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ON ISOMORPHIC POWER SERIES RINGS

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Introduction

Let A and B be commutative rings with an identity. In this paper we investigate the following question raised by M.J. O'Malley [4]. Can there be an isomorphism of A onto B whenever the formal power series rings $A[[X_1, \cdots, X_n]]$ and $B[[Y_1, \cdots, Y_n]]$ are isomorphic? We shall say that A is n-power invariant if whenever C is a ring and $A[[X_1, \cdots, X_n]] \cong C[[Y_1, \cdots, Y_n]]$, then we have $A \cong C$. A ring A will be said to be strongly n-power invariant if whenever C is a ring and φ is an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $C[[Y_1, \cdots, Y_n]]$, then there exists a C-automorphism ψ of $C[[Y_1, \cdots, Y_n]]$ such that $\varphi(X_i) = \psi(Y_i)$ for $1 \leqq i \leqq n$. The present paper consists of three parts. In the first part we shall give a characterization of A-automorphisms of $A[[X_1, \cdots, X_n]]$. The second part will deal with higher derivations on a complete local ring and we shall determine a necessary and sufficient condition in order that a complete local ring A is isomorphic to a formal power series ring $A_0[[X]]$. M.J. O'Malley has proved that semisimple rings (the Jacobson radical$=(0)$) are strongly 1-power invariant [4]. In the last part we shall show that semisimple rings are strongly n-power invariant for any positive integer n. In particular an affine domain over a field is strongly n-power invariant for any n. Next we shall prove that if A and B are local rings which may not be noetherian (see [2], p. 13) and $A[[X_1, \cdots, X_n]] \cong B[[Y_1, \cdots, Y_n]]$ under φ, then there is either a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leqq i \leqq n$ or A(resp. B) is isomorphic to a formal power series ring $A_0[[X]]$ (resp. $B_0[[Y]]$). From this we shall easily conclude that a local ring A which may not be noetherian is either strongly n-power invariant for any n, or A is isomorphic to a formal power series ring $A_0[[X]]$. Furthermore we shall show that any noetherian local ring is n-power invariant for any n.

Throughout this paper all rings are assumed to be commutative and contain an identity.

1. A-automorphisms of $A[[X_1, \cdots, X_n]]$

We denote the Jacobson radical of a ring A by $\mathfrak{J}(A)$. In this section let us suppose that a ring A satisfies the condition $\bigcap_{m=1}^{\infty}\mathfrak{J}(A)^m=(0)$. As is well-known we have $\bigcap_{m=1}^{\infty}\mathfrak{J}(A)^m=(0)$ when A is noetherian.

Proposition 1. *Let B be a ring and let φ be an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$. Let $\varphi(X_i)=b_i+b_{i1}Y_1+\cdots+b_{in}Y_n+\cdots$ for $1\leqq i\leqq n$, where $b_i, b_{ij}\in B$. We set $\mathfrak{B}=(b_1, \cdots, b_n)$, the ideal of B generated by $b_1, \cdots, b_n$. Then we have*

(1) $\bigcap_{m=1}^{\infty}\mathfrak{J}(B)^m=(0)$ *and* $\mathfrak{B}\subset\mathfrak{J}(B)$,

(2) *B is complete in the $\mathfrak{B}$-adic topology,*

(3) *for any power series $\sum a_{i_1\cdots i_n}X_1^{i_1}\cdots X_n^{i_n}\in A[[X_1, \cdots, X_n]]$, $\sum\varphi(a_{i_1\cdots i_n})\varphi(X_1)^{i_1}\cdots\varphi(X_n)^{i_n}$ is a well defined power series in $B[[Y_1, \cdots, Y_n]]$ and we have $\varphi(\sum a_{i_1\cdots i_n}X_1^{i_1}\cdots X_n^{i_n})=\sum\varphi(a_{i_1\cdots i_n})\varphi(X_1)^{i_1}\cdots\varphi(X_n)^{i_n}$.*

