

A NECESSARY AND SUFFICIENT CONDITION FOR A 3-MANIFOLD TO HAVE GENUS g HEEGAARD SPLITTING (A PROOF OF HASS-THOMPSON CONJECTURE)

Tsuyoshi KOBAYASHI and Haruko NISHI

(Received July 9, 1992)

1. Introduction

R.H. Bing had shown that a closed 3-manifold M is homeomorphic to S^3 if and only if every knot in M can be ambient isotoped to lie inside a 3-ball [1]. In [5], J. Hass and A. Thompson generalize this to show that M has a genus one Heegaard splitting if and only if there exists a genus one handlebody V embedded in M such that every knot in M can be ambient isotoped to lie inside V. Moreover, they conjectures that this can be naturally generalized for genus $g(>1)$. The purpose of this paper is to show that this is actually true. Namely we prove:

Main Theorem. *Let M be a closed 3-manifold. There exists a genus g handlebody V such that every knot in M can be ambient isotoped to lie inside V if and only if M has genus g Heegaard splitting.*

The proof of this goes as follows. First we generalize Myers' construction of hyperbolic knots in 3-manifolds [14] to show that, for each integer $g(\geq 1)$, every closed 3-manifold has a knot whose exterior contains no essential closed surfaces of genus less than or equal to g (Theorem 4.1). Knots with this property will be called g-characteristic knots. Then we show that, for each integer $h(\geq 1)$, there exists a knot K in M such that K cannot be ambient isotoped to a 'simple position' in any gensu h handlebody which gives a Heegaard splitting of of M. This is carried out by using good pencil argument of K. Johannson [9] (, and we note that this also can be proved by using inverse operation of type A isotopy argument of M. Ochiai [15]). By using this very complicated knot in M, we can show that if M contains a genus g handlebody as in Main Theorem, then M admits a Heegaard splitting of genus g.

This paper is organized as follows. In Section 2, we slightly generalize

This work was supported by Grant-in-Aid for Scientific Research. The Ministry of Education, Science and Culture.

results of Johannson in [8], which will be used in Sections 3 and 5. In Section 3, we generalize the concept of prime tangles [13] to 'height g' tangles, and show that there are many height g tangles. In Section 4, we show that, by using these tangles, there are infinitely many g-characteristic knots in M. In Section 5, we show that there are non-simple position knots by using these g-characterisisc knots. In Section 6, we prove Main Theorem.

The second author would like to express her thanks to Prof. Mitsuyoshi Kato for his constant encouragement.

2. Preliminaries

Throughout this paper, we work in the piecewise linear category. All submanifolds are in general position unless otherwise specified. For a subcomplex H of a complex K, $N(H, K)$ denotes a regular neighborhood of H in K. When K is well understood, we often abbreviate $N(H, K)$ to $N(H)$. Let N be a manifold embedded in a manifold M with $\dim N = \dim M$. Then $\mathrm{Fr}_M N$ denotes the frontier of N in M. For the definitions of standard terms in 3-dimensional topology, we refer to [6], and [7].

An arc a properly embedded in a 2-manifold S is *inessential* if there exists an arc b in ∂S such that $a \cup b$ bounds a disk in S. We say that a is *essential* if it is not inessential. A *surface* is a connected 2-manifold. Let E be a 2-sided surface properly embedded in a 3-manifold M. We say that E is *essential* if E is incompressible and not parallel to a subsurface of ∂M. We say that E is *∂-compressible* if there is a disk Δ in M such that $\Delta \cap E = \partial\Delta \cap E = \alpha$ is an essential arc in E, and $\Delta \cap \partial M = \partial\Delta \cap \partial M = \beta$ is an arc such that $\alpha \cup \beta = \partial\Delta$. We say that E is *∂-incompressible* if it is not *∂-compressilee.*

Let F be a closed surface of genus g. A *genus g compression body W* is a 3-manifold obtained from $F \times [0, 1]$ by attaching 2-handles along mutually disjoint simple closed curves in $F \times \{1\}$ and attaching some 3-handles so that $\partial_- W = \partial W - \partial_+ W$ has no 2-sphere components, where $\partial_+ W$ is a component of ∂W which corresponds to $F \times \{0\}$. It is known that W is irreducible ([2, Lemma 2.3]). We note that W is a handlebody if $\partial_- W = \emptyset$.

A *complete disk system D* for a compression body W is a disjoint union of disks $(D, \partial D) \subset (W, \partial_+ W)$ such that W cut along D is homeomorphic to

$$\begin{cases} \partial_- W \times [0, 1], & \text{if } \partial_- W \neq \emptyset, \\ B^3, & \text{if } \partial_- W = \emptyset. \end{cases}$$

Note that for any handle decomposition of W as above, the union of the cores of the 2-handles extended vertically to $F \times [0, 1]$ contains a complete disk system for W.

Let M be a compact 3-maniifold such that ∂M has no 2-sphere components. A *genus g Heegaard splitting* of M is a pair (V, W) where V, W are genus g compression bodies such that $V \cup W = M$, $V \cap W = \partial_+ V = \partial_+ W$. Then the purpose of this section is to give a generalization of some results of Johannson [8] to the above Heegaard splittings.

The next lemma can be proved by using the above complete eisk system, and the proof is left to the reader (cf. [2, Lemma 2.3]).

Lemma 2.1. *Let S be an incompressible and ∂-incompressible surface properly embedded in a compression body W. Then S is etiher a closed surface parallel to a component of $\partial_- W$, disk D with $\partial D \subset \partial_+ W$, or an annulus A, where one component of ∂A lies in $\partial_+ W$ and the other in $\partial_- W$.*

The annulus A as in Lemma 2.1 is called *vertical.*

Let S be an essential surface in a 3-manifold M, and (W_1, W_2) a Heegaard splitting of M. We say that S is *normal* with respect to (W_1, W_2) if:

(1) each component of $S \cap W_1$ is an essential disk or a vertical annulus, and

(2) $S \cap W_2$ is an essential surface in W_2.

By using the incompressibility of S and Lemma 2.1, we see that if M is irreducible then S is ambient isotopic to a normal surface. Suppose that S is normal. Let $S_2 = S \cap W_2$, and b an arc properly embedded in S_2. We say that b is a *compression arc* (for S_2), if b is essential in S_2, and there exists a disk Δ in W_2 such that $\partial\Delta = b \cup b'$, where $b' = \Delta \cap \partial_+ W_2$ (and, possibly, $\mathrm{Int}\,\Delta \cap S_2 \neq \emptyset$). Let M, (W_1, W_2), and S be as above. Let $\mathcal{D}$ be a complete disk system for W_2. We say that S is *strictly normal* (with respect to $\mathcal{D}$), if:

(1) S is normal with respect to (W_1, W_2), and

(2) for each component D_i of $\mathcal{D}$, we have; (i) each component of $S_2 \cap D_i$ (if exists) is an essential arc in S_2 and (ii) if b is an arc of $S_2 \cap D_i$ such that ∂b is contained in mutually different components C_1, C_2 of ∂S_2, and that C_1 or C_2 is a boundary of a disk component E of $S \cap W_1$, then for each (open arc) component $\partial D_i - \partial b$, say a_1, a_2, we have have $a_i \cap \partial E \neq \emptyset$.

Then the next proposition is a generalization of [8, 2.3].

Proposition 2.2. *Let M, (W_1, W_2) be as above. Let S be an essential surface in M which is normal with respect to (W_1, W_2). Then we have either:*

(1) S is strictly normal, or

(2) S is ambient isotopic to a surface S' in M such that; (i) S' is normal with respect to (W_1, W_2), and (ii) $\#\{S' \cap W_1\} < \#\{S \cap W_1\}$.

The proof of this is essentially contained in [8, Sect. 2]. However, for the convenience of the reader, we give the proof here.

Lemma 2.3. *Let M, (W_1, W_2), and S be as in Proposition 2.2. Let b be a* *compression arc for* $S \cap W_2$, *with a disk* Δ *in* W_2 *such that* $\partial\Delta = b \cup b'$, *where* $b' = \Delta \cap \partial_+ W_2$ *and* $\partial b = \partial b'$. *Suppose that there is a disk componnet* E *of* $S \cap W_1$ *such that* $b' \cap E = \partial b' \cap \partial E$ *consists of a point. Then* S *is ambient isotopic to a surface* S' *in* M *such that;*

(1) S' *is normal with respect to* (W_1, W_2), *and*

(2) $\#\{S' \cap W_1\} = \#\{S \cap W_1\} - 1$.

Proof. Note that b joins mutually different components of $S \cap W_1$, one of them is E and the other is D, say. Let E_+ be one of the components of $\mathrm{Fr}_{W_1} N(E, W_1)$ which meets b'. We note that ∂E_+ meets b' in one point. Let $B = N(E_+, N(E, W_1)) \cup N(\Delta, W_2)$. Then B is a 3-ball in M since $\partial E_+ \cap \partial\Delta$ is a point. Move W_1 by an ambient isotopy along B so that the image W_1' has the following form: $W_1' = \mathrm{cl}(W_1 - N(E_+, N(E, W_1))) \cup N(b, W_2)$.

Let $W_2' = \mathrm{cl}(M - W_1')$. Then clearly (W_1', W_2') is a Heegaard splitting of M which is ambient isotopic to (W_1, W_2). Note that $S \cap W_1'$ is a system of essential disks and vertical annuli which has the number of components one less than that of $S \cap W_1$, because E is connected with D by the band $S \cap N(b, W_2)$. Moreover, $S \cap W_2'$ is an essential surface since b is essential in S_2. It follows that there exists an ambient isotopy of M which push S into S' so that S' is normal with respects to (W_1, W_2) and $\#\{S' \cap W_1\} = \#\{S \cap W_1\} - 1$. ■

Proof of Proposition 2.2. Let $\mathcal{D} = \cup D_i$ be a complete disk system for W_2. Suppose that S is not strictly normal. Since S_2 is incompressible and W_2 is irreducible, by standard innermost disk argument, we may assume that $S_2 \cap D_i$ has no circle components. If there exists an inessential arc component b of $S_2 \cap D_i$ in S_2, then without loss of generality, we may assume that there exists a disk Δ in S_2 such that $\Delta \cap \mathcal{D} = b$, and $\Delta \cap \partial_+ W_2$ is an arc b' such that $\partial b = \partial b'$, and $b \cup b' = \partial\Delta$. We note that $\mathrm{Fr}_{W_2} N(D_i \cup \Delta, W_2)$ consists of three disks E_0, E_1, E_2 such that E_0 is parallel to D_i. Then it is easy to see that either $(\mathcal{D} - D_i) \cup E_1$ or $(\mathcal{D} - D_i) \cup E_2$ is a complete disk system for W_2. Moreover this complete disk system intersects S_2 in less number of components. Continuing in this way, we can finally get the complete disk system for W_2 which intersects S_2 in all essential arcs.

