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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR SCHRÖDINGER OPERATORS WITH HOMOGENEOUS MAGNETIC FIELDS, II

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Introduction

The present paper is a continuation to [9] in which we have studied the asymptotic distribution of eigenvalues (bound state energies) below the bottom of essential spectrum for Schrödinger operators of one particle systems in homogeneous magnetic fields. In this paper we consider a similar problem for Schrödinger operators of many particle systems.

We work in the $3N$ dimensional space $\boldsymbol{R}^{3N}$ with generic point $x=(x^1, \cdots, x^N)$, $x^j=(x_1^j, x_2^j, x_3^j)\in\boldsymbol{R}^3$. Consider N particles of mass μ_j and charge e_j, $1\leqq j\leqq N$, interacting with each other through static potentials $V_{ij}(x^j-x^i)$, $1\leqq i<j\leqq N$, and subjected to external potentials $V_{0j}(x^j)$, $1\leqq j\leqq N$, and to a homogeneous magnetic field $B=(0, 0, b)$, $b\neq 0$. Under a suitable normalization of units, the energy Hamiltonian H for such a system takes the following form:

$$H = \textstyle\sum_{j=1}^N\{(1/2\mu_j)T_j^2+V_{0j}(x^j)\}+\sum_{1\leqq i<j\leqq N} V_{ij}(x^j-x^i)\,, \tag{0.1}$$

where T_j is defined by

$$T_j = -i\nabla_j+A_j(x^j)\,, \quad A_j = (e_j/2)B\times x^j\,, \tag{0.2}$$

∇_j being the gradient with respect to x^j. Let $\sigma_{ess}(H)$ be an essential spectrum and let $\Sigma(H)=\inf\sigma_{ess}(H)$ be the bottom of essential spectrum. We denote by $N(\lambda)$, $\lambda>0$, the number of eigenvalues not exceeding $\Sigma(H)-\lambda$ of H with repetition according to multiplicities. The aim of this paper is to study the asymptotic behavoir as $\lambda\to 0$ of $N(\lambda)$ when the Hamiltonian H has an infinite number of eigenvalues below the bottom $\Sigma(H)$. Such a problem has been already studied by [5] and [8] in the case $B=0$. The method used here is in principle the same as that in these works, but the obtained result is quite different from that in the case $B=0$.

The paper consists of nine sections. We formulate tbe main theorem in section 5. The precise formulation requires several notations and assumptions. The first four sections have the character of preliminaries. In section 1 we disucss the self-adjointness problem for H formally defined by (0.1) and in section 2 we formulate the HVZ (Hunziker-Van Winter-Zhislin) theorem on the location of the bottom $\Sigma(H)$. For a cluster decomposition $D=\{C_0, C_1, \cdots, C_k\}$ of $\{0, 1, 2, \cdots, N\}$ with $0\in C_0$, we denote by $|D|$ the number of clusters in D and for a nonempty cluster C, we difine the Hamiltonian $h(C)$ as

$$h(C)=\sum_{j\in C\setminus\{0\}}(1/2\mu_j)T_j^2+\sum_{i,j\in C}V_{ij}(x^j-x^i) \tag{0.3}$$

with $x^0=(0, 0, 0)\in\boldsymbol{R}^3$, where $h(C)$ is considered as an operator acting on $L^2(R^{3l})$, l being the number of elements in $C\setminus\{0\}$. If $C=\{0\}$, then $h(C)$ is defined as zero and is considered as an operator acting on the scalar field $\boldsymbol{C}$. Let $\sigma(h(C))$ denote the spectrum of $h(C)$ and let $\Lambda(h(C))$ be defined by $\Lambda(h(C))=\inf\sigma(h(C))$. Then the HVZ theorem gives

$$\Sigma(H)=\min\{\textstyle\sum_{j=0}^k\Lambda(h(C_j));\ D=\{C_0, \cdots, C_k\},\ |D|\geqq 2\}\ .$$

We make the basic assumption that the bottom $\Sigma(H)$ is determined only by single cluster (2–cluster) decompositions $D=\{C_0, C\}$. Throughout the discussion, we use the terminology "single cluster decomposition" in the sense of decomposition into two clusterrs. Under this assumption, we will see intuitively that the asymptotics as $\lambda\to 0$ of $N(\lambda)$ is determined by the interaction betwen sufficiently distant clusters C_0 and C. This is the main idea used in the works [5] and [8] in the case $B=0$. Indeed, such an asymptotics with $B=0$ has been shown to coincide with the asymptotics as $\lambda\to 0$ of $N_D(\lambda)$ (=number of eigenvalues less than $-\lambda$) of the two particle Hamiltonian $-(1/2\mu(C))\Delta+V_D(x)$, $x\in\boldsymbol{R}^3$, acting on $L^2(\boldsymbol{R}^3)$, where $\mu(C)=\sum_{j\in C}\mu_j$ is the total mass of cluster C and $V_D(x)=\sum V_{ij}(x)$ is the intercluster potential between C_0 and C, the sum being taken over pairs (i, j) such that i and j are in different clusters. In the case $B\neq 0$, such a reduced Hamiltonian takes a different form. Roughly speaking, this is represented as a pseudodifferential operator of the form

$$-(1/2\mu(C))(\partial/\partial x_3)^2+V_D^W(y, D_y, x_3) \tag{0.4}$$

acting on $L^2(\boldsymbol{R}^2)$, $(y, x_3)\in\boldsymbol{R}^2$, where $V_D^W(y, D_y, x_3)$ is defined by the Weyl formula. To see this, we make, in section 3, a separation of the center of mass for the Hamiltonian $h(C)$, $0\notin C$, with translation invariant interactions under the assumption that the total charge $e(C)=\sum_{j\in C}e_j$ of cluster C is not zero. In section 4 we prove the rapidly decaying property of ground state for such a Hamiltonian obtained by removing the center of mass.

After these preparations in sections 1~4, we formulate the main theorem in section 5 and prove it in sections 6~8. As stated above, the proof is done by reducing the problem under consideration to that of eigenvalue asymptotics for pseudodifferential operators of the form (0.4) and by applying to such operators the result obtained in [9] for one particle systems. In section 9 we mention some simple examples to which the main theorem can be applied.

We conclude the introduction by making some comments on the notations accepted in this paper. (1) For a self-adjoint operator A, we denote by $\sigma(A)$ and $\sigma_{ess}(A)$ the spectrum and essential spectium of A, respectively. We also define $\Lambda(A)$ and $\Sigma(A)$ by $\Lambda(A)=\inf\sigma(A)$ and $\Sigma(A)=\inf\sigma_{ess}(A)$. (2) For a cluster decomposition D, we write iDj if i and j are in the same cluster. If they are in different clusters, we write $\sim iDj$. We also use the notation $\sum_{ijD}$ (resp. $\sum_{\sim iDj}$) to denote the sum over pairs (i, j), $i<j$, with property iDj (resp. $\sim iDj$). (3) We work in the various L^2 spaces $L^2(\boldsymbol{R}^n)$, $1\leqq n\leqq 3N$. The scalar product and norm in $L^2(\boldsymbol{R}^n)$ are specified by writing them as $(\ ,\ )_{(n)}$ and $\|\cdot\|_{(n)}$ respectively. If there is no fear of confusion, then we use, for notational convenience, the same notaions $(\ ,\ )$ and $\|\cdot\|$ to denote the scalar product and norm in these spaces.

1. Self-adjoint realization

In this section we discuss the self-adjointness problem for the Hamiltonian H formally defined by (0.1). This problem has been already studied by many authors (see [7] and references there). The result here does not contain any new ones.

For brevity, we fix B as $B=(0, 0, 1)$ throughout the entire discussion, and we write $V_{0j}(x^j)=V_{0j}(x^j-x^0)$ with $x^0=(0, 0, 0)$. Then the energy Hamiltonian H under consideration is written as

$$H=\sum_{j=1}^{N}(1/2\mu_j)T_j^2+\sum_{0\leqq i<j\leqq N}V_{ij}(x^j-x^i), \tag{1.1}$$

where T_j is defined by (0.2) with $B=(0, 0, 1)$. We now make the following assumption on $V_{ij}(x)$, $x\in R^3$.

Assumption (*V*). $V_{ij}(x)$ is a real function belonging to the class $L^{3/2}(\boldsymbol{R}^3)+L^\infty(\boldsymbol{R}^3)$ and $V_{ij}(x)\to 0$ as $|x|\to\infty$.

Assume $V(x)$ to belong to $L^{3/2}(\boldsymbol{R}^3)$. Then, for any $\delta>0$ small enough, there exists K_δ such that

$$(|V|\phi, \phi)\leqq\delta\sum_{j=1}^{3}\|D_j\phi\|^2+K_\delta\|\phi\|^2,\quad D_j=-i\partial/\partial x_j, \tag{1.2}$$

for $\phi\in C_0^\infty(\boldsymbol{R}^3)$. The next simple but useful lemma can be found in the book [7], p. 213, although the essential idea of proof is due to [4].

Lemma 1.1. *Assume* (1.2). *Let* $b_j\in L^\infty_{\mathrm{loc}}(\boldsymbol{R}^3)$, $1\leqq j\leqq 3$, *be a real function. Then*

$$(|V|\phi, \phi)\leqq\delta\sum_{j=1}^{3}\|(D_j+b_j)\phi\|^2+K_\delta\|\phi\|^2$$

for $\phi\in C_0^\infty(\boldsymbol{R}^3)$ *with the same* K_δ *as in* (1.2).

The above lemma will be used for another purpose in section 4. As an immediate conseqence of Lemma 1.1, it follows that the multiplication $V=\sum_{0\leqq i<j\leqq N} V_{ij}$ is relatively form bounded with respect to the unperturbed Hamiltonian $H_0=\sum_{j=1}^{N}(1/2\mu_j)T_j^2$ with bound less than one. Thus we obtain the following

Theorem 1.2. *Assume* (V). *Let* H *be defined by* (1.1) *with the form domain* $C_0^\infty(\boldsymbol{R}^{3N})$. *Then* H *admits a unique self-adjoint realization in* $L^2(\boldsymbol{R}^{3N})$ (*denoted by the same notation* H) *with the domain*

$$\mathcal{D}(H) = \{u\in L^2(R^{3N}):\ |V_{ij}|^{1/2}u,\ T_ju,\ Hu\in L^2(R^{3N})\},$$

where T_j *and* H *act on* u *in the distribution sense.*

2. The HVZ theorem

In this section we formulate the HVZ theorem on the location of the bottom $\Sigma(H)$ of essential spectrum $\sigma_{ess}(H)$. This theorem has been already proved by many authors in the case $B=0$ and similar arguments will apply to the case $B\neq 0$. For completeness and for latei reference, we here sketch a proof by making use of the geometric spectral method due to Agmon [1].

Theorem 2.1 (HVZ theorem). *Let the notations be as in the introduction. Assume* (V). *Then*

$$\Sigma(H) = \min\{\textstyle\sum_{j=0}^{k}\Lambda(h(C_j));\ D=\{C_0,\cdots,C_k\},\ |D|\geqq 2\}.$$

Before proving the theorem, we introduce several notations and definitions. We follow the notations in [1]. Let $\boldsymbol{S}^{3N-1}=\{\omega\in\boldsymbol{R}^{3N};\ |\omega|=1\}$. For $\omega\in\boldsymbol{S}^{3N-1}$, $0<\varepsilon\ll 1$ and $L\gg 1$, we define

$$\Gamma_\omega^{\varepsilon,L} = \{x\in\boldsymbol{R}^{3N};\ \langle x,\omega\rangle>|x|\cos\varepsilon,\ |x|>L\}\ ,$$
$$\Sigma^{\varepsilon,L}(\omega;H) = \{\inf(H\phi,\phi)_{(3N)};\ \|\phi\|=1,\ \phi\in C_0^\infty(\Gamma_\omega^{\varepsilon,L})\},$$
$$K(\omega;H) = \lim_{\varepsilon\downarrow 0}\lim_{L\uparrow\infty}\Sigma^{\varepsilon,L}(\omega;H)\,,$$

where $\langle\ ,\ \rangle$ denotes the scalar product in $\boldsymbol{R}^{3N}$.

Lemma 2.2. *Let the notatins be as above. Then*:
(i) $K(\omega;H)$ *is a lower semi-continuous function of* ω.
(ii) $\Sigma(H)=\min\{K(\omega;H);\ \omega\in S^{3N-1}\}$.

The lemma is proved in exactly the same way as in the proof of Theorem 5.2, [1].

Lemma 2.3. *Let* $V(x)=\sum_{0\leqq i<j\leqq N} V_{ij}(x^j-x^i)$ *with* $x=(x_1,\cdots,x^N)\in\boldsymbol{R}^{3N}$. *If* $V(x)=V(x+\tau\omega_0)$ *for some* $\omega_0\in\boldsymbol{S}^{3N-1}$ *and all* $\tau\in R^1$, *then* $K(\omega_0;H)=\Sigma(H)=\Lambda(H)$.

Proof. Define $Z=(z^1, \cdots, z^N)\in R^{3N}$ with $z^j=(e_j/2)B\times x^j\in \boldsymbol{R}^3$, $B=(0, 0, 1)$, and set

$$\phi_\tau(x) = \phi(x+\tau\omega_0) \exp(-i\tau\langle\omega_0, Z\rangle)$$

for $\phi\in C_0^\infty(\boldsymbol{R}^{3N})$. Then $(H\phi_\tau, \phi_\tau)=(H\phi, \phi)$. This relation enables us to prove the lemma in the same way as in the proof of Lemma 6.1, [1]. □

Let Π_{ij}: $\boldsymbol{R}^{3N}\to\boldsymbol{R}^3$, $0\leqq i<j\leqq N$, be difined by $\Pi_{ij}x=x^j-x^i$ for $x=(x^1, \cdots, x^N)\in\boldsymbol{R}^{3N}$. For given $\omega\in\boldsymbol{S}^{3N-1}$, we difine

$$H_\omega = \sum_{j=1}^N (1/2\mu_j)T_j^2+\sum_{\Pi_{ij}\omega=0} V_{ij}\,,$$

where the sum of V_{ij} is taken over pairs (i, j) with property $\Pi_{ij}\omega=0$.

Lemma 2.4. *Let H_ω be as above. Then*

$$K(\omega; H) = K(\omega; H_\omega) = \Sigma(H_\omega) = \Lambda(H_\omega)\,.$$

This lemma is also proved in the same way as in the proof of Theorem 6.3, [1], by making use of Lemma 2.3.

We now proceed to prove Theorem 2.1.