Therefore, it S is not stricity normal, we may assume that it does not satisfy (ii) of the definition. Then, there exists an arc component b of $\mathcal{D} \cap S_2$ such that ∂b is contained in mutually different components C_1, C_2 of ∂S, and one of them, say C_2, is a boundary of a disk component E of $S \cap W_1$, and for one of open arc components a of $\partial D_i - \partial b$, $a \cap \partial E = \emptyset$. Note that b is a compression arc for S_2, and $b \cap E = \partial b \cap \partial E$ is a point. Hence by Lemma 2.3, S can be ambient isotoped to a 2-manifold S' which is normal with respects to (W_2, W_1), and $\#\{S' \cap W_1\} < \#\{S \cap W_1\}$. ■

3. Height h tangles

An *n-string tangle* is a pair (B, t), where B is a 3-ball, and t is a union of mutually disjoint n arcs properly embedded in B. We note that for each tangle (B, t) there is a (unique) 2-fold branched cover of B with branch set t. We say that a tangle (B, t) has *height h* if the 2-fold branched cover of B over t contains no essential surface S with $-\chi(S) \leq h$. We note that 2-string tangles with height -1 are called prime tangles in [13]. We say that a tangle (B, t) has *property I* if $X = \mathrm{cl}(B - N(t, B))$ is ∂-irreducible, i.e. ∂X is incompressible in X. The purpose of this section is to show that a height h tangle actually exists. Namely we prove:

Proposition 3.1. *For each even integer $g(\geq 2)$, and for each integer $m(\geq -1)$. there exists a g-string tangle (B, t) with height m. Moreover if we suppose that $2g-4 > m \geq 0$, then we can take (B, t) to have proprety I.*

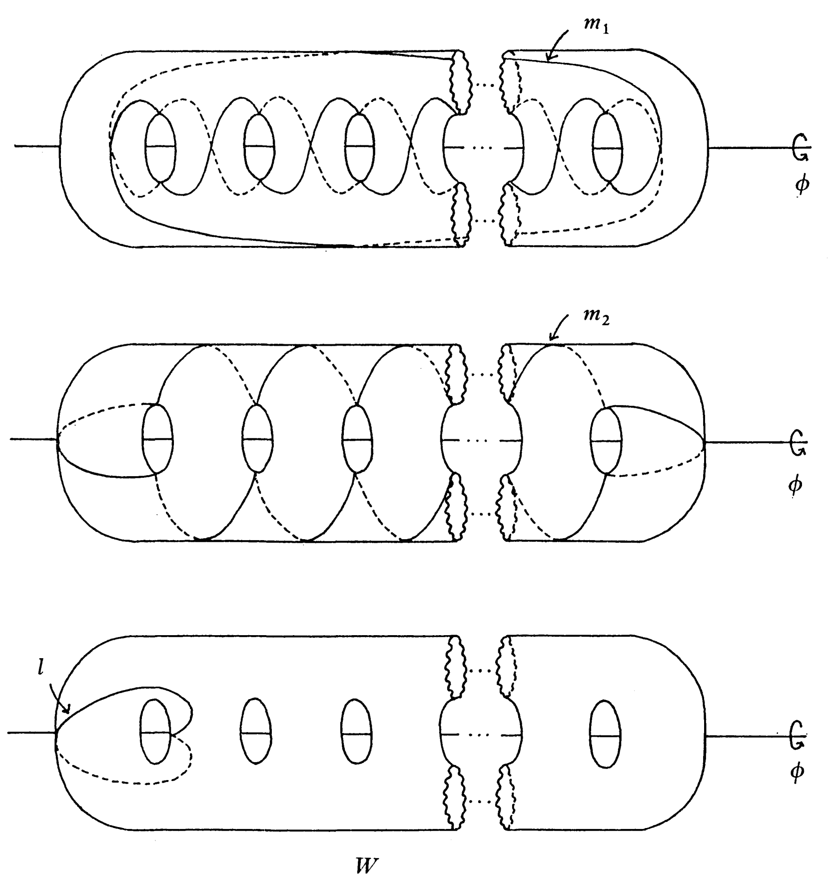

Figure 3.1

For the proof of Proposition 3.1, we recall some definitions and results from [12]. Let W be a compression body and $l(\subset\partial_+W)$ a simple closed curve. Then the *height* of l for W, denoted by $h_W(l)$, is defined as follows [12].

$$h_W(l)=\min\{-\chi(S)\,|\,S \text{ is an essential surface in } W \text{ such that } \partial S\cap l=\emptyset\}.$$

Let W be a handlebody of genus $g(\geq 2)$, and m_1, m_2, l simple closed curves on ∂W as in Figure 3.1. Then for a sufficiently large integer q we let f be an automorphism of ∂W such that $f=T_{m_1}\circ T_{m_2}^{2q}$, where T_{m_i} denotes a right hand Dehn twist along the simple closed curve m_i. By sections 2, 3 of [12] we have:

Proposition 3.2. *For each $m(\geq -1)$, there exists a constant $N(m)$ such that if $p>N(m)$, then $h_W(\bar{l})>m$ for each simple closed curve $\bar{l}$ on ∂W which is disjoint from $f^p(l)$ and not contractible in ∂W.*

Let N be the 3-manifold obtained from W by attaching a 2-handle along the simple closed curve $f^{N(m)+1}(l)$. By Proposition 3.2 and the handle addition lemma (see, for example [3]), we see that N is irreducible. We note that W admits an orientation preserving involution ϕ as in Figure 3.1. Then we have:

Lemma 3.3. *The involution ϕ extends to an involution $\bar{\phi}$ of N. Moreover, the quotient space of N under $\bar{\phi}$ is a 3-ball B, and the singular set t in B consists of a union of g arcs properly embedded in B.*

Proof. We note that m_1, m_2, and l are invariant under ϕ. Hence we may suppose that $f^{N(m)+1}(l)$ is invariant under ϕ. Hence the involution ϕ naturally extends to the 2-handle $D^2\times[0,1]$, where the quotient space of $D^2\times[0,1]$ is a 3-ball and the singular set in $D^2\times[0,1]$ is an arc α properly embedded in $D^2\times\{1/2\}$. We note that W/ϕ is a 3-ball, the singular set consists of $g+1$ arcs s, and $N(f^{N(m)+1}(l), \partial W)/\phi$ is a 2-disk. Moreover it is easy to see that the components of $\partial\alpha$ are contained in mutually different components of s. Hence we see that B is a 3-ball and t consists of g arcs properly embedded in B. ■

Let B, t be as above, and we regard (B, t) as a g-string tangle. Then we show that (B, t) is a height m tangle (the first half of Proposition 3.1) by using good pencil argument of Johannson used in [9].

Lemma 3.4. (B, t) *has hight m.*

Proof. Let $C=N(\partial W, W)\cup(\text{a 2-handle})$. Let E be a disk properly embedded in C, which is obtained by extending the core of the 2-handle vertically to $N(\partial W, W)$ $(\cong\partial W\times[0,1])$. Then C is a genus g compression body, and E is a complete disk system for C. We regard $\mathrm{cl}(N-C)$ as W. Then we note that (C, W) is a Heegaard splitting of N.

Let $C'=\mathrm{cl}(C-N(E,C))$, then C' is homeomorphic to $\partial_-C\times[0,1]$, where ∂_-C corresponds to $\partial_-C\times\{0\}$. Let E^+, E^- be the disks in $\partial_-C\times\{1\}$ corresponding to $\mathrm{Fr}_C N(E,C)$.

Claim 1. Let D be an essential disk in C which is non-separating in C. Then D is ambient isotopic to E in C.

Proof. Since C is irreducible, by standard innermost disk argument, we may suppose that $D\cap E$ has no circle components. Suppose that $D\cap E=\emptyset$. Then ∂D bounds a disk D' in $\partial_-C\times\{1\}$ such that D is parallel to D'. Since ∂D is essential in ∂_+C and non-separating in ∂_+C, we see that D' contains exactly one of E^+, E^-. Hence D is parallel to E in C. Suppose that $D\cap E\neq\emptyset$. Let Δ be an outermost disk in D, i.e. $\alpha=\Delta\cap E=\partial\Delta\cap E$ an arc, $\beta=\Delta\cap\partial D$ an arc such that $\alpha\cup\beta=\partial\Delta$ and $\alpha\cap\beta=\partial\alpha=\partial\beta$. Then we see that $\Delta\cap C'$ is a properly embedded disk in C'. Without loss of generality, we may suppose that $\partial(\Delta\cap C')\cap E^-=\emptyset$. Then there is a disk Δ' in $\partial_-C\times\{1\}$ such that $\partial\Delta'=\partial(\Delta\cap C')$. If Δ' does not contain E^-, then by moving D by an ambient isotopy, we can remove α from $D\cap E$. Suppose that Δ' contains E^-. Then, by tracing $\mathrm{cl}(\partial D-\beta)$ from one endpoint to the other, we see that there exists a subarc β' in $\partial D-\beta$ such that $\beta'\cap E^+=\emptyset$, $\beta'\subset\Delta'$, and $\partial\beta'\subset\partial E$. Hence, by moving D by an ambient isotopy, we can reduce the number of components of $D\cap E$. Then by the induction on $\#\{D\cap E\}$, we have the conclusion. ■

Claim 2. Let D be an essential disk in C which is separating in C. Then D can be ambient isotoped so that D is disjoint from E. Moreover, D splits C into a solid tours containing E, and a manifold homeomorphic to $\partial_-C\times[0,1]$.

Proof. Since C is irreducible, by standard innermost disk arguement, we may assume that $D\cap E$ has no circle components. Suppose that $D\cap E\neq\emptyset$. Let Δ be an outrmost disk in D such that $\Delta\cap E=\alpha$ and $\beta=\Delta\cap\partial D$. Then $\Delta\cap C'$ is a properly embedded disk in C'. Without loss of generality, we may assume that $\partial(\Delta\cap C')\cap E^-=\emptyset$. Then there is a disk Δ' in $\partial_-C\times\{1\}$ such that $\partial\Delta'=\partial(\Delta\cap C')$. If Δ' does not contain E^-, then by moving D by an ambient isotopy, we can remove α from $D\cap E$. Suppose that Δ' contains E^-. Then, by tracing $\mathrm{cl}(\partial D-\beta)$ from one endpoint to the other, we see that there exists a subarc β' in $\partial D-\beta$ such that $\beta'\cap E^+=\emptyset$, $\beta'\subset\Delta'$, and $\partial\beta'\subset\partial E^-$. Hence, by moving D by an ambient isotopy, we can reduce the number of components of $D\cap E$. Then by the induction on $\#\{D\cap E\}$, we have the first conclusion of Claim 2. Hence we may assume that $D\cap E=\emptyset$.

Let T be the closure of the component of $C-D$ which contains E, and T' the closure of the other componnet. By [2, Corollary B.3], we see that T, T' are compression bodies. Since T contains a non-separating disk E, and $\partial_-C\subset T'$, we see that T is a handlebody. Then, by Claim 1, we see that

T' is a solid torus. This shows that $\partial_- T'(=\partial_- C)$ is homeomorphic to $\partial_+ T'$, so that T' is homeomorphic to $\partial_- C \times [0, 1]$. ■

By Claim 2, we immediately have:

Claim 3. Let D_1, D_2 be essential disks in C such that D_1 and D_2 are both separating, and mutually disjoint in C. Then D_1 is parallel to D_2.

Next, we show:

Claim 4. Let A be a vertical annulus in C. Then A can be ambient isotoped so that it is disjoint from E.