Proof of Theorem 2.1. For given cluster decomposition D, we define the subset Ω_D of $\boldsymbol{S}^{3N-1}$ by

$$\Omega_D = \{\omega\in\boldsymbol{S}^{3N-1};\ \Pi_{ij}\,\omega = 0 \text{ if } iDj,\ \Pi_{ij}\,\omega\neq0 \text{ if } \sim iDj\}, \tag{2.1}$$

Ω_D being defined as empty if $|D|=1$. Conversely, for given $\omega\in\boldsymbol{S}^{3N-1}$, we can construct a cluster decomposition D uniquely so that $\omega\in\Omega_D$. If ω_1 and ω_2 are in the same Ω_D, then it follows from Lemma 2.4 that $K(\omega_1; H)=K(\omega_2; H)$. Thus, for $D=\{C_0, \cdots, C_k\}$, we can define $\kappa(D; H)$ as

$$\kappa(D; H) = K(\omega; H_\omega) = \Lambda(H_\omega)\,, \qquad \omega\in\Omega_D\,,$$

and also this is written as $\kappa(D; H)=\sum_{j=0}^k \Lambda(h(C_j))$. Hence the theorem follows from Lemmas 2.2 and 2.4. □

Let $\Sigma_k(H)=\min\{\kappa(D; H);\ |D|=k\}$, $2\leqq k\leqq N+1$. Since $K(\omega; H)$ is lower semi-continuous by Lemma 2.2, it follows that $\Sigma_k(H)$ is non-decreasing in k and hence we have $\Sigma(H)=\Sigma_2(H)$. Let

$$\Sigma_0 = \{D;\ \kappa(D; H) = \Sigma(H)\}\,. \tag{2.2}$$

We now make the following basic assumption.

ASSUMPTION (Σ). If $D\in\Sigma_0$, then $|D|=2$.

Proposition 2.5. *Assume* (V) *and* (Σ). *Let* $D=\{C_0, C\}\in\Sigma_0$, $0\in C_0$. *Then* *the Hamiltonian $h(C_0)$ has an eigenvalue of finite multiplicities (ground state energy) at the bottom in its spectrum. In other words, $\Lambda(h(C_0))$ is the ground state energy of $h(C_0)$.*

Proof. The proposition is an immediate consequence of the HVZ theorem. In fact, if $\Lambda(h(C_0))=\Sigma(h(C_0))$, then we have $\Sigma(H)=\Sigma_3(H)$ by applying the HVZ theorem to $h(C_0)$. This contradicts the assumption (Σ). $\square$

In section 3, we will prove that $\Lambda(h(C))$, $0\notin C$, is also determined as the ground state energy of the Hamiltonian obtained by removing the center of mass from $h(C)$.

3. Separation of center of mass

In this section we make a separation of the center of mass (*c.m.*) for the Hamiltonian $h(C)$, $0\notin C$, with translation invariant interactions. For notational brevity, we fix C as $C=\{N-l+1, \cdots, N\}$ for some l, $1\leqq l\leqq N$. We assume that $e(C)=\sum_{j\in C} e_j\neq 0$. If $e(C)=0$, then a different analysis is required and we do not deal with this case here.

We follow the method in [2] to represent $h(C)$ in terms of the new coordinates introduced below. Let $r^j=(r_1^j, r_2^j, r_3^j)\in \boldsymbol{R}^3$, $1\leqq j\leqq l-1$, be the Jacobi coordiantes defined by

$$r^j = x^{N-l+j+1}-(\textstyle\sum_{k=1}^j \mu_{N-l+k})^{-1}\sum_{k=1}^j \mu_{N-l+k}x^{N-l+k} \tag{3.1}$$

and let $R=(R_1, R_2, R_3)\in \boldsymbol{R}^3$ be defined by

$$\begin{cases} R_1 = -(1/e(C))\sum_{k=1}^l e_{N-l+k}x_2^{N-l+k}\,, \\ R_2 = (1/e(C))\sum_{k=1}^l e_{N-l+k}x_1^{N-l+k}\,, \\ R_3 = (1/\mu(C))\sum_{k=1}^j \mu_{N-l+k}x_3^{N-l+k}\,, \end{cases} \tag{3.2}$$

where $\mu(C)=\sum_{j\in C}\mu_j$. It is easy to make a separation of the *c.m.* motion in the direction parallel to B.

Lemma 3.1. *Decompose $L^2(\boldsymbol{R}^{3l})$ as $L^2(\boldsymbol{R}^{3l})=L^2(\boldsymbol{R}^{3l-1})\otimes L^2(\boldsymbol{R}^1)$. Then there exists an operator $h_\perp(C)$ acting on $L^2(\boldsymbol{R}^{3l-1})$ for which*

$$h(C) = -I\otimes(1/2\mu(C))(\partial/\partial R_3)^2+h_\perp(C)\otimes I\,,$$

I being the identity operator.

We further analyze the operator $h_\perp(C)$. Let $p=(p^{N-l+1}, \cdots, p^N)$, $q=(q^1, \cdots, q^{l-1})$ and $Q=(Q_1, Q_2, Q_3)$ be the coordinates dual to $x=(x^{N-l+1}, \cdots, x^N)$, $r=(r^1, \cdots, r^{l-1})$ and $R=(R_1, R_2, R_3)$, respectively. Then a simple calculation shows that:

$$(3.3)\qquad \begin{cases} x_1^j = R_2+a_1^j(r_1^1, \cdots, r_1^{l-1}), \\ x_2^j = -R_1+a_2^j(r_2^1, \cdots, r_2^{l-1}), \\ p_1^j = (e_j/e(C))Q_2+b_1^j(q_1^1, \cdots, q_1^{l-1}), \\ p_2^j = -(e_j/e(C))Q_1+b_2^j(q_2^1, \cdots, q_2^{l-1}) \end{cases}$$

for j, $N-l+1\leqq j\leqq N$, when a_k^j and b_k^j, $1\leqq k\leqq 2$, are represented as linear functions of $(r_k^1, \cdots, r_k^{l-1})$ and $(q_k^1, \cdots, q_k^{l-1})$, respectively. For later reference, we here note that the $l\times l-1$ matrix B_k, $1\leqq k\leqq 2$: $(q_k^1, \cdots, q_k^{l-1})\to(b_k^{N-l+1}, \cdots, b_k^N)$ has the property

$$(3.4)\qquad \operatorname{rank} B_k = l-1\,.$$

We now introduce the new coordinates as follows:

$$(3.5)\qquad \begin{cases} y = |e(C)|^{-1/2}Q_1+(1/2)|e(C)|^{1/2}R_2\,, \\ \eta = |e(C)|^{-1/2}Q_2-(1/2)|e(C)|^{1/2}R_1\,, \\ z = |e(C)|^{-1/2}Q_2+(1/2|)e(C)|^{1/2}R_1\,, \\ \zeta = |e(C)|^{-1/2}Q_1-(1/2)|e(C)|^{1/2}R_2\,. \end{cases}$$

As is easily seen, the transformation above is symplectic;

$$dQ_1\wedge dR_1+dQ_2\wedge dR_2 = d\eta\wedge dy+d\zeta\wedge dz\,.$$

Assume that $e(C)>0$. Then we have by (3.3) that

$$(3.6)\qquad \begin{aligned} p_1^j-(e_j/2)x_2^j &= |e(C)|^{-1/2}e_jz+b_1^j-(e_j/2)a_2^j\,, \\ p_2^j+(e_j/2)x_1^j &= -|e(C)|^{-1/2}e_j\zeta+b_2^j+(e_j/2)a_1^j \end{aligned}$$

for j, $N-l+1\leqq j\leqq N$, a_k^j and b_k^j being as in (3.3). If $e(C)<0$, then we obtain the same relations as above with z and ζ replaced by $-\eta$ and $-y$, respectively. Thus we have the following lemma.

Lemma 3.2. *Decompose $L^2(\boldsymbol{R}^{3l-1})$ as $L^2(\boldsymbol{R}^{3l-1})=L^2(\boldsymbol{R}^{3l-2})\otimes L^2(\boldsymbol{R}^1)$. Then there exists an operator $h_r(C)$ acting on $L^2(\boldsymbol{R}^{3l-2})$ for which $h_\perp(C)=h_r(C)\otimes I$.*

We shall call $h_r(C)$ a Hamiltonian with *c.m.* removed. The *HVZ* throrem for $h_r(C)$ is formulated as follows (Theorem 6.1, [2]).

Theorem 3.3. *Assume* (V). *Let α be a partition of C into disjoint nonempty clusters C_1^α and C_2^α. Then*

$$\Sigma(h_r(C)) = \min\{\textstyle\sum_{k=1}^2 \Lambda(h(C_k^\alpha));\ \alpha\}\,.$$

We will be able to prove the theorem above by making use of the geometric spectral method as in the proof of Theorem 2.1, but we do not go into details here.

Let Σ_0 be defined by (2.2). In addition to (V) and (Σ), we make the following assumption.

Assumption (E). $e(C)=\sum_{j\in C} e_j\neq 0$ for $D=\{C_0, C\}\in\Sigma_0$.

Proposition 3.4. *Assume* (V), (Σ) *and* (E). *Let* $D=\{C_0, C\}\in\Sigma_0$ *and let* $h_r(C)$ *be the Hamiltonian obtained by removing the center of mass from* $h(C)$. *Then* $h_r(C)$ *has a ground state energy at the bottom in its spectrum.*

Proof. The proposition follows immediately from Theorem 3.3 by use of the same argument as in the proof of Proposition 2.5. □

4. Decaying property of ground states

Let $D=\{C_0, C\}\in\Sigma_0$. In the previous section we have shown that the Hamiltonian $h_r(C)$ has a ground state energy at the bottom in its spectrum. In this section we will prove that the eigenfunction (ground state) associated with the ground state energy has the rapidly decaying property.

As in section 3, we fix C as $C=\{N-l+1, \cdots, N\}$, $1\leqq l\leqq N$. For brevity, we assume that $e(C)>0$. Then the Hamiltonian $h_r(C)$ with *c.m.* removed is represented in terms of the coordinates $(r, z)\boldsymbol{R}^{3l-2}$, z being defined in (3.5).

Proposition 4.1. *Let the notations be as above. Then the ground state* $u=u(r, z)\in L^2(\boldsymbol{R}^{3l-2})$ *of* $h_r(C)$ *has the following property*; $(1+|r|+|z|)^K(\partial/\partial z)^M u\in L^2(\boldsymbol{R}^{3l-2})$ *for any nonnegative integers* K *and* M.

Remark. In the case $B=0$, it has been proved by many authors (see [1] and references there) that eigenfunctions associated with eigenvalues below the bottom of essential spectrum have the (pointwise) exponentially decaying property. It may be possible to prove such a sharp result in the case $B\neq 0$ also. Howevr, only a weak bound as in the proposition is sufficient to the later application.

Before proving the above proposition, we mention several properties of $h_r(C)$. Let $h_{0r}(C)$ be the Hamiltonian obtained by removing the center of mass mass from $h_0(C)$ with no interactions; $V_{ij}=0$ for $i, j\in C$. Then $h_r(C)=h_{0r}(C)+\sum_{i,j\in C} V_{ij}$. By (3.6), Hamiltonian $h_{0r}(C)$ takes the form

$$h_{0r}(C) = \sum_{j=1}^{3l-1} S_j(r, z, D_r, D_z)^2\,,$$

where $S_j(r, z, q, \zeta)$ is represented as a linear function of (r, z, q, ζ). We see from (3.4) that q_k^j, $1\leqq k\leqq 3$, is written as

$$q_k^j = \sum_{m=1}^{3l-1} \alpha_{km}^j S_m(r, z, q, \zeta)+c_k^j(r)$$

with some linear function $c_k^j(r)$ of r. This, together with Lemma 1.1, implies that

$$(4.1) \qquad (|V_{ij}|\phi, \phi) \leqq \delta \textstyle\sum_{j=1}^{3l-1} \|S_j\phi\|^2 + K_\delta\|\phi\|^2, \qquad 0<\delta\ll 1,$$

for $\phi \in C_0^\infty(\boldsymbol{R}^{3l-2})$ and hence the domain $\mathcal{D}(h_r(C))$ is given by

$$\mathcal{D}(h_r(C)) = \{u \in L^2(\boldsymbol{R}^{3l-2});\ |V_{ij}|^{1/2}u,\ S_ju,\ h_r(C)u \in L^2(\boldsymbol{R}^{3l-2})\}.$$

Furthermore, it follows from (4.1) that for $\phi \in \mathcal{D}(h_r(C))$

$$(4.2) \qquad \textstyle\sum_{j=1}^{3l-1} \|S_j\phi\|^2 + \sum_{i,j\in C}(|V_{ij}|\phi, \phi) \leqq K_1(h_r(C)\phi, \phi) + K_2\|\phi\|^2.$$

We can also show by use of (3.4) and (3.6) that

$$\textstyle\sum_{j=1}^{3l-1} S_j(0, z, q, \zeta)^2 \geqq \gamma(|q|^2 + z^2 + \zeta^2), \qquad \gamma>0.$$

This implies that multiplications by bounded functions with compact support are relatively compact with respect to $h_{0r}(C)$ and hence it follows from the Persson theorem ([6]) that

$$(4.3) \qquad \Sigma(h_r(C)) = \lim_{L\uparrow\infty} \inf\,(h_r(C)\phi, \phi)_{(3l-2)},$$

where the infimum is taken over $\phi \in \mathcal{D}(h_r(C))$, $\|\phi\|=1$, vanishing on B_L, B_L being the ball in $\boldsymbol{R}^{3l-2}$ centered at the origin with radius L. Let $\lambda_C = \Lambda(h_r(C))$ and let $\gamma_C = \Sigma(h_r(C)) - \lambda_C > 0$. Then it follows from (4.3) that there exists $L \gg 1$ such that

$$(4.4) \qquad ((h_r(C)-\lambda_C)\phi, \phi) \geqq (\gamma_C/2)(\phi, \phi)$$

for $\phi \in \mathcal{D}(h_r/(C))$ vanishing on B_L.

We now proceed to prove Proposition 4.1.

Proof of Proposition 4.1. Define $\rho_\varepsilon(\tau)$, $0 \leqq \varepsilon \ll 1$, by

$$\rho_\varepsilon(\tau) = (1+\tau^2)^{1/2}(1+\varepsilon\tau^2)^{-1/2}, \qquad \tau \in \boldsymbol{R}^1.$$

We write $Y=(r, z) \in \boldsymbol{R}^{3l-2}$ and denote by [,] the commutator notation.

We shall show that $\rho_0(|Y|)u \in \mathcal{D}(h_r(C))$. Since the L^2 norm of $[h_r(C), \rho_\varepsilon(|Y|)]u$ is bounded uniformly in $\varepsilon>0$, we have by (4.4) that $\rho_0(|Y|)u \in L^2$ $(=L^2(\boldsymbol{R}^{3l-2}))$ and also it follows from (4.2) that both $S_j\rho_0(|Y|)u$ and $|V_{ij}|^{1/2}\rho_0(|Y|)u$ are in L^2. This proves that $\rho_0(|Y|)u \in \mathcal{D}(h_r(C))$. By repeated use of the above arguments, we obtain that $\rho_0(|Y|)^K u \in \mathcal{D}(h_r(C))$ for any $K \geqq 0$ and hence the decaying property with $M=0$ follows at once. To prove this property for the case $M \geqq 1$, we first note that the coefficients of $h_r(C)$ are smooth in z and hence the L^2 norm of the term $\rho_0(|Y|)^K[h_r(C), \rho_\varepsilon(D_z)]u$, $K \gg 1$, is bounded uniformly in $\varepsilon>0$. Thus the decaying property with $M \geqq 1$ is proved by applying the same argument as above to $\rho_\varepsilon(D_z)u$. □

Similarly we can prove that the ground state of $h(C_0)$ has the rapidly decaying property.