Proof. Since C is irreducible and A is incompressible in C, by standard innermost disk argument, we may suppose that $E \cap A$ has no circle components. Suppose that $A \cap E \neq \emptyset$. Then each component of $E \cap A$ is an arc whose endpoints are contained in $\partial_+ C$. Let Δ be an outermost disk in A, such that $\Delta \cap E = \alpha$ an arc and $\beta = \Delta \cap \partial A$ an arc in $\partial A \cap \partial_+ C$. Then, $\Delta \cap C'$ is a properly embedded disk in C'. Without loss of generality, we may assume that $\partial(\Delta \cap C') \cap E^- = \emptyset$. Then there is a disk Δ' in $\partial_- C \times \{1\}$ such that $\partial \Delta' = \partial(\Delta \cap C')$. If Δ' does not contain E^-, then by moving A by an ambient isotopy, we can remove α from $A \cap E$. Suppose that Δ' contains E^-. Then, by tracing $\mathrm{cl}(\partial A \cap \partial_+ C - \beta)$ from one endpoint to the other, we see that there exists a subarc β' in $(\partial A \cap \partial_+ C) - \beta$ such that $\beta' \cap E^+ = \emptyset$, $\beta' \subset \Delta'$, and $\partial \beta' \subset \partial E^-$. Hence, by moving A by an ambient isotopy, we can reduce the number of components of $A \cap E$. Then by the induction on $\sharp\{A \cap E\}$, we have the conclusion. ■

Let S be an essential surface properly embedded in N and chosen to minimize $-\chi(S)$. In the rest of this proof, we show that $-\chi(S) > m$. By moving S by an ambient isotopy, we may assume that S is normal with respect to (C, W) (Sect. 2). Then $S \cap C \neq \emptyset$, and each component of $S \cap C$ is an essential disk or a vertical annulus in C. Let p be the number of the disk components of $S \cap C$, and suppose that p is minimal among all the essential surfaces $\bar{S}$ such that $-\chi(\bar{S}) = -\chi(S)$, and $\bar{S}$ is normal with respect to (C, W). Let $S^* = S \cap W$.

Suppose that $S \cap C$ has no disk components. Let A be any annulus component of $S \cap C$. Then, by Claim 4, we may asume that A is disjoint from E. Therefore $(S^*, \partial S^*) \subset (W, \partial W - \partial E) = (W, \partial W - f^{N(m)+1}(l))$. Since $f^{N(m)+1}(l)$ has height m, we have $-\chi(S) = -\chi(S^*) > m$.

Now suppose that $S \cap C$ has a disk component. By the argument of the proof of Proposition 2.2, there exists a complete disk system $\mathcal{D}$ of W such that each component of $\mathcal{D} \cap S^*$ is an essential arc in S^*. Let α be an outermost arc component of $\mathcal{D} \cap S^*$, i.e. there exists a disk Δ in $\mathcal{D}$ such that $\Delta \cap S^* = \partial \Delta \cap S^*$

$=\alpha$ an essential arc in S^*, and $\Delta\cap\partial W=\partial\Delta\cap\partial W=\beta$ an arc such that $\alpha\cup\beta=\partial\Delta$.

Assume that $\partial\beta$ is contained in mutually different components of ∂S^*, and one of which is a boundary of a disk component E^* of $S\cap C$. Then S is not strictly normal since $\mathrm{Int}\,\beta\cap\partial E^*=\emptyset$. Hence, by Proposition 2.2, S is ambient isotopic to a normal surface S' with respect to (C, W), and S' intersects W in less number of disk components than that of S, contradicting the minimality of p.

Therefore we have the following four cases.

Case 1. Both endopoints of β are contained in the boundaries of annulus components of $S\cap C$.

By Claims 1, and 2, we may suppose that $\beta\cap\partial E=\emptyset$. Let $\Delta_1=\beta\times[0,1]\subset C'(\cong\partial_-C\times[0,1])$ be a disk in C such that $\beta\times\{1\}$ corresponds to β, and $\partial\beta\times[0,1]=\Delta_1\cap(S\cap C)$. Let $\tilde{\Delta}=\Delta\cup\Delta_1$. Let $\tilde{S}$ be the 2-manifold obtained by ∂-compressing S along $\tilde{\Delta}$. If $\tilde{S}$ is disconnected, choose one essential component of $\tilde{S}$ and we denote it by $\tilde{S}$ again. Then $\tilde{S}$ is an essential surface in N and $-\chi(\tilde{S})\leq-\chi(S)-1<-\chi(S)$. This contradicts the minimality of $-\chi(S)$.

Case 2. Both endopints of β are contained in the boundary of one non-separating disk component D of $S\cap C$.

Let S' be an essential surface obtained by moving S by an ambient isotopy along Δ. Then $S'\cap C$ has an annulus component A', which is obtained from D by attaching a band produced along β. Let $\partial A'=\{\alpha_1, \alpha_2\}$. By Claim 1, we may suppose that $\partial D\cap\partial E=\emptyset$, hence, that $\alpha_1\cap E=\emptyset$ $(i=1, 2)$. Let $A_i=\alpha_i\times[0,1]\subset\partial_-C\times[0,1]$ be a vertical annulus in C. Let $\tilde{S}=(S'-A')\cup A_1\cup A_2$. If $\tilde{S}$ is disconnected, choose one essential component, and denote it by $\tilde{S}$ again. Then $\tilde{S}$ is an essential surface in N, and $-\chi(\tilde{S})\leq-\chi(S)$. Moreover $\tilde{S}$ is normal with respect to (C, W), and the number of the disk components of $\tilde{S}\cap C$ is less than p. This contradicts the minimality of p.

Case 3. Both endpoints of β are contained in the boundary of one separating disk component D of $S\cap C$, and β does not lie in the solid torus T_0 splitted by D from C.

Let S' be as in Case 2. Then there exists an annulus A' in $S'\cap C$ such as in Case 2. Let $\partial A'=\{\alpha_1, \alpha_2\}$. Then, by Claim 2, we may assume that D is disjoint from E. Hence $\alpha_i\cap E=\emptyset$ $(i=1, 2)$. Then, by the same argument as in Case 2, we have a contradiction.

Case 4. Both endpoints of β are contained in the boundary of one separating disk component D of $S\cap C$, and β lies in the solid torus T_0 splitted by

D from C.

Let S', A' be as in Case 2.

Claim 5. A' is incompressible in C.

Proof. Assume that A' is compressible in C. Since S' is incompressible, the core curve of A' is contractible in S'. Hence there is a planar surface P in S^* such that $\partial P = l_0 \cup l_1 \cup \cdots \cup l_r$, where $r \geq 1$, $l_0 \cap D = l_0 \cap \partial D$ an arc, $l_1, \cdots, l_r$ are boundary of disk components of $S' \cap C$. See Figure 3.2. Since $\mathscr{D}$ is a complete disk system for W, each component of $P - (\mathscr{D} \cap P)$ is simply connected. This shows that there is a component b of $\mathscr{D} \cap P (\subset \mathscr{D} \cap S^*)$ which satisfies the assumption of Lemma 2.3, contradicting the minimality of p. ∎

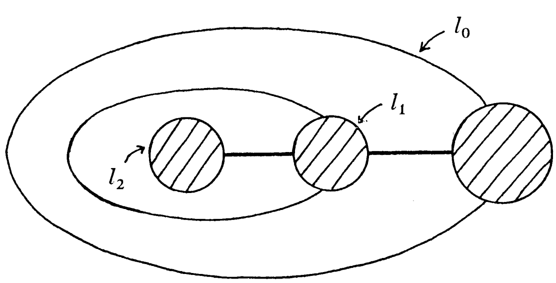

Figure 3.2

By Claims 1, and 5, we see that $S \cap C$ has no non-separating disk component. Let $\{D_1, D_2, \cdots, D_q\}$ be the system of disk components of $S \cap C$ which lies in this order. Then, by Claim 3, these components are mutually parallel in C. Let A be an annulus in $\partial_+ C$ such that A contains $\partial D_1 \cup \cdots \cup \partial D_q$, and each ∂D_i is ambient isotopic in A to a core of A. We suppose that $\sharp\{\partial \mathscr{D} \cap \partial D_i\}$ is minimal in the ambient isotopy class of $\partial \mathscr{D}$ in $\partial W (= \partial_+ C)$, and hence, $I = \partial \mathscr{D} \cap A$ is a system of essential arcs in A. We lable the points $\partial D_i \cap I$ by i, then in each component of I, they lie in this order.

Claim 6. There exists a subsystem P of $\mathscr{D} \cap S^*$ such that there exists a component I_0 of I which satisfies the following.

(1) Every arc of P has one of its endpoints in I_0.

(2) Every arc of $\mathscr{D} \cap S^*$ which has one of its endpoints in I_0 belongs to P.

(3) Every arc t of P joins I_0 with one of components of I which are neighbouring of I_0 in $\partial \mathscr{D}$, i.e. if s_1, s_2 are subarcs of $\partial \mathscr{D}$ such that $(\operatorname{Int} s_i) \cap I = \emptyset$, and one of its endpoints lies in ∂I_0 and the other in the boundary of a component I_i of I, say, then one of the endpoints of t lies in $I_1 \cup s_1 \cup s_2 \cup I_2$ (Figure 3.3).

Proof. Let I_1 be a component of I. Suppose that I_1 does not satisfy the conclusions of Claim 6. Then there is an arc t_1 of $\mathscr{D} \cap S^*$ such that one of its

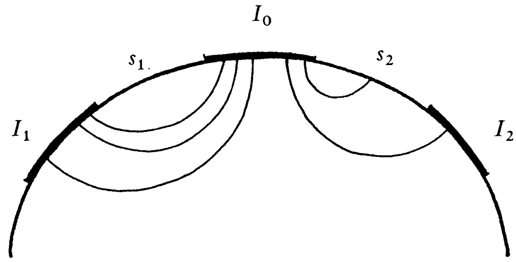

Figure 3.3

endpoints lies in I_1 and does not join two neighbouring components of I. Let E_1 be the closure of a component of $\mathscr{D}-t_1$, and I_2 a component of I contained in ∂E_1. If I_2 does not satisfy the conclusions of Claim 6, then there is an arc t_2 of $E_1 \cap S^*$ such that one of its endpoints lies in I_2 and does not join two neighbouring of I. Let E_2 be the closure of the components of $\mathscr{D}-t_2$ such that $E_2 \subset E_1$. By continueing in this way, it is easy to see that we finally obtain a component of I satisfying the conclusion of Claim 6. ■

Claim 7. For each component of P in Claim 6, both of its endpoints are contained in I, and have the same label.

Proof. Assume that there exists an arc α such that it has one of its endpoints in I_0 and the other not in I. Then α satisfies the assumption of Lemma 2.3, contradciting the minimality of p. Let a_1, a_2 be the closures of the components of $\partial\mathscr{D}-\partial P$ which contains s_1, s_2 respectively. Since $D_1, \cdots, D_q$ are mutually parallel separating disks in C, we see that the points ∂a_i are contained in either ∂D_1 or ∂D_q. This immediately shows that, for each component α of P, the endpoints of α have the same label (Figure 3.4). ■

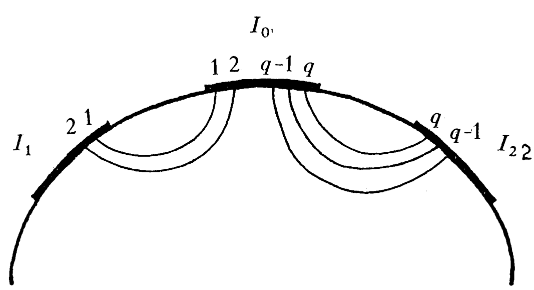

Figure 3.4

Claim 8. $\partial P \subset I_0 \cup I_1$, *say* (Figure 3.5).