Proposition 4.2. *Let $l \geqq 1$ be the number of elements in $C_0 \setminus \{0\}$. Assume that the Hamiltonian $h(C_0)$ has a ground state energy at the bottom in its spectrum. Then the ground state $u=u(x)$, $x \in \boldsymbol{R}^{3l}$, of $h(C_0)$ has the property $(1+|x|)^K u \in L^2(\boldsymbol{R}^{3l})$ for any integer $K \geqq 0$.*

5. Formulation of main theorem

In tihs section we formulate the main theorem. To do this, we furhter have to assume a condition which guarantees that H has an infinite number of eigenvalues below the bottom $\Sigma(H)$ of essential spectrum $\sigma_{ess}(H)$. Let $D=\{C_0, C\} \in \Sigma_0$, Σ_0 being defined by (2.2). In addition to (V), (Σ) and (E), we make the following assumptions on $V_{ij}(x)$, $x \in \boldsymbol{R}^3$, with $\sim iDj$ and the inter-cluster potential $V_D(x)=\sum_{\sim iDj} V_{ij}(x)$.

Assumption $(\mathrm{A})_\rho$. (A.0) $V_{ij}=V_{ij}^{(0)}+V_{ij}^{(1)} \in L^{3/2}(\boldsymbol{R}^3)+L^\infty(\boldsymbol{R}^3)$.
(A.1) $V_{ij}^{(0)}$ is of compact support.
(A.2) $V_{ij}^{(1)}$ is C^∞-smooth and there exists $\rho>0$ such that

$$|\partial_x^\alpha V_{ij}^{(1)}| \leqq K_\alpha (1+|x|)^{-\rho-|\alpha|}$$

for all multi-indices α.
(A.3) For $|x|>L_0 \gg 1$, $V_D(x)<0$ and

$$K^{-1}(1+|x|)^{-\rho} \leqq |V_D(x)| \leqq K(1+|x|)^{-\rho}, \qquad K \geqq 1,$$

for the same as ρ above.

In the above assumption, we have assumed that the same ρ is chosen for all $D \in \Sigma_0$. In general, such a choice depends on $D \in \Sigma_0$. The result below can be easily extended to this general case.

We introduce the notations. For $D=\{C_0, C\} \in \Sigma_0$, we define the Hamiltonian h_D acting on $L^2(\boldsymbol{R}^{3N-2})$ by

$$h_D = h_r(C) \otimes I + I \otimes h(C_0). \tag{5.1}$$

By Propositions 2.5 and 3.4, h_D has the eigenvalue $\Sigma(H)=\Lambda(h_r(C))+\Lambda(h(C_0))$ as a ground state energy. We denote by $m(D)$ its multiplicity.

Theorem 5.1. *Assume (V), (Σ), (E) and $(A)_\rho$ with $\rho \neq 2$. Let $N(\lambda)$, $\lambda>0$, be the number of eigenvalues less than $\Sigma(H)-\lambda$ of H (counting multiplicities). Then $N(\lambda)$ obeys the following asymptotic formula as $\lambda \to 0$:*

$$N(\lambda) = \{\textstyle\sum_{D \in \Sigma_0} m(D) N_0(\lambda;\, V_D)\}(1+o(1)),$$

the sum being over $D=\{C_0, C\} \in \Sigma_0$, where the leading term $N_0(\lambda\colon V_D)$ is defined as follows: If $0<\rho<2$, then

(5.2) $\quad N_0(\lambda; V_D) = (2\pi)^{-2}|e(C)|\,\mathrm{vol}\,[\{(p, x)\in \boldsymbol{R}^1\times \boldsymbol{R}^3;\ k_D(p, x)<-\lambda\}]$

with

$$k_D(p, x) = (1/2\mu(C))p^2+V_D(x)$$

and if $\rho>2$, *then*

(5.3) $\quad N_0(\lambda; V_D) = (2\pi)^{-1}|e(C)|\,\mathrm{vol}\,[\{w\in \boldsymbol{R}^2;\ W_D(w)<-(2\mu(C)^{-1}\lambda)^{1/2}\}]$

with

$$W_D(w) = \int_{-\infty}^{\infty} V_D(w, x_3)dx_3, \qquad x = (w, x_3)\in \boldsymbol{R}^3\,.$$

REMARKS. (i) In the case $B=0$, Simon [8] has considered the special case in which Σ_0 consists of only one single cluster decomposition and Ivrii [5] has considered the general case in which Σ_0 is not necessarily assumed to consist of only one sigle cluster decomposition. In [5], the asymptotic formulas with sharp remainder estimates have been also obtained, although the detailed proof has not been given. (ii) As is easily seen, $N(\lambda)$ behaves like $O(\lambda^{1/2-3/\rho})$, $0<\rho<2$, and like $O(\lambda^{-1/(\rho-1)})$, $\rho>2$, as $\lambda\to 0$. In particular, $N(\lambda)\to\infty$ as $\lambda\to 0$ even in the case $\rho>2$. This is one of main differences between the cases $B=0$ and $B\neq 0$.

6. Partition of unity and variational principle

As the first step toward the proof of Theorem 5.1, we start by a simple localization formula, which has been effectively used in proving the Mourre estimate for N-body Schrödinger operators ([3]).

Lemma 6.1. *Let* H *be defined by* (1.1) *with* $B=(0, 0, 1)$. *Let* $\Psi=\{\psi_\alpha\}$, $\psi_\alpha\geq 0$, $1\leq\alpha\leq k$, *be a smooth partition of unity normalized by* $\sum_{\alpha=1}^k \psi_\beta^2=1$, $\psi_\alpha\in C^\infty(\boldsymbol{R}^{3N})$. *Then*

$$H = \sum_{\beta=1}^k \psi_\alpha H\psi_\alpha - J(x; \Psi)\,,$$

where

(6.1) $$J(x; \Psi) = \sum_{j=1}^N (1/2\mu_j)\sum_{\alpha=1}^k |\nabla_j\psi_\alpha|^2\,,$$

∇_j *being teh gradient with respect to* $x^j\in\boldsymbol{R}^3$.

Proof. The proof is an easy calculation.

Now, assume that Σ_0 consists of m single cluster decompositions $D_\alpha=\{C_0^\alpha, C^\alpha\}$, $1\leq\alpha\leq m$, with $0\in C_0^\alpha$. Let Ω_α be defined by (2.1) with $D=D_\alpha$. We take a normalized partition of unity, $\Psi_0=\{\psi_0, \psi^\alpha, \psi_1\}$, $1\leq\alpha\leq m$, with the following properties: (i) ψ_0 has support in $\{x\in \boldsymbol{R}^{3N};\ |x|<L\}$ and $\psi_0=1$ for $|x|<L/2$. (ii) ψ^α has support in a small conical neighborhood of

$$\Gamma_\alpha^L = \{x=|x|\omega\in \boldsymbol{R}^{3N};\ |x|>L/2,\ \omega\in\Omega_\alpha\}\ . \tag{2.6}$$

(iii) ψ_1 vanishes in a small conical neighborhood of $\Gamma^L=\cup_{1\leqq\alpha\leqq m}\Gamma_\alpha^L$. We further take $\chi^\alpha\in C^\infty(\boldsymbol{R}^{3N})$, $1\leqq\alpha\leqq m$, to satisfy $\chi^\alpha\psi^\alpha=\psi^\alpha$. We may assume that the support of χ^α is contained in a small conical neighborhood of Γ_α^L and dose not intersect with each other.

Let $J_0(x)=J(x;\ \Psi_0)$ be defined by (6.1) with $\Psi=\Psi_0$. By definition, $J_0(x)$ vanishes on $\cup_{1\leqq\alpha\leqq m}\Gamma_\alpha^{2L}\cup\{x;\ |x|<L/2\}$. We may assume that

$$|\partial_x^\beta J_0(x)|\leqq K_\beta(1+|x|)^{-2-|\beta|} \tag{6.3}$$

for K_β independent of $L\gg 1$.

We now define the Hamiltonian H_1^α, $1\leqq\alpha\leqq m$, by

$$H_1^\alpha = H^\alpha+\textstyle\sum_{\sim iD_\alpha j}\chi^\alpha V_{ij}\chi^\alpha-\chi^\alpha J_0\chi^\alpha\ , \tag{6.4}$$

where

$$H^\alpha = \textstyle\sum_{j=1}^N (1/2\mu_j)T_j^2+\sum_{iD_\alpha j}V_{ij}\ .$$

We denote by $N(\lambda;\ H_1^\alpha)$, $\lambda>0$, the number of eigenvalues less than $\Sigma(H)-\lambda$ of H_1^α. We assert that

$$\limsup_{\lambda\to 0} N(\lambda)/\textstyle\sum_{\alpha=1}^m N(\lambda;\ H_1^\alpha)\leqq 1\ , \tag{6.5}$$

if $N(\lambda;\ H_1^\alpha)\to\infty$ as $\lambda\to 0$.
Let $\Psi_0=\{\psi_0,\ \psi^\alpha,\ \psi_1\}$ be as above and let $\Lambda_j=\mathrm{supp}\,\psi_j$, $0\leqq j\leqq 1$. To prove (6.5), we evaluate the maximal dimension $N_j(\lambda)$, $0\leqq j\leqq 1$, of subspaces in $C_0^\infty(\Lambda_j)$ such that

$$(H\phi,\ \phi)-(J_0\phi,\ \phi)<(\Sigma(H)-\lambda)(\phi,\ \phi)\ ,\qquad \phi\in C_0^\infty(\Lambda_j)\ .$$

Since ψ_0 is of compact support, we can easily obtain $N_0(\lambda)\leqq K_L$ for K_L independent of λ. $N_1(\lambda)$ is also easy to evaluate. Recall the notation $K(\omega;\ H)$ in section 2. If ω is not in $\Omega=\cup_{1\leqq\alpha\leqq m}\Omega_\alpha$, then $K(\omega;\ H)>\Sigma(H)$ strictly. By property (iii), ψ_1 vanishes in a conic neighborhood of Γ^L and hence $N_1(\lambda)=0$ for $L\gg 1$. Thus, by the min-max principle, (6.5) follows from Lemma 6.1.

Next we shall evaulate the lower bound for $N(\lambda)$ as $\lambda\to 0$. Let ψ_0, ψ^α and χ^α, $1\leqq\alpha\leqq m$, be as above and let $\Lambda^\alpha=\mathrm{supp}\,\psi^\alpha$. We denote by $N^\alpha(\lambda)$ the maximal dimension of subspaces in $C_0^\infty(\Lambda^\alpha)$ such that

$$(H\phi,\ \phi)<(\Sigma(H)-\lambda)(\phi,\ \phi),\qquad \phi\in C_0^\infty(\Lambda^\alpha)\ .$$

Then we have $N(\lambda)\geqq\sum_{\alpha=1}^m N^\alpha(\lambda)$. We now take a normalized partition of unity $\Psi^\alpha=\{\psi_0, \psi^\alpha, \zeta_\alpha\}$ and define $J_\alpha(x)=J(x; \Psi^\alpha)$ by (6.1) with $\Psi=\Psi^\alpha$. J_α has a property similar to J_0. Let

$$H_2^\alpha = H^\alpha+\textstyle\sum_{\sim iD_\alpha j} \chi^\alpha V_{ij}\chi^\alpha+J_\alpha . \tag{6.6}$$

We denote by $N(\lambda; H_2^\alpha)$, $\lambda>0$, the number of eigenvalues less than $\Sigma(H)-\lambda$ of H_2^α, H_2^α being considered as an operator acting on $L^2(\boldsymbol{R}^{3N})$. We claim that

$$\liminf_{\lambda\to 0} N^\alpha(\lambda)/N(\lambda; H_2^\alpha)\geqq 1 , \tag{6.7}$$

if $N(\lambda; H_2^\alpha)\to\infty$ as $\lambda\to 0$. If this is proved, then

$$\liminf_{\lambda\to 0} N(\lambda)/\textstyle\sum_{\alpha=1}^m N(\lambda; H_2^\alpha)\geqq 1 . \tag{6.8}$$

Set

$$H_0^\alpha = H^\alpha+\textstyle\sum_{\sim iD_\alpha j} \chi^\alpha V_{ij}\chi^\alpha .$$

To prove (6.7), we apply Lemma 6.1 with $\psi=\psi^\alpha$ to H_0^α, so that

$$H_2^\alpha = \psi_0 H_0^\alpha\psi_0+\psi^\alpha H_0^\alpha\psi^\alpha+\zeta_\alpha H_0^\alpha\zeta_\alpha .$$

We first note that $\psi^\alpha H_0^\alpha\psi^\alpha=\psi^\alpha H\psi^\alpha$. The multiplication operator by $\sum_{\sim iD_\alpha j}\chi^\alpha V_{ij}\chi^\alpha$ is relatively compact with respect to H^α and hence $K(\omega; H_0^\alpha)=K(\omega; H^\alpha)$ for all $\omega\in \boldsymbol{S}^{3N-1}$. Under assumption ($\Sigma$), we can also show that $K(\omega; H^\alpha)\geqq\Sigma(H)$ and $K(\omega; H^\alpha)=\Sigma(H)$ for $\omega\in\Omega_\alpha$ only. Since ζ_α vanishes in a small conical neighborhood of Γ_α^L, (6.7) is obtained by making use of the same argument as used to prove (6.5).

7. Reduction to pseudodifferential operators

We keep the same notations as in section 6. The problem is now reduced to the study on the asymptotic behavior as $\lambda\to 0$ of $N(\lambda; H_1^\alpha)$ and $N(\lambda; H_2^\alpha)$. We fix one of D_α, $1\leqq\alpha\leqq m$, and denote it by $D=\{C_0, C\}$. For this fixed α, we also write χ, H_1 and H_2 for χ^α, H_1^α and H_2^α, respectively. For notational brevity, we further write C_0 and C as $C_0=\{0, 1, \cdots, N-l\}$ and $C=\{N-l+1, \cdots, N\}$ for some l, $1\leqq l\leqq N$, and assume that $e(C)=\sum_{i\in C} e_j$ and $\mu(C)=\sum_{j\in C}\mu_j$ are normalized as $e(C)=\mu(C)=1$.

We study the asymptotic behavior as $\lambda\to 0$ of $N(\lambda; H_1)$ only. A similar argument applies to $N(\lambda; H_2)$. Define

$$U_D(x) = \chi(x)\{\textstyle\sum_{\sim iDj} V_{ij}(x^j-x^i)-J_0(x)\}\chi(x) .$$

Recall that χ vanishes on $\{x\in\boldsymbol{R}^{3N};\ |x|<L/2\}$ and $\chi=1$ for $x=|x|\omega$, $|x|>L$, with $\omega\in\Omega_D$. Hence, if we take L large enough, then it follows from $(A)_\rho$ that U_D is C^∞-smooth and

(7.1) $$|\partial_x^\beta U_D| \leqq K_\beta(1+|x|)^{-\kappa-|\beta|}, \qquad \kappa = \min(\rho, 2),$$

for K_β independent of $L \gg 1$. Let $r=(r^1, \cdots, r^{l-1}) \in \boldsymbol{R}^{3l-3}$ and $R=(R_1, R_2, R_3) \in \boldsymbol{R}^3$ be defined by (3.1) and (3.2), respectively. We now write $X=(x^1, \cdots, x^{N-l}, r) \in \boldsymbol{R}^{3N-3}$ and denote by $U_D(R, X)$ the representation for $U_D(x)$ in terms of the coordinates $(R, X) \in \boldsymbol{R}^{3N}$. Then we have

(7.2) $$U_D(R, X)|_{X=0} = V_D(R_2, -R_1, R_1) = \textstyle\sum_{\sim iDj} V_{ij}(R_2, -R_1, R_3)$$

for $|R|>L$.