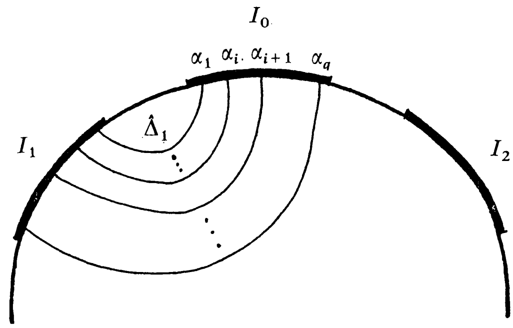

Figure 3.5

Proof. Let α_i be the component of P such that one of its endpoints contained in I_0 is labelled by i. Assume that one endpoint of α_1 is contained in I_1, and that there exists α_i such that one endpoint of α_i is contained in I_2. Then by Claim 7, $\partial\alpha_q$ is contained in ∂D_q, and one endpoint of α_q is contained in I_2. Let Δ be a disk in $\mathcal{D}$ which is splitted by α_q and does not contain $\alpha_1 \cup \cdots \cup \alpha_{q-1}$. We may suppose that $\Delta \cap \partial_+ C$ is not contained in the solid torus splitted by D_q from W. Assume that there exists a component α of $\mathcal{D} \cap S^*$ in $\Delta - \alpha_q$. Then $\partial\alpha$ is contained in annulus components of $S \cap C$. Hence it reduces to Case 1, and we have a contradiction. Therefore $\Delta \cap S^* = \alpha_q$. Let $\beta_q = \Delta \cap \partial\mathcal{D}$. Since β_q cannot lie in the solid torus T_0, it reduces to Case 3, a contradiction. ■

Let $P = \{\alpha_1, \cdots, \alpha_q\}$ be as above. Let Δ_1 be the disk in $\mathcal{D}$ splitted by α_1 and does not contain $\alpha_2 \cup \cdots \cup \alpha_q$, and $\Delta_i (2 \leq i \leq q)$ the closure of the component of $\mathcal{D} - \alpha_i$ such that $\Delta_i \supset \Delta_1$. By moving S by an ambient isotopy along Δ_i successively, we obtain a surface S'' which intersects C in annuli, and in particular, there exist q annuli which are mutually parallel in C. Let $\bar{l}$ be one of the components of ∂A. Then $\bar{l}$ is a simple closed curve in ∂W, and by Claim 2, we may assume that $\bar{l}$ is disjoint from $f^{N(m)+1}(l)$ $(=\partial E)$. Let $\tilde{S}$ be an essential component of $S'' \cap W$. Then $(\tilde{S}, \partial\tilde{S}) \subset (W, \partial W - \bar{l})$. By Proposition 3.2, we see that $-\chi(S) \geq -\chi(\tilde{S}) > m$. This completes the proof. ■

Now we give the proof of the latter half of Proposition 3.1. Let W' be a genus g compression body with $\partial_- W'$ a genus $g-1$ closed surface, m_1', m_2', l' simple closed curves on $\partial_+ W'$ as in Figure 3.1. Then by applying the above argument to W' and $f' = T_{m_2'} \circ T_{m_2'}^{2q'}$ together with Sect. 6 of [12] we have:

Proposition 3.2'. *For each $m (\geq -1)$, there exists a constant $N'(m)$ such that if $p > N'(m)$, then $h_{W'}(\bar{l}) > m$ for each simple closed curve $\bar{l}$ on $\partial_+ W'$ which is disjoint from $f^p(l')$ and not contractible in $\partial^+ W'$.*

Let ϕ' be the involution on W' as in Figure 3.6. Let N' be a 3-manifold obtained from W' by attaching a 2-handle along $f^{N'(m)+1}(l')$. Then we have:

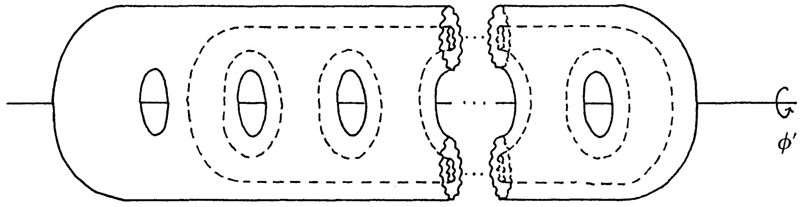

Figure 3.6

Lemma 3.3'. *The involution ϕ' extends to the involution $\bar{\phi}'$ of N'. Moreover, the quotient space of N' under $\bar{\phi}'$, denoted by B', is homeomorphic to (2-sphere)$\times[0, 1]$, and the singular set t' in B' consists of a union of $2g$ arcs such that the endpoints of each component of t' are contained in pairwise different components of $\partial B'$.*

Moreover, by applying the argument of the proof of Lemma 3.4 to N', we have:

Lemma 3.4'. *Let S be an essential surface in N'. Then we have $-\chi(S)>m$.*

The proofs of these are essentially the same as above, and we omit them.

Proof of the latter half of Proposition 3.1. Let $(\tilde{B}, \tilde{t})$ be a tangle which is obtained from (B, t) by capping off (B', t') so that ∂t is joined with $\partial t'$ in a component of $\partial B'$. Then the 2-fold branched cover $\tilde{N}$ of $\tilde{B}$ branched over $\tilde{t}$ is regarded as a union of N and N'. Let $F=N\cap N'$, then F is a closed orientable surface of genus $g-1$.

Claim. $\tilde{N}$ is irreducible and F is incompressible in $\tilde{N}$.

Proof. Since $h_W(f^{N(m)+1}(l))>m$, $\partial_+W-f^{N(m)+1}(l)$ is incompressible in W. We note that W is irreducible. Then by the handle addition lemma, we see that N is irreducible and ∂N is incompressible in N. Similarly, N' is irreducible and $\partial N'$ is incompressible in N'. Hence $\tilde{N}$ is irreducible and F is incompressible in $\tilde{N}$. ■

First we show that $(\tilde{B}, \tilde{t})$ has height m. Let S be an essential surface in $\tilde{N}$, chosen to minimize $-\chi(S)$. Suppose that $S\cap F=\emptyset$. If S is boundary-parallel in N or N', then $-\chi(S)=2g-4>m$. If S is not boundary-parallel (hence, essential) in N, then by Lemma 3.4, $-\chi(S)>m$. If S is not boundary-parallel (hence, essential) in N', then by Lemma 3.4', we see that $-\chi(S)>m$.

Suppose that $S\cap F\neq\emptyset$ and $S\cap F$ has the minimal number of the components among all the essential surfaces in $\tilde{N}$ ambient isotopic to S. Then, by the irreducibility of N, we see that each component of $S\cap N$ is incompressible in N. Moreover, by using the minimality of $\sharp\{S\cap F\}$ again, we see that each component of $S\cap N$ is an essential surface in N. Hence we have $-\chi(S\cap N)>m$, by Lemma 3.4. On the other hand, since F is incompressible in N', $S\cap N'$ has no disk components. Therefore $\chi(S\cap N')\leq 0$, and, hence, $-\chi(S)=-(\chi(S\cap N)+\chi(S\cap N'))\geq -\chi(S\cap N)>m$.

Next, we show that $(\tilde{B}, \tilde{t})$ has Property I. Let $\tilde{X}=\mathrm{cl}(\tilde{B}-N(\tilde{t}, \tilde{B}))$ be the tangle space and $X=\tilde{X}\cap B$, $X'=\tilde{X}\cap B'$. Let $P=X\cap X'$. Then P is a planar surface properly embedded in $\tilde{X}$. By Propositions 3.2 and 3.2', it is easy to see that P is incompressible in X and X'. Suppose that there exists a compressing disk D for $\partial\tilde{X}$, and $\sharp\{D\cap P\}$ is minimal among all the compressing disks for $\partial\tilde{X}$.

If $D\cap P=\emptyset$, then $D\subset X'$ and $\partial D\subset\partial X'-P$. Hence by moving D by a rel P ambient isotopy of X', we may suppose that $\partial D\subset\partial X'\cap\partial\tilde{B}$. Since $\partial X'\cap\partial\tilde{B}$ is incompressible in X', we see that ∂D bounds a disk in $\partial X'\cap\partial\tilde{B}$, a contradiction.

Suppose that $D\cap P\neq\emptyset$. Since P is incompressible in $\tilde{X}$, and $\tilde{X}$ is irreducible, by standard innermost disk argument, we may suppose that $D\cap P$ has no circle components. Moreover, by the minimality of $\sharp\{D\cap P\}$, we see that $D\cap P$ has no inessential components in P. Let α be an outermost arc component of $D\cap P$ in D, i.e. there exists a disk Δ in D such that $\Delta\cap P=\alpha$, $\Delta\cap\partial D=\beta$ an arc such that $\partial\Delta=\alpha\cup\beta$ and $\partial\alpha=\partial\beta$. Then Δ is properly embedded in either X or X'. The first case contradicts the incompressibility of P in X. Then we consider the second case. Suppose that the endpoints of α are contained in different boundary components of P, say d_1, d_2. Let t_1', t_2' be the components of t' such that $N(t_i', B')\cap P=d_i\,(i=1, 2)$. Let $A=\mathrm{Fr}_{X'}N(N(t_1', B')\cup\Delta\cup N(t_2', B'), X')$. Recall that $N'\to B'$ is the 2-fold branched cover with $\bar{\phi}'$ generating the group of covering translation. Let $\tilde{A}$ be the lift of A in N'. Then $\tilde{A}$ consists of two annuli. If $\tilde{A}$ is compressible in N', then by equivariant loop theorem ([10]), there exists a compressing disk $\tilde{D}$ such that $\phi(\tilde{D})\cap\tilde{D}=\emptyset$ or $\phi(\tilde{D})=\tilde{D}$. The first case contradicts the incompressibility of A. Since ϕ exchanges the components of $\tilde{A}$, the second case does not occur. Therefore $\tilde{A}$ is incompressible in N'. Since $\tilde{A}$ is not boundary parallel, $\tilde{A}$ is essential in N' with $\chi(\tilde{A})=0$. This contradicts Lemma 3.4.' Suppose that $\partial\alpha$ lies in one component of ∂P, say α_0. Let t_0' be the component of t' such that $N(t_0', B')\cap P=\alpha_0$. Let A be the component of $\mathrm{Fr}_{X'}N(N(t_0', B')\cup\Delta)$ such that each component of $P-(A\cap P)$ contains even components of ∂P. Then we have a contradiction as above, completing the proof. ■

4. Characteristic knots

Let M be a closed 3-manifold throughout this section.

Two knots K_0 and K_1 in M are *equivalenet* if there exists an ambient isotopy h_t $(0\leq t\leq 1)$ of M such that $h_0=\mathrm{id}$, and $h_1(K_0)=K_1$. We say that K_0 and K_1 are *inequivalent* if they are not equivalent. Let g be an inetger such that $g\geq 1$. A knot K in M is a *g-characteristic knot* if the exterior of K has no 2-sided closed incompressible surfaces of genus less than or equal to g except for boundary-parallel tori.

In this section, we prove the following theorem. The proof of this is a generalization of a construction of simple knots in [14] (see also [5]).

Theorem 4.1. *For each integer $g(\geq 1)$, every closed orientable 3-manifold M contains infinitely many, mutually inequivalent g-characteristic knots.*

REMARK. We note that if $\operatorname{rank} H_1(M; Q)\geq 2$, then, for each knot K in M, there exists a non-separating closed incompressible surface in $E(K)$.

Proof. First we recall a *special handle decomposition* of M from [14]. A handle decomposition $\{h_i^k\}$ of M is *special* if;

(1) The intersection of any handle with any other handle is either empty or connected.

(2) Each 0-handle meets exactly four 1-handles and six 2-hanles.

(3) Each 1-handle meets exactly two 0-handles and three 2-handles.

(4) Each pair of 2-handles either

(a) meets no common 0-handle or 1-handle, or

(b) meets exactly one common 0-handle and no common 1-handle, or

(c) meets exactly one common 1-handle and two common 0-handles.

(5) The complement of any 0-handles in H is connected, where H is the union of the 0-handles and the 1-handles.

(6) The union of any 0-handle with H' is a handlebody, where H' is the union of the 2-handles and the 3-handles.

Note that every closed orientable 3-manifold has a special handle decomposition [14, Lemma 5.1].

Now we fix a special handle decomposition $\{h_i^k\}$ of M. For each 1-handle h_j^1, we identify h_j^1 with $D\times[0, 1]$, where D is a disk and $D\times[0, 1]$ meets 0-handles in $D\times\{0, 1\}$. Let g be an integer such that $g\geq 1$. Let α_j be a system of $2g+2$ arcs properly embedded in h_j^1 such that each arc is identified with {one point} $\times[0, 1]$ $(\subset D^2\times[0, 1])$. Let $\tau_i=(B_i, t_i)$ be a copy of $(4g+4)$-string tangle with height $4g-4$ and Property I (Progosition 3.1). Identify each 0-handle h_i^0 with B_i in a way that ∂t_i is joined with the boundary of the arcs $\alpha_{j_i(1)}$, $\alpha_{j_i(2)}$, $\alpha_{j_i(3)}$, $\alpha_{j_i(4)}$, where $h_{j_i(1)}^1, \cdots, h_{j_i(4)}^1$ are the four 1-handles which meet the 0-handle h_i^0, and $(\cup_i t_i)\cup(\cup_j\alpha_j)$ becomes a knot K where the unions are taken over all the 0-handles and 1-handles of the handle decomposition.

Let $V=(\cup_i h_i^0)\cup(\cup_j h_j^1)$ and $V'=M-\text{Int}\,V$. Then we note that (V, V') is a Heegaard splitting of M.

Assertion 1. *The above knot K in M is a g-characteristic knot.*

Proof. Let $V_1=\text{cl}(V-N(K))$, $V_2=V'$, $X_i^0=V_1\cap h_i^0$, and $X_i^1=V_1\cap h_i^1$. Then $X_i^1\cap(\cup X_i^1)$ consists of four disk-with-(2g+2)-holes properly embedded in V_1, say $P_{i1}, P_{i2}, P_{i3}, P_{i4}$.

Claim 1. Each P_{ij} is incompressible in V_1, and V_1 is irreducible.

Proof. Suppose that $X_k^0\cap X_l^1=P_{kj}$. Since the height of τ_i is greater than -1, we see that P_{kj} is incompressible in X_k^0. Since (X_l^1, P_{kj}) is homeomorphic to $(P_{kj}\times[0, 1], P_{kj}\times\{0\})$, we see that P_{kj} is incompressible in X_l^1. From these facts, it is easy to see that each P_{kj} is incompressible in V_1. Then the irreducibility of each X_k^0, X_l^1, and the incompressibility of each P_{ij} imply that V_1 is irredicible. ■

Let $Q_i=\partial X_i^0\cap\partial B_i$. Then Q_i is an $(8g+8)$-punctured sphere properly embedded in $E(K)$.

Claim 2. Each Q_i is incompressible in $E(K)$, and $E(K)$ is irreducible.

Proof. Let $W=\text{cl}(V-\cup_j X_j^1)$ and $W'=V'\cup(\cup_j X_j^1)$ (Figure 4.1). Then we note that W, W' are handlebodies.

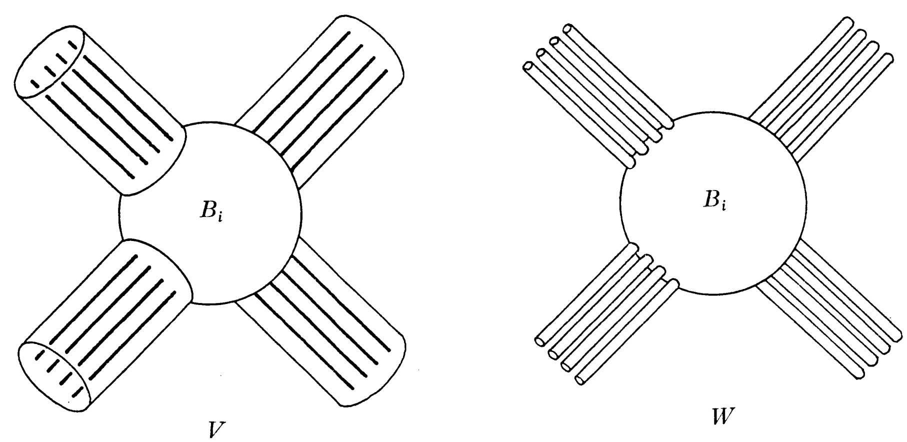

Figure 4.1

Suppose that there exists a compresing disk D for Q_i in $E(K)$. Since (B_i, t_i) has height $4g-4$, we see that Int D is not contained in h_i^0. Let D' be a disk in ∂h_i^0 such that $\partial D'=\partial D$. We note that $V'\cup h_i^0$ is a handlebody by the definition of a special handle decomposition (6). Then it is easy to see that $W'\cup h_i^0$ is a handlebody. Hecne $W' \cup h_i^0$ is irreducible, and the 2-sphere $D \cup D'$ bounds a 3-ball B in $W' \cup h_i^0$. Since $V - h_i^0$ is connected by the definition of a special handle decomposition (5), we see that $W - h_i^0$ is connected. Since $\partial D = \partial D' \subset Q_i$, and $W - h_i^0$ is not contained in B, this implies that ∂D bounds a disk in Q_i. Hence Q_i is incompressible. Since $E(K) = W' \cup (\cup_i X_i^0)$, $W' \cap X_i^0 = Q_i$, by the irreducibility of W', X_i^0, and the incompressibility of Q_i, we see that $E(K)$ is irreducible. ■

Let S be a closed incompressible surface in the exterior $E(K)$ of K in M which is not a boundary parallel torus in $E(K)$. Then S must intersect V_1 since V_2 is a handlebody. We suppose that $\#\{S \cap \partial V_1\}$ is minimal among all surfaces which is ambient isotopic to S in $E(K)$.

Claim 3. $S \cap V_1$ is incompressible in V_1, and there exists X_i^0 such that $X_i^0 \cap (S \cap V_1) \neq \emptyset$.

Proof. By the irreducibility of $E(K)$ (Claim 2), and the minimality of $\#\{S \cap \partial V_1\}$, we see that $S \cap V_1$ is incompressible in V_1. Assume that $X_i^0 \cap (S \cap V_1) = \emptyset$ for each i, i.e. $S \cap V_1 \subset \cup X_j^1$. Suppose that $X_j^1 \cap (S \cap V_1) \neq \emptyset$. Let $S_j = X_j^1 \cap (S \cap V_1)$. Then, by [4, Sect.8 Lemma], we see that each component of S_j is an annulus which is parallel to an annulus in $X_j^1 \cap \partial V_2$, contradicting the miniimality of $\#\{S \cap \partial V_1\}$. ■

Now we suppose that $\#\{(S \cap V_1) \cap (\cup_i Q_i)\}$ is minimal among the ambient isotopy class of $S \cap V_1$ in V_1. Let X_i^0 be the tangle space in a 0-handle h_i^0 such that $X_i^0 \cap (S \cap V_1) \neq \emptyset$, and $S_i = X_i^0 \cap (S \cap V_1)$. Let $p: N \to B_i$ be the 2-fold branched cover of B_i over t_i with ϕ generating the group of the covering translation. Let $\tilde{S}_i = p^{-1}(S_i)$. If $\tilde{S}_i$ is compressible in N, there exists a compressing disk D for $\tilde{S}_i$ in N such that either $\phi(D) \cap D = \emptyset$ or $\phi(D) = D$ [10]. However the first case contradicts the incompressibility of S_i. Hence $\phi(D) = D$ and $p(D)$ is a disk in B_i meeting t_i in one point. Then compress S_i by $p(D)$ (hence, the surface intersects K in two points). By repeating this step finitely many times for all i such that $X_i^0 \cap (S \cap V_1) \neq \emptyset$, we finally get a 2-manifold S' in M such that each component of $\tilde{S}_i' = p^{-1}(S_i')$ is incompressible in N, where $S_i' = B_i \cap (S' \cap V_1)$. Then we have the following two cases.

Case 1. There exists i such that $\tilde{S}_i'$ has a non-boundary-parallel component.

Then $\tilde{S}_i'$ has an essential component F in N. Since (B_i, t_i) has height $4g-4$, $-\chi(F) > 4g-4$. Suppose that $p(F)$ does not intersect with the singular set. Then either $p(F)$ is homeomorphic to F, or $p: F \to p(F)$ is a regular covering, and, hence, we have either $\chi(F) = \chi(p(F))$, or $\chi(p(F)) = \chi(F)/2$. By the minimality of $\#\{(S \cap V_1) \cap (\cup_i Q_i)\}$, incompressibility of Q_i, and Claim 2, we see that each component of $\partial p(F)$ is essential in S. Hence we have $-\chi(S) \geq$

$-\chi(F)>2g-2$, and the genus of S is greater than g. Suppose that F intersects the singular set in $q(\geq 1)$ points. Then we have $\chi(p(F)-K)=(\chi(F)-q)/2<(\chi(F))/2<2-2g$. By the same reason as above, we see that each component of $\partial p(F)$ is essential in S. Hence we see that $-\chi(S)=-\chi(S'-K)\geq-\chi(p(F)-K)>2g-2$. Hence the genus of S is greater than g.

Case 2. For every i, each component of $\tilde{S}_i'$ is boundary-parallel in N.

Move $\tilde{S}_i'$ by an equivariant ambient isotopy along those parallelisms so that S_i' is pushed off B_i. By Claim 3, we see that S' meets K. Let $A_j=\partial h_j^1-(\cup_i \partial h_i^0)$. Assume that $S'\cap(\cup_j A_j)=\emptyset$. Then $S'\subset \mathrm{Int}(\cup_j h_j^1)$. Then, by [4, Sect. 8 Lemma], we see that each component of S' is a 2-sphere intersecting exactly one component of α_j in two points. This implies that S is a boundary-parallel torus, contradicting our assumption. Therefore $S'\cap(\cup_j A_j)\neq\emptyset$. Since S is incompressible in $E(K)$, and $E(K)$ is irreducible (Claim 2), the minimality of $\#\{S\cap\partial V_1\}$ implies that $S'\cap(\cup_j A_j)$ has no inessential components in $\cup_j A_j$. Hence, by [4, Sect. 8 Lemma], we see that each component of $S'\cap h_j^1$ is a horizontal disk in $h_j^1\cong D\times[0, 1]$. It follows that S' meets all the components of α_j. Since α_j consists of $2g+2$ arcs, this shows that for each component F' of S', we have $\chi(F'-K)\leq 2-(2g+2)=-2g$. Hence $\chi(S)=\chi(S'-K)\leq-2g$. Then we conclude that the genus of S is greater than g. ■

Let n be the number of 0-handles of $\{h_j^i\}$. Let $F_i(i=1, \cdots, n)$ be a closed surface of genus $4g+4$ in $E(K)$ obtained by pushing ∂X_i^0 slightly into $\mathrm{Int}\, E(K)$.

Assertion 2. *$F_1, \cdots, F_n$ are incompressible in $E(K)$ and F_i is not parallel to F_j for each $i\neq j$.*

Proof. Assume that there is a compressing disk D for F_i in $E(K)$. Since the tangle τ_i has Property I, D lies in $\mathrm{cl}(E(K)-X_i)$. Let $\mathcal{A}$ be the union of $4g+4$ annuli in $\mathrm{cl}(E(K)-X_i)$ such that one boundary component of each annulus is contained in F_i and the other boundary component is a union of core curves of the annuli in $\partial E(K)$ corresponding to $\mathrm{Fr}_{B_i} N(t_i, B_i)$ (Figure 4.2).