Let $w=(y, z) \in \boldsymbol{R}^2$ and $\xi=(\eta, \zeta) \in \boldsymbol{R}^2$ be defined by (3.5) with $e(C)=\mu(C)=1$, so that $R_1=z-\eta$ and $R_2=y-\zeta$. We write $s \in \boldsymbol{R}^1$ for the variable R_3. Then, in the coordinate system (w, s, X), the multiplication operator by $U_D(R, X)$ acts as the pseudodifferential operator $A_D=a_D^W(w, D_w, s, X)$ defined by the Weyl formula

$$A_D f = (2\pi)^{-2} \iint e^{i\langle w-w, \xi\rangle} a_D((w+w')/2, \xi, s, X) f(w', s, X) dw' d\xi$$

with the symbol

(7.3) $$a_D(w, \xi, s, X) = U_D(z-\eta, y-\zeta, s, X),$$

where the intrgration with no domain attached is taken over the whole space.

We further introduce the coordinates $\theta=(z, X) \in \boldsymbol{R}^{3N-2}$ and decompose $L^2(\boldsymbol{R}^{3N})$ as

$$L^2(\boldsymbol{R}^{3N}) = L^2(\boldsymbol{R}_y^1) \otimes L^2(\boldsymbol{R}_s^1) \otimes L^2(\boldsymbol{R}_\theta^{3N-2}).$$

Then the operator H_1 defined by (6.4) is represented as

$$H_1 = I \otimes I \otimes h_D - I \otimes (1/2)(\partial/\partial s)^2 \otimes I + A_D$$

or

$$H_1 = h_D - (12)(\partial/\partial s)^2 + A_D$$

in the simplified form, where h_D is defined by (5.1). Recall that h_D has the ground state energy $\Sigma(H)$. We now assume that $\Sigma(H)$ is a simple eigenvalue (multiplicity $m(D)=1$). This assumption is not essential. At the end of this section, we make a brief comment on modifications to be made in the case that $\Sigma(H)$ is m-fold degenerate, $m=m(D)>1$.

Now, let $\phi_0(\theta)=\phi_0(z, X)$ be the normalized ground state of h_D associated with $\Sigma(H)$. We define the projection P: $L^2(\boldsymbol{R}^{3N}) \to L^2(\boldsymbol{R}^{3N})$ by

$$(Pf)(y, s, \theta) = (f(y, s, \cdot), \phi_0(\cdot))_{(3N-2)} \phi_0(\theta)$$

and Q by $Q=I-P$. Let

$$E_D^{\pm} = P(-(1/2)(\partial/\partial s)^2 + A_D \pm \delta^{-1} A_D^2)P$$

for $\delta > 0$ small enough, δ being fixed. Then we have

$$E_D^- + F_D^- + \Sigma(H) \leqq H_1 \leqq E_D^+ + F_D^+ + \Sigma(H) \tag{7.4}$$

in the form sense, where

$$F_D^{\pm} = Q(h_D - (1/2)(\partial/\partial s)^2 + A_D \pm \delta)Q .$$

This follows from the operator inequality

$$PA_D Q + QA_D P \leqq \delta Q + \delta^{-1} PA_D^2 P .$$

Since $U_D(x) = O(L^{-\kappa})$ by (7.1), A_D is a bounded operator with bound $O(L^{-\kappa})$, when considered as an operator from $L^2(\boldsymbol{R}^{3N})$ into itself. Hence, if we take δ small enough and L large enough, then $F_D^{\pm}$ has no eigenvalues below the bottom $\Sigma(H)$. Let $N(\lambda; E_D^{\pm})$, $\lambda > 0$, be the number of eigenvalues less than $-\lambda$ of $E_D^{\pm}$. By (7.4), we have that $N(\lambda; E_D^+) \leqq N(\lambda; H_1) \leqq N(\lambda; E_D^-)$.

We proceed to analyze the operator $E_D^{\pm}$. If we write $Pf = g(y, s)\phi_0(\theta)$ with $g = (f(y, s, \cdot), \phi_0(\cdot))$, then

$$(PA_D Pf)(y, s, \theta) = (B_D g)(y, s)\phi_0(\theta) , \tag{7.5}$$

where

$$B_D g = (2\pi)^{-1} \iint e^{i(y-y')\eta} b_D((y+y')/2, \eta, s) g(y', s) dy' d\eta \tag{7.6}$$

with the symbol $b_D(y, \eta, s)$ defined by

$$b_D = (2\pi)^{-1} \iiint e^{i(z-z')\zeta} a_D(y, (z+z')/2, \eta, \zeta, s, X)\phi_0(z', X)\bar{\phi}_0(\theta) dz' d\zeta d\theta .$$

DEFINITION 7.1. We denote by S^d, $d \in R^1$, the class of all smooth symbols $a(y, \eta, s)$ such that

$$|\partial_y^k \partial_\eta^m a| \leqq K_{km}(1 + |y| + |\eta| + |s|)^{d-k-m}$$

for K_{km} independent of s.

We define the operator $a^W(y, D_y, s)$ with symbol $a(y, \eta, s)$ by the Weyl formula (7.6) and denote by OPS^d the class of such operators with symbols in S^d.

Lemma 7.2. *Let $B_D = b_D^W(y, D_y, s)$ be defined by* (7.6). *Then B_D is of class $OPS^{-\rho}$, ρ being as in $(A)_\rho$, and*

$$b_D(y, \eta, s) = U_D(z-\eta, y-\zeta, s, X)|_{z=\zeta=X=0} \quad (\text{mod } S^{-\rho-1}) . \tag{7.7}$$

Proof. The proof is done by use of the standard asymptotic expansion method for oscillatory integrals, so we giev only a sketch for the proof.

We first note that the ground state $\phi_0(z, X)$ is smooth in z but is not necessarily smooth in X, although any serious difficulty does not come from this fact. We write

$$b_D = \int c(y, z, \eta, s, X)\bar{\phi}_0(\theta)d\theta ,$$

where

$$c = (2\pi)^{-1}\iint e^{-iu\zeta} a_D(y, z+u/2, \eta, \zeta, s, X)\phi_0(z+u, X)dud\zeta .$$

The symbol c is asymptotically expanded as

$$c = \sum_{j=0}^{N-1} \gamma_j \partial_\zeta^j \partial_z^j [a_D(y, z, \eta, \zeta, s, X)\phi_0(z, X)]|_{\zeta=0}+c_N$$

with some constant γ_j, $0\leqq j\leqq N-1$, $(\gamma_0=1)$. By partial integration, it follows from (7.1) that the remainder term $c_N=c_N(y, z, \eta, s, X)$ satisfies

$$|c_N|\leqq K_N \int_0^1\iint (1+u^2+\zeta^2)^{-N} d_{\tau N}(u, y, z, \eta, \zeta, s, X)dud\zeta d\tau ,$$

where

$$d_{\tau N} = (1+|y-\zeta|+|\eta-z-\tau u/2|+|s|+|X|)^{-N} \sum_{j=0}^{3N}|(\partial/\partial z)^j\phi_0(z+\tau u, X)| .$$