If $D\cap\mathcal{A}=\emptyset$, by moving D by an ambient isotopy of $E(K)$, we may assume that ∂D lies in $Q_i=\partial B_i\cap X_i$. This contradicts the incompressibility of Q_i in $E(K)$ (Claim 2 in the proof of Theorem 4.1). Hence we have $D\cap\mathcal{A}\neq\emptyset$. Then we suppose that $\#\{D\cap\mathcal{A}\}$ is minimal among all compressing disks for F_i. Since $\mathrm{cl}(E(K)-X_i)$ is irreducible, we see that $D\cap\mathcal{A}$ has no circle components, by standard innermost disk argument. Let α be an outermost arc component of $D\cap\mathcal{A}$ in $\mathcal{A}$, i.e. there exists a disk Δ in $\mathcal{A}$ such that $\Delta\cap D=\alpha$, $\Delta\cap\partial\mathcal{A}=\beta$ an arc such that $\partial\Delta=\alpha\cup\beta$ and $\partial\alpha=\partial\beta$. Then by compressing D along Δ toward F_i we have two disks D', D'' such that $\partial D'\subset F_i$, $\partial D''\subset F_i$. Since D is a compressing disk for F_i, we see that one of D', D'' is a compressing disk for F_i, contradicting the minimality of $\#\{D\cap\mathcal{A}\}$. Hence F_i is incompressible in $E(K)$.

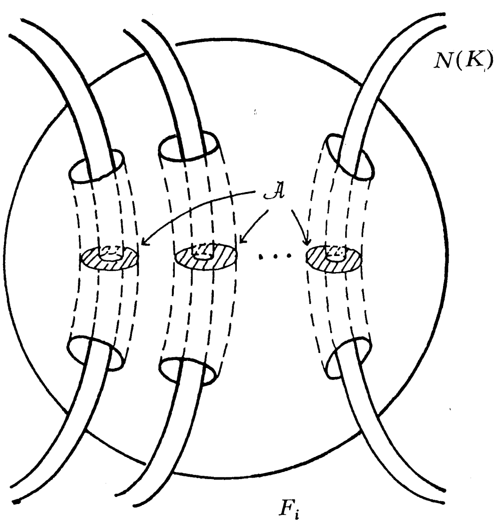

Figure 4.2

Next suppose that F_i and F_j are parallel in $E(K)$ for some $i \neq j$. Then $n=2$, and contradicting the fact that $\{h_i^j\}$ is special (cf. [5, Fact 1 of Proposition 3]). ■

For the proof of Theorem 4.1, we need the following theorm which is due to Haken.

Theorem 4.2. ([4], [6]). *Let M be a compact, orientable 3-manifold. There is an integer $n(M)$ such that if $\{F_1, \cdots, F_k\}$ is any collection of mutually disjoint incompressible closed surfaces in M, then either $k<n(M)$, or for some $i \neq j$, F_i is parallel to F_j in M.*

Completion of the Proof of Theorem 4.1. First we note that for every non-negative integer h, there exists a special handle decomposition of M with more than h 0-handles [5, Fact 2 of Proposition 3].

Let $K_0=K$ be a g-characteristic knot in M obtained by the above construction (Assertion 1). Let $M_0=M-\operatorname{Int} N(K_0)$. Then we find a special handle decomposition of M with h 0-handles, where $h>n(M_0)$. Let K_1 be a g-characteristic knot constructed as above by using this handle decomposition. Then $M_1=M-\operatorname{Int} N(K_1)$ contains h incompressible, mutually non-parallel closed surfaces (Assertion 2). Then, by Theorem 4.2, we see that M_1 is not homeomorphic to M_0. Hence K_0 and K_1 are inequivalent. Continuing in this way, we obtain infinitely many inequivalent g-characteristic knots in M. ■

5. Existence of a non-simple position knot

Let H be a handlebody, and k a knot in H. We say that k is in a *simple* *position* in H if there exists a disk D properly cmbedded in H such that $D \cap k = \emptyset$, and D splits a solid torus V from H such that $k \subset V$ and k is a core curve of V (Figure 5.1). We note that k is in a simple position in H if and only if $\mathrm{cl}(H - N(k))$ is a compression body.

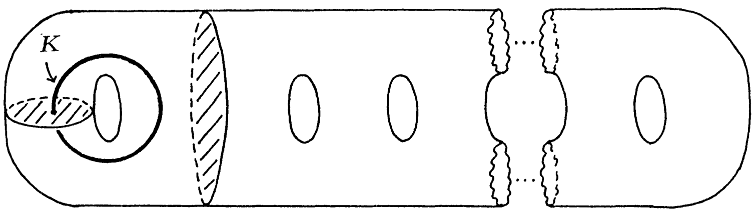

Figure 5.1

Then the prupose of this section is to prove:

Thoerem 5.1. *Suppose that a closed, orientable 3-manifold M admits a Heegaard splitting of genus h. Then for each integer $g \geq 1$, there exists a g-characteristic knot K in M such that, for any genus h Heegaard splitting (V, W) of M, K is not ambient isotopic in M to a simple position knot in V.*

Proof. Let $\{h_j^i\}$ be a special handle decomposition of M with n 0-handles, where $n \geq 8(3h-3)+1$. By applying the argument of Sect. 4 to this handle decomposition, we get a g-characteristic knot K whose complement contains a system of mutually disjoint, non-parallel incompressible closed surfaces of genus $4g+4$, denoted by $\mathcal{F} = \{F_1, \cdots, F_n\}$ (Sect. 4 Assertion 2).

We show that this knot K satisfies the conclusion of Theorem 5.1.

Assume that there is a genus h Heegaard splitting (V, W) of M such that K is in a simple position in V. Let $V_1 = \mathrm{cl}(V - N(K))$ and $V_2 = W$. Then V_1 is a genus h compression body with $\partial_- V_1$ is a torus. We note that (V_1, V_2) is a Heegaard splitting of $E(K)$. Then, by the irreducibility of $E(K)$, $\mathcal{F}$ can be ambient isotoped to be normal with respect to (V_1, V_2) (see Sect. 2). We suppose that $\#\{\mathcal{F} \cap V_1\}$ is minimal in the ambient isotopy class of $\mathcal{F}$ in $E(K)$.

First we show that there exists a system $\mathcal{F}'$ of surfaces which is ambient isotopic to $\mathcal{F}$ in $E(K)$ and $\mathcal{F}' \cap V_1$ has at least five annulus components $A_1, \cdots, A_5$ which are mutually parallel in V', and essential in $\mathcal{F}'$.

Let $\mathcal{F}_i = \mathcal{F} \cap V_i (i=1, 2)$. Then we note that since ∂V_i can contain at most $3h-3$ parallel classes of mutually disjoint essential simple closed curves, there exists a system of mutually parallel disk components $\{D_1, \cdots, D_q\}$ of $\mathcal{F}_1$ which lies in this order in V_1, where $q \geq 9$.

By the argument of the proof of Proposition 2.2, there exists a complete disk system $\mathcal{D}$ for V_2 such that each component of $\mathcal{D} \cap \mathcal{F}_2$ is an essential arc in $\mathcal{F}_2$. Let A be an annulus in $\partial_+ V_1$ such that A contains $\partial D_1 \cup \cdots \cup \partial D_q$, and each

∂D_i is isotopic to a core of A. We suppose that $\#\{\partial\mathcal{D}\cup\partial D_i\}$ is minimal in the ambient isotopy class of $\partial\mathcal{D}$ in $\partial V_2(=\partial_+ V_1)$, and hence, $I=\partial\mathcal{D}\cap A$ is a system of essential arcs in A. We label the points $\partial D_i\cap I$ by i, then, in each component of I, they lie in this order. Let D be a component of $\mathcal{D}$ such that $D\cap A\neq\emptyset$. Then by applying the argument of Claim 6 of Lemma 3.4, we see that there exists a subsystem P of $D\cap\mathcal{F}_2$ such that there exists a component I_0 of I which satisfies the following.

(1) Every arc of P has one end-point in I_0.
(2) Every arc of $D\cap\mathcal{F}_2$ which has one end point in I_0 belongs to P.
(3) Every arc t of P joints I_0 with one of components of I which are neighbouring of I_0 in ∂D.

Moreover, by the argument of Claim 7 of Lemma 3.4, for each component of P, both of its endpoints are contained in I. Then, by using Lemma 2.3, we see that the endpoints of each component of P have the same label. Hence P consists of at most two subsystems each of which contains all arcs of P joing two components of I. Therefore by labelling "1, 2, $\cdots$, q" instead of "q, $q-1$, $\cdots$, 1" if necessary, we may assume that there exists a subsystem of at least five arcs $\{\alpha_1,\cdots,\alpha_p\}$ $(p\geq 5)$ of $D\cap\mathcal{F}_2$ such that α_i joints two points in I_0 and I_1, say. Let Δ_1 be the disk in D splitted by α_1 and does not contain $\alpha_2\cup\cdots\cup\alpha_p$, and Δ_i $(2\leq i\leq p)$ the closure of the component of $D-\alpha_i$ such that $\Delta_i\supset\Delta_1$. Move $\mathcal{F}$ by an ambient isotopy along Δ_i successively, and denote the image by $\mathcal{F}'$. Then we see that $\mathcal{F}'\cap V_1$ has p mutually parallel annuli $\{A_1,\cdots,A_p\}$ in V_1. By the argument of the proof of Claim 5 of Lemma 3.4, we see that A_i is incompressible, hence essential in V_1.

Now in these parallelisms $A_i\times[0,1]$ in V_1 where $A_i\times\{0\}=A_i$, $A_i\times\{1\}=A_{i+1}(1\leq i\leq p-1)$, there exist annuli Λ_i such that each Λ_i corresponds to $C_i\times[0,1]$ where C_i is a core curve of $A_i(i=1,\cdots,p-1)$ (Figure 5.2).

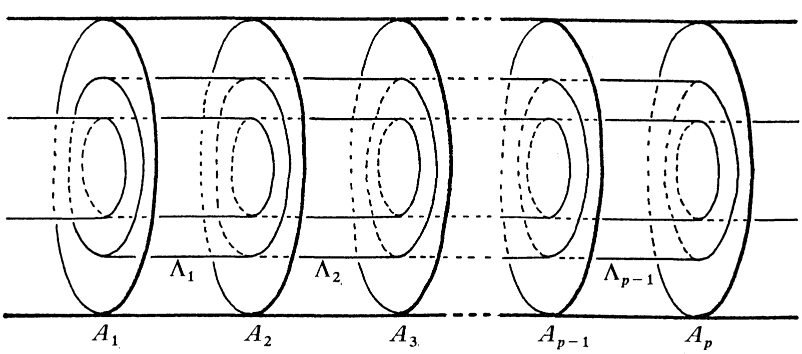

Figure 5.2

Let $E(K)=X_0\cup X_1\cup\cdots\cup X_n$ where X_j corresponds to the 'inside' of F_j

(hence $X_0 \cap X_j = F_j, j=1, \cdots, n$). Then Λ_i is an annulus properly embedded in X_k, for some k. Assume that there exists a compressing disk D for Λ_i in X_j. Let Λ be a subannulus in Λ_i cobounded by ∂D and C_i. Move the disk $D \cup \Lambda$ slightly by an ambient isotopy so that $D \cup \Lambda$ becomes a properly embedded disk in X_k. This contradicts the incompressibility of $\mathscr{F}$ in $E(K)$. Hence, Λ_i is incompressible in X_k. We have either $\Lambda_1 \subset X_0$ or $\Lambda_2 \subset X_0$. If $\Lambda_1 \subset X_0$, then we have $\Lambda_3 \subset X_0$, and if $\Lambda_2 \subset X_0$, then we have $\Lambda_4 \subset X_0$. Now we suppose that $\Delta_1 \subset X_0$, $\Lambda_2 \subset X_1$, and $\Lambda_3 \subset X_0$. (The case of Λ_2, $\Lambda_4 \subset X_0$ is essentially the same.)