By Propositions 4.1 and 4.2, we have $(1+|z|+|X|)^K(\partial/\partial z)^M \phi_0\in L^2(\boldsymbol{R}_\theta^{3N-2})$ for any nonnegative integers K and M, and hence

$$\int|c_N(y, z, \eta, s, X)\bar{\phi}_0(\theta)|d\theta = O((1+|y|+|\eta|+|s|^{-N})) .$$

We again use Propositions 4.1 and 4.2. Then the Taylor expansion formula yields

$$b_D = a_D(y, z, \eta, \zeta, s, X)|_{z=\zeta=X=0}+O((1+|y|+|\eta|+|s|)^{-\rho-1}) .$$

The same argument as above applies to $\partial_y^k\partial_\eta^m b_D$ and we obtain that b_D is of class $S^{-\rho}$. Relation (7.7) follows immediately from (7.3). Thus the proof is complete. $\square$

The operator PA_D^2P is also represented in the form (7.5);

$$(PA_D^2Pf)(y, s, \theta) = (C_Dg)(y, s)\phi_0(\theta) .$$

It follows from Lemma 7.2 that C_D is of class $OPS^{-2\rho}$. In view of (7.2), we have $U_D(-\eta, y, s, X)|_{X=0}=V_D(y, \eta, s)$ for $|y|+|\eta|+|s|>L$. We now define the symbol $e_D\in S^{-\rho}$ to satisfy $e_D(y, \eta, s)=V_D(y, \eta, s)$ for (y, η, s) as above. Then the operator $E_D^\pm$ takes the following form:

$$E_D^\pm = -(1/2)(\partial/\partial s)^2+e_D^W(y, D_y, s)+e_\pm^W(y, D_y, s)$$

with $e_\pm \in S^{-\sigma}$, $\sigma = \min(2\rho, \rho+1)$, when it is considered as an operator acting on $L^2(\boldsymbol{R}^2_{y,s}) \simeq \mathrm{Range}\, P$.

We conclude this section by making a brief comment on modifications in the case that the ground state energy $\Sigma(H)$ is m-fold degenerate. In this case, $E_D^\pm$ is considered as an operator acting on the space $\Sigma \oplus L^2(\boldsymbol{R}^2_{y,s})$, m summands, and has the following matrix representation:

$$E_D^\pm = -(1/2)(\partial/\partial s)^2 + e_D^W(y, D_y, s) + e_\pm^W(y, D_y, s),$$

where $e_\pm$ is a $m \times m$ matrix with components in $S^{-\sigma}$. The argument below applies to such a system case without any essential changes.

8. Completion of proof

In this section we complete the proof of Theorem 5.1. The problem is now reduced to the study on eigenvalue asymptotics for the pseudodifferential operator $E_D^\pm$ and the proof is completed by deriving the asymptotic formula for $N(\lambda; E_D^\pm)$ as $\lambda \to 0$.

To do this, we consider the Hamiltonian T_D (acting on $L^2(\boldsymbol{R}^3)$) for one particle system in the homogeneous magnetic field $B = (0, 0, 1)$;

$$T_D = (1/2)(-i\nabla + (B/2) \times x)^2 + V_D(x).$$

Under assumption $(A)_\rho$, T_D has essential spectrum beginning at $1/2$; $\sigma_{ess}(T_D) = [1/2, \infty)$, and an infinite number of eigenvalues below the bottom $\Sigma(T_D)$ $(=1/2)$. Let $N(\lambda; T_D)$, $\lambda > 0$, be the number of eigenvalues less than $1/2 - \lambda$ of T_D. Then, in the first paper [9], we have proved that $N(\lambda; T_D)$ obeys the asymptotic formula

$$N(\lambda; T_D) = N_0(\lambda; V_D)(1 + o(1)), \qquad \lambda \to 0,$$

where the leading term $N_0(\lambda; V_D)$ is defined by (5.2), $0 < \rho < 2$, and (5.3), $\rho > 2$, with $e(C) = \mu(C) = 1$. In [9], we have also shown implicitly that

$$\lim_{\lambda \downarrow 0} N(\lambda; E_D^\pm)/N(\lambda; T_D) = 1$$

and hence it follows that

$$N(\lambda; H_1) = N_0(\lambda; V_D)(1 + o(1)), \qquad \lambda \to 0.$$

By a similar argument, we obtain

$$N(\lambda; H_2) = N_0(\lambda; V_D)(1 + o(1)), \qquad \lambda \to 0.$$

This proves the theorem in the case that $e(C) = \mu(C) = 1$ and the ground state energy $\Sigma(H)$ of h_D is simple. The arguments can be easily extended to the case without normalization or to the case that $\Sigma(H)$ is degenerate. Thus the proof of the main theorem is now complete. $\square$

9. Examples

We mention two simple examples to which the main theorem (Theorem 5.1) can be applied.

EXAMPLE 9.1. Consider two particles of charge $e_j \neq 0$, $1 \leqq j \leqq 2$, interacting with each other through static potentials and subjected to external electrostatic and (homogeneous) magnetic fields. The energy Hamiltonian for such a system is of the form

$$H = \sum_{j=1}^{2} \{(1/2\mu_j)T_j^2 + V_{0j}(x^j)\} + V_{12}(x^2 - x^1),$$

where $T_j = -\nabla i_j + (e_j/2)B \times x^j$, $1 \leqq j \leqq 2$. Assume V_{ij}, $0 \leqq i < j \leqq 2$, to satisfy the assumption (V). If $V_{0j}(x)$, $x \in \boldsymbol{R}^3$, behaves like $V_{0j} \sim -\gamma_j |x|^{-\rho}$, $\rho > 0$, with $\gamma_j > 0$, as $|x| \to \infty$, then the Hamiltonian $(1/2\mu_j)T_j^2 + V_{0j}$ for one particle system has a ground state energy at the bottom in its spectrum. Hence it is easily seen that the basic assumption (Σ) is satisfied. If $V_{12} \geqq 0$, then the bottom $\Sigma(H)$ is determined by the single cluster decomposition $D_\alpha = \{C_0^\alpha, C^\alpha\}$ with $C^\alpha = \{\alpha\}$, $1 \leqq \alpha \leqq 2$. If $V_{12} \leqq 0$ and the single cluster decomposition $D_0 = \{\{0\}, \{1, 2\}\}$ determines the bottom $\Sigma(H)$, then we have to assume that $\sum_{j=1}^{2} e_j \neq 0$. In any case, the main theorem can be applied to this Hamiltonian, if the intercluster potential V_α for D_α, $0 \leqq \alpha \leqq 2$, satisfies the assumption $(A)_\rho$, $\rho \neq 2$.

EXAMPLE 9.2. Consider a nucleus fixed at the origin with positive charge Ne and N moving particles with negative charge $-e$ subject to Coulomb potentials. The corresponding energy Hamiltonian H takes the form

$$H = \sum_{j=1}^{N} \{(1/2\mu_j)\} T_j^2 + V_{0j}(x^j)\} + \sum_{1 \leqq i < j \leqq N} V_{ij}(x^j - x^i),$$

where $T_j = -i\nabla_j - (e/2)B \times x^j$, $V_{0j}(x^j) = -Ne^2/|x^j|$ and $V_{ij}(x^j - x^i) = e^2/|x^j - x^i|$. Since $V_{ij} \geqq 0$, $1 \leqq i < j \leqq N$, a simple inductive argument on N proves that the bottom $\Sigma(H)$ of essential spectrum is determined only by N single cluster decompositions $D_\alpha = \{C_0^\alpha, C^\alpha\}$ with $C^\alpha = \{\alpha\}$, $1 \leqq \alpha \leqq N$, and hence the basic assumption (Σ) is satisfied. The other assumptions (V), (E) and $(A)_\rho$, $\rho \neq 2$, are easy to check. Thus the main theorem can be applied to this example.

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