Claim. We have either one of:
(1) Λ_1 is boundaiy-parallel in X_0, or
(2) Λ_2 is boundary-parallel in X_1, or
(3) Λ_3 is boundary-parallel in X_0.

Proof. Recall that Q_i is a planar surface in ∂X_i, which corresponds to $\partial X_i \cap \partial B_i$ (Sect. 4). Let $\mathscr{A}$ be a disjoint union of annuli properly embedded in X_0, which is defined in the proof of Assertion 2 of Sect. 4 (Figure 4.2). We suppose that $\#\{\Lambda_1 \cap \mathscr{A}\}$ is minimal among the ambient isotpy class of Λ_1 in X_0. Suppose that $\Lambda_1 \cap \mathscr{A} \neq \emptyset$. If there are inessential arc components of $\Lambda_1 \cap \mathscr{A}$ in Λ_1, let α be the outermost arc component of $\Lambda_1 \cap \mathscr{A}$ in Λ_1, i.e. there exists a disk Δ in Λ_1 such that $\Delta \cap \mathscr{A} = \alpha$, $\Delta \cap \partial \Lambda_1 = \alpha$ an arc in $\partial \Lambda_1$ such that $\partial \Delta = \alpha \cup \beta$ and $\partial \alpha = \partial \beta = \alpha \cap \beta$. Let Δ' be the disk in $\mathscr{A}$ such that $\mathrm{Fr}_{\mathscr{A}} \Delta' = \alpha$. Then, by moving $\Lambda \cup \Delta'$ in a neighborhood of $\mathscr{A}$ by an ambient isotopy of X_0, we get a disk properly embedded in X_0, whose boundary contained in Q_1. Since Q_1 is incompressible in $E(K)$ and X_0 is irreducible, we see that this disk is parallel to a disk in Q_1, This shows that $\alpha \cap Q_1$ is an ineseential arc in Q_1. Therefore there is an ambient isotopy which removes α from $\Lambda_1 \cap \mathscr{A}$, contradicting the minimality of $\#\{\Lambda_1 \cap \mathscr{A}\}$. Suppose that every component of $\Lambda_1 \cap \mathscr{A}$ is an essential arc in Λ_1. Let Π be a disk in Λ_1 which is bounded by two arcs a_1 a_2, of $\Lambda_1 \cap \mathscr{A}$ and two arcs in $\partial \Lambda_1$ such that $\mathrm{Int}\, \Pi \cap \mathscr{A} = \emptyset$. Let Δ_i be a disk in $\mathscr{A}$ such that a_i bounds Δ_i with an arc in $\partial \mathscr{A} (i=1, 2)$. Assume that one of Λ_i is contained in the other. Without loss of generality, we may assume that $\Delta_1 \subset \Lambda_2$. Then by moving $\Pi \cup \Delta_1$ by rel a_2 isotopy, we get a disk Π' in X_0 such that $\Pi' \cap \mathscr{A} = a_2$, $\Pi' \cap \partial X_0 = \mathrm{cl}(\partial \Pi' - a_2)$, and $(\Pi' \cap \partial X_0) \cap Q_1 = \beta'$ an arc. By the above argument, we see that β' is an inessential arc in Q_1 (i.e. there is a disk Δ^* in Q_1 such that $\mathrm{Fr}_{Q_1} \Delta^* = \beta'$). Since Π is reproduced by adding a band to Π' along an arc γ such that $\gamma \cap \Delta^* \neq \emptyset$, we wee that $\Pi \cap Q_1$ consists of two inessential arcs in Q_1, contradicting the minimality of $\#\{\Lambda_1 \cap \mathscr{A}\}$. Hence $\Delta_1 \cap \Delta_2 = \emptyset$. Let $E = \Pi \cup \Delta_1 \cup \Delta_2$. Then, by moving the disk E in a neighborhood of $\mathscr{A}$ by an ambient isotopy of X_0, we may assume that E is a disk properly embedded in X_0 and ∂E in Q_1. Then by the above argument we see that E is parallel to a disk in Q_1 The same is hold for any pair of neighbouring arcs of $\Lambda_1 \cap \mathcal{A}$ Then we conclude that Λ_1 is boundary parallel in X_0. Similarly, if every component of $\Lambda_3 \cap \mathcal{A}$ is an essential arc in Λ_3, Λ_3 is boundary-parallel in X_0.

Now suppose that $\partial\Lambda_i \cap \partial\mathcal{A} = \emptyset$ $(i=1, 3)$ (hence $\Lambda_i \cap \mathcal{A} = \emptyset$ or each component of $\Lambda_i \cap \mathcal{A}$ is an essential circle in Λ_i). Then $\partial\Lambda_2 \cap \partial\mathcal{A} = \emptyset$. Assume that Λ_2 is not boundary-parallel in X_1. Let p: $N \to B_1$ be the 2-fold branched cover over $t_1 = K \cap B_1$ with ϕ generating the group of covering translation. Let $\tilde{\Lambda}_2 = p^{-1}(\Lambda_2)$. Since the tangle (B_1, t_1) has height $4g-4$, $\tilde{\Lambda}_2$ is compressible in N. Then there exists a compressing disk $\tilde{D}$ for $\tilde{\Lambda}_2$ in N such that $\phi(\tilde{D}) \cap \tilde{D} = \emptyset$ or $\phi(\tilde{D}) = \tilde{D}$ ([10]). The first case contradicts the incompressibility of Λ_2 in X_1. In the second case, $D = p(\tilde{D})$ meets t_1 in one point. Let D_1 and D_2 be disks obtained by compressing Λ_2 by D. Since the height of (B_1, t_1) is greater than -1, there is a closure of a component of $B_1 - D_i$, say B^i, such that $(B^i, B^i \cap t_1)$ is a 1-string trivial tangle. Then we have either $B^1 \cap B^2 = \emptyset$, or one of B^1, B^2 is contained in the other (Figure 5.3). In the first case, we see that Λ_2 is parallel to an annulus in ∂X_0 corresponding to a component of $\mathrm{Fr}_{B_1} N(t_1, B_1)$. In the second case, we see that Λ_2 is parallel to an annulus in Q_1. Hence we have the conclusion (2) of Claim. ■

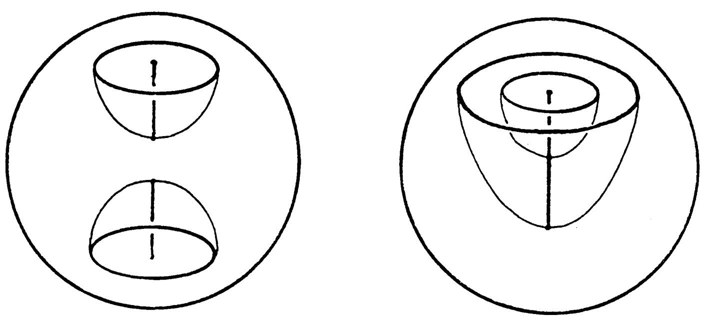

Figure 5.3

Now we may assume that Λ_i is boundary-parallel in X_j for some i and j. By extending the ambient isotopy along this parallelism, we can remove two annuli A_i and A_{i+1} from $\mathcal{F}' \cap V_1$. Denote this image by $\mathcal{F}''$. Then moving $\mathcal{F}''$ by an ambient isotopy, which corresponds to the reverse that of $\mathcal{F}$ to $\mathcal{F}'$, we obtained a system of surfaces $\mathcal{F}'''$ which intersects V_1 in essential disks and the number of the components of $\mathcal{F}''' \cap V_1$ is less than that of $\mathcal{F} \cap V_1$. This contradicts the minimality of the number of the components of $\mathcal{F} \cap V_1$, completing the proof. ■

6. Proof of Main Theorem

In this section, we give a proof of Hass-Thompson conjecture. First we prepare the follwing lemma.

Lemma 6.1. ([3]). *Let (W_1, W_2) be a Heegaard splitting of a 3-manifold M. Let $\mathcal{S}$ be a disjoint union of essential 2-spheres and disks in M. Then, there exists a disjoint union of essential 2-spheres and disks $\mathcal{S}'$ in M such that*

(1) *$\mathcal{S}'$ is obtained from $\mathcal{S}$ by ambient 1-surgery and isotopy,*

(2) *each component of $\mathcal{S}'$ meets $\partial_+ W_1 - \partial_+ W_2$ in a circle,*

(3) *there exists complete disk systems $\mathcal{D}_i$ for W_i, such that $\mathcal{D}_i \cap \mathcal{S}' = \emptyset$ $(i=1, 2)$.*

(4) *if M is irreducible, then $\mathcal{S}'$ is actually isotopic to $\mathcal{S}$.*

Let M be a compact, orientable 3-manifold such that ∂M has no 2-sphere components. A Heegaard splitting (V, W) of M is of *type T(unnel)*, if W is a handlebody (hence V is a compression body with $\partial_- V = \partial M$). Then we define the *T-Heegaard genus* of M, denoted by $g^T(M)$, as the minimal genus of the type T Heegaard splittings. Then for the proof of Main Theorem, we first show:

Proposition 6.2. *Let M be a connected 3-manifold such that ∂M has no 2-sphere components. Suppose that there exists a compressing disk for ∂M in M. Let $\bar{M}$ be a 3-manifold obtained by cutting M along D. Then*

$$g^T(\bar{M}) = \begin{cases} g^T(M)\,, & \text{if } \bar{M} \text{ is disconnected,} \\ g^T(M)-1\,, & \text{if } \bar{M} \text{ connected} \end{cases}$$

Proof. First we note that the T-Heegaard genus is additive under connected sum [3]. Let S be a system of 2-spheres which gives a prime decomposition of M. By standard innermost disk argument, we may assume that D is disjoint from S. Therefore we may assume, without loss of generality, that M is irreducible.

Case 1. D is separating in M.

Let $\bar{M} = M_1 \cup M_2$ where M_i $(i=1, 2)$ is a connected component of $\bar{M}$. Then M is a boundary connected sum of M_1 and M_2, i.e. $M = M_1 \natural M_2$. Hence, the fact that $g^T(\bar{M}) = g^T(M)$ follws from Lemma 6.1 (for the detailed argument, see [3]).

Case 2. D is non-separating in M.

Let (V, W) be a minimal genus type T Heegaard splitting of M. Then, by Lemma 6.1, we may assume that D meets ∂W in a circle. Let $\bar{D} = D \cap W$ and $\bar{A} = D \cap V$. Then $\bar{D}$ is an essential disk in W and $\bar{A}$ is an essential annulus in V. Let $\bar{W} = \mathrm{cl}(W - N(\bar{D}, W))$, and N a sufficiently small regular neighborhood of D in M such that $N \cap \bar{W} = \emptyset$. We identity $\bar{M}$ to $\mathrm{cl}(M-N)$, and let $\bar{V} = \mathrm{cl}(\bar{M} - \bar{W})$. Then we see that $(\bar{V}, \bar{W})$ is a type T Heegaard splitting of $\bar{M}$. Hence $g^T(\bar{M}) \leq g(\partial \bar{W}) = g^T(M) - 1$.

Next suppose that $(\bar{V}, \bar{W})$ is a type T Heegaard splitting of $\bar{M}$ which realizes T-Heegaard genus of $\bar{M}$. By considering dual picture, we identify $\bar{V}$ to $\partial_-\bar{V}\times I\cup$(1-handles). We identify $N(D, M)$ as $D\times[0, 1]$, then $M=\bar{M}\cup(D\times[0, 1])$. Let α be an arc obtained by extendiog the core of $D\times[0, 1]$ vertically to $\partial_-\bar{V}\times[0, 1]$. By general position argument, we may suppose that $\alpha\cap$(1-handles)$=\emptyset$ (hence, α is properly embedded in $\mathrm{cl}(M-\bar{W})$). Let N' be a regular neighborhood of α in $\mathrm{cl}(M-\bar{W})$, $W=\bar{W}\cup N'$, and $V=\mathrm{cl}(M-W)$. Then it is easy to see that W is a handlebody in $\mathrm{Int}M$, and V is a compression body in M. Therefore (V, W) is a type T Heegaard splitting of M. Hence $g^T(M)\leq g(\partial W)=g(\partial\bar{W})+1=g^T(\bar{M})+1$. Therefore $g^T(\bar{M})=g^T(M)-1$. ■

Proof of Main Theorem. The 'if' part of Main Theorem is clear. Hence we give a proof of 'only if' part. Let M, V be as in Main Theorem. Let $E=\mathrm{cl}(M-V)$. If E is a handlebody, then we are done. Hence we suppose that E is not a handlebody. Let $\bar{g}$ be an integer such that V can be extended to a genus $\bar{g}$ Heegaard splitting of M $(\bar{V}, \bar{W})$, i.e. there exists a system of mutually disjoint $\bar{g}-g$ arcs $\mathcal{A}$ properly embedded in E such that $\bar{V}=V\cup N(\mathcal{A}, E)$, $\bar{W}=\mathrm{cl}(M-\bar{V})$ are handlebodies. Let K be a g-characteristic knot in M which is not ambient isotopic to a simple position in any genus $\bar{g}$ handlebody giving Heegaard splittings of M (Theorem 5.1). Then take a handlebody V_* in M with the following properties; (i) V_* contains K, (ii) V_* can be extended to a genus $\bar{g}$ Heegaard splitting, and (iii) the genus of V_*, denoted by g_*, is minimal among all the handlebodies in M satisfying the above conditions (i), and (ii). We note that V satisfies the above conditions (i), and (ii), and, hence, $g_*\leq g$. Let $E_*=\mathrm{cl}(M-V_*)$. Then in the rest of this sectoin, we show that E_* is a handlebody, which completes the proof of Main Theorem.

Now assume that E_* is not a handlebody. Since $E(K)$ is irreducible and $E_*\subset E(K)$, E_* is irreducible. Hence there exists a maximal compression body W_* for ∂E_* in E_* unique up to ambient isotopy [2]. Since E_* is not a handlebody, $\partial_-W_*\neq\emptyset$. Let $Y=V_*\cup W_*$, then (V_*, W_*) is a Heegaard splitting of Y. We note that ∂_-W_* lies in $E(K)$, and the sum of the genus of components of ∂_-W_* is less than or equal to g_*. Then, by the property of g-characteristic knot K, each component of ∂_-W_* is a boundary-parallel torus or a compressible closed surface in $E(K)$. Hence we have the following two cases.

Case 1. Each component of ∂_-W_* is a boundary-parallel torus in $E(K)$.

Assume that ∂_-W_* has more than one components $T_1, \cdots, T_n (n\geq 2)$. Let $P_i (i=1, \cdots, n)$ be the paralleisms between T_i and $\partial E(K)$. By exchanging the suffix if necessary, we may suppose that $P_i\subset P_j$ if $i<j$. Then we have $P_1\supset W_*$. On the other hand, we have $\partial W_*=\partial V_*\cup\partial_-W_*=\partial V_*\cup T_1\cup T_2\cdots\cup T_n$. Hence $P_1\supset T_2, \cdots, T_n$, a contradiction.

Therefore ∂_-W_* consists of one boundary-parallel torus in $E(K)$. Then we see that $Y=V_* \cup W_*$ is a solid torus. Let D be a meridian disk of Y. Since Y is irreducible, by moving D by an ambient isotopy, we may suppose that D meets ∂V_* in a circle (Lemma 6.1). By considering dual picture, we identify W_* to $\partial_- W_* \times [0, 1] \cup$ (1-handles). Then, by Lemma 6.1 (3), we may suppose that $D \cap W_*$ is disjoint from the 1-handles. Let $\alpha_1, \cdots, \alpha_{g_*-1}$ be arcs properly embedded in W_* obtained by extending the cores of the 1-handles vertically to $\partial_- W_* \times [0, 1]$ (hence $\partial_- W_* \cup \alpha_1 \cup \cdots \cup \alpha_{g_*-1}$ is a deformation retract of W_*). Let $Q=N(Y, M)$. Then, move K by an ambient isotopy in Q so that $K \subset \partial Y$, $N(K, Q) \cap N(\alpha_i, Y)=\emptyset$, and $K \cap D=K \cap \partial D$ consists of one point. Let $Y^*=Y \cup N(K, Q)\ (\cong Y)$, and identify $\mathrm{cl}(Q-Y^*)$ with the product of a torus $T(=\partial Y^*)$ and an interval $T\times[0, 1]$. Then, we may view W_*, V_* as follows: $W_*=(T\times[0, 1]) \cup (\cup_i N(\alpha_i, Y))$, $V_*=\mathrm{cl}(Y^*-(\cup_i N(\alpha_i, Y))$.

Let $\Delta=\mathrm{Fr}_{Y^*}(N(K, Q) \cup N(D, Y))$ be a disk properly embedded in V_*. Then Δ splits a solid torus $N(K, Q) \cup N(D, Y)$ from V_*, and K lies in it as a core curve. This implies that K is in a simple position in V_*. Since V_* can be extended to a genus $\bar{g}$ Heegaard splitting, which is ambient isotopic to $(\bar{V}, \bar{W})$, we see that K is ambient isotopic to a simple position in $\bar{V}$, a contradiction.

Case 2. There exists a component of $\partial_- W_*$ which is compressible in $E(K)$.

Let D be a compressing disk for $\partial_- W_*$. Since W_* is a maximal compression body for ∂E_* in E_*, we see that $D \subset Y$. Let $\bar{Y}$ be the 3-manifold obtained by cutting Y along D. Then, by the proof of Proposition 6.2, there exists a minimal genus Heegaard spltting (V^*, W^*) of Y such that $V^* \cap D$ is an essential disk in V^*. We note that since $D \subset E(K)$, K is disjoint from D. Moreover, by moving K by an ambient isotopy in $\bar{Y}$, we may suppose that $K \subset V^* -(D \cap V^*)$. If $g(V^*)<g_*$, attach $g_*-g(V^*)$ trivial 1-handles in W^* disjoint from D to V^*. We denote the new genus g_* Heegaard splitting of Y by (V^*, W^*), again. Then (V^*, W^*) is a genus g_* Heegaard splitting of Y such that V^* cotains K and there exists an essential disk $D^*=V^* \cap D$ in V^* which is disjoint from K.

Let $E^*=\mathrm{cl}(M-Y) \cup W^*$. Since W_* and W^* are compression bodies such that $\partial_- W_*=\partial_- W^*=\partial Y$, and $\partial_+ W_* \cong \partial_+ W^*$ a genus g_* closed surface, W_* is homeomorphic to W^*. Hence $E_*=\mathrm{cl}(M-V_*)=\mathrm{cl}(M-Y) \cup W_* \cong \mathrm{cl}(M-Y) \cup W^*=E^*$ i.e, E_* is homeomorphic to E^*.

By the assumption, V_* can be extended to a genus $\bar{g}$ Heegaard splitting $(\bar{V}_*, \bar{W}_*)$ of M. Let $V'_*=\mathrm{cl}(N(\bar{V}_*, M)-V_*)$, and $W'_*=\mathrm{cl}(E_*-V'_*)$. Then (V'_*, W'_*) is a genus $\bar{g}$ type T Heegaard splitting of E_*. Since E^* is homeomorphic to E_*, there is a genus $\bar{g}$ type T Heegaard splitting $(V^{*\prime}, W^{*\prime})$ of E^* corresponding to (V'_*, W'_*). We note that since $\partial V^{*\prime} \cap V^*=\partial_- V^{*\prime}=\partial V^*$,

$V^{*\prime} \cup V^*$ is a handlebody in M. Hence $(V^{*\prime} \cup V^*, W^{*\prime})$ is a genus $\bar{g}$ Heegaard splitting of M. Let $\tilde{V}$ be a component of $V^* - N(D^*)$ which contains K inside. Then $\tilde{V}$ is a handlebody of genus less than g_* and it can be extended to a genus $\bar{g}$ Heegaard splitting $(V^{*\prime} \cup V^*, W^{*\prime})$ of M. This contradicts the minimality of g_*.

This completes the proof of Main Theorem. ■

Referenes

[1] R.H. Bing: *Necessary and sufficient condition that a 3-manifold be S^3*, Ann. of Math., **68** (1958), 17–37.

[2] F. Bonahon: *Cobordism of automorphisms of surfaces*, Ann. Sc. Éc. Norm. sup. (4) **16** (1983), 237–270.

[3] A.J. Casson and C.McA Gordon: *Reducing Heegaard splittings*, Topology Appl. **27** (1987), 275–283.

[4] W. Haken: *Some results on surfaces in 3-manifolds*, Studies in Modern topology (1968), 34–98, Math. Assos. Amer., Prentice-Hall.

[5] J. Hass and A. Thompson: *A necessary and sufficient condition for a 3-manifold to have Heegaard gensus one*, Proc. Amer. Math. Soc. **107** (1989), 1107–1110.

[6] J. Hemple: 3-manifolds, Princeton University Press, Princeton N.J., 1976.

[7] W. Jaco: Lectures on three-manifold topology, C.B.M.S., Regional Conf. Ser. in Math., 1980.

[8] K. Johannson: *On surfaces and Heegaard surfaces*, Trans. Amer. Math. Soc. **325** (1991), 573–591.

[9] K. Johannson: *Computations in 3-manifolds*, preprint.

[10] P.K. Kim and J.L. Tollefson: *Splitting the P.L. involutions of nonprime 3-manifolds*, Michigan Math. J. **27** (1980), 259–274.

[11] T. Kobayashi: *Structures of full Haken manifolds*, Osaka J. Math. **24** (1987), 173–215.

[12] T. Kobayashi: *Heights of simple loops and pseudo-Anosov homeomorphisms*, Contemp. Math. **78** (1988), 327–338.

[13] W.B.R. Lickorish: *Prime knots and tangles*, Trans. Amer. Math. Soc. **267** (1981), 321–332.

[14] R. Myers: *Simple knots in compact, orientable 3-manifolds*, Trans. Amer. Math Soc. **273** (1982), 75–91.

[15] M. Ochiai: *On Haken's theorem and its extension*, Osaka J. Math. **20** (1983), 461–468.

[16] F. Waldhausen: *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

Tsuyoshi Kobayashi
Department of Mathematics
Nara Women's University
Kitauoya Nishimachi, Nara, 630
Japan

Haruko Nishi
Department of Mathematics
Kyushu University 33
Fukuoka 812
Japan