Proof. (1) Since $\mathfrak{J}(A[[X_1, \cdots, X_n]])=\mathfrak{J}(A)[[X_1, \cdots, X_n]]+(X_1, \cdots, X_n)$ and $\bigcap_{m=1}^{\infty}\mathfrak{J}(A)^m=(0)$, we get $\bigcap_{m=1}^{\infty}\mathfrak{J}(A[[X_1, \cdots, X_n]])^m=(0)$. On the other hand $\varphi(\mathfrak{J}(A[[X_1, \cdots, X_n]]))=\mathfrak{J}(B[[Y_1, \cdots, Y_n]])$ and hence $\bigcap_{m=1}^{\infty}\mathfrak{J}(B[[Y_1, \cdots, Y_n]])^m=(0)$. Then it is easy to see that $\bigcap_{m=1}^{\infty}\mathfrak{J}(B)^m=(0)$. In order to show $\mathfrak{B}\subset\mathfrak{J}(B)$, we have only to prove that $b_i\in\mathfrak{J}(B)$ for $1\leqq i\leqq n$. For each $b\in B$, $1+\varphi^{-1}(b)X_i$ is a unit of $A[[X_1, \cdots, X_n]]$ and hence $\varphi(1+\varphi^{-1}(b)X_i)=(1+bb_i)+bb_{i1}Y_1+\cdots+bb_{in}Y_n+\cdots$ is a unit of $B[[Y_1, \cdots, Y_n]]$. Therefore $1+bb_i$ is a unit of B for each $b\in B$ and so $b_i\in\mathfrak{J}(B)$ as asseretd. If B is $\mathfrak{B}$-adic complete, $B[[Y_1, \cdots, Y_n]]$ is complete in the $(\mathfrak{B}[[Y_1, \cdots, Y_n]]+(Y_1, \cdots, Y_n))$-adic topology. Then the assertion (3) is obvious. Thus it is sufficient to prove (2). (2) We set $\mathfrak{B}_k=(b_1^k, \cdots, b_n^k)$, the ideal of B generated by $b_1^k, \cdots, b_n^k$. The sequence of ideals $\{\mathfrak{B}_k\}$ defines a topology on B which is equivalent to the $\mathfrak{B}$-adic topology on B. Let $\{c_k\}$ be a Cauchy sequence of B in the $\mathfrak{B}$-adic topology. Then $\{c_k\}$ is a Cauchy sequence with respect to the topology defined by $\{\mathfrak{B}_k\}$. It is therefore immediate to see that there exists a subsequence $\{d_k\}$ of $\{c_k\}$ such that $d_k=\sum_{i=0}^{k}(r_{i1}b_1^i+\cdots+r_{in}b_n^i)$ for each k, where $r_{ij}\in B$. Let $f_{ij}=\varphi^{-1}(r_{ij})\in A[[X_1, \cdots, X_n]]$ and we set $f=\sum_{i=0}^{\infty}(f_{i1}X_1^i+\cdots+f_{in}X_n^i)$ which is a well defined power series in $A[[X_1, \cdots, X_n]]$. If B^* is the $\mathfrak{B}$-adic completion of B, then we have the canonical injection ι: $B[[Y_1, \cdots, Y_n]]\to B^*[[Y_1, \cdots, Y_n]]$. We shall identify $B[[Y_1, \cdots, Y_n]]$ with the subring $\iota(B[[Y_1, \cdots, Y_n]])$ of $B^*[[Y_1, \cdots, Y_n]]$ and for $h\in B[[Y_1, \cdots, Y_n]]$ we shall denote $\iota(h)$ by h. The sequence $\{\sum_{i=0}^{k}(r_{i1}\varphi(X_1)^i+\cdots+r_{in}\varphi(X_n)^i)\}_k$ is obviously a Cauchy sequence of $B[[Y_1, \cdots, Y_n]]$ under the $(\mathfrak{B}[[Y_1, \cdots, Y_n]]+(Y_1, \cdots, Y_n))$-adic toplogy. Hence $\sum_{i=0}^{\infty}(r_{i1}\varphi(X_1)^i+\cdots+r_{in}\varphi(X_n)^i)$ is a well defined power series in $B^*[[Y_1, \cdots, Y_n]]$. On the other hand we have

$$\begin{aligned}
&\varphi(f)-\textstyle\sum_{i=0}^{k}(r_{i1}\varphi(X_1)^i+\cdots+r_{in}\varphi(X_n)^i)\\
&=\varphi(f)-\varphi(\textstyle\sum_{i=0}^{k}(f_{i1}X_1^i+\cdots+f_{in}X_n^i))\\
&=\varphi(\textstyle\sum_{i=k+1}^{\infty}(f_{i1}X_1^i+\cdots+f_{in}X_n^i))\\
&=\varphi(X_1)^{k+1}\varphi(\textstyle\sum_{i=k+1}^{\infty}f_{i1}X_1^{i-k-1})+\cdots+\varphi(X_n)^{k+1}\varphi(\textstyle\sum_{i=k+1}^{\infty}\\
&\quad f_{in}X_n^{i-k-1})\in(\mathfrak{B}[[Y_1,\cdots,Y_n]]+(Y_1,\cdots,Y_n))^{k+1}
\end{aligned}$$

in $B[[Y_1, \cdots, Y_n]]$. Hence we get

$$\begin{aligned}
\varphi(f)&=\textstyle\sum_{i=0}^{\infty}(r_{i1}\varphi(X_1)^i+\cdots+r_{in}\varphi(X_n)^i)\\
&=\textstyle\sum_{i=0}^{\infty}(r_{i1}b_1^i+\cdots+r_{in}b_n^i)+g
\end{aligned}$$

in $B^*[[Y_1, \cdots, Y_n]]$, where $g\in B^*[[Y_1, \cdots, Y_n]]$ and g has no constant term. Hence we see that $\{d_k\}\to c$, the constant term of $\varphi(f)$. Since $\varphi(f)\in B[[Y_1, \cdots, Y_n]]$ we have $c\in B$. Thus $\{c_k\}\to c$ and it follows that B is complete in its $\mathfrak{B}$-adic topology.

Theorem 2. *Let $Y_i=a_i+a_{i1}X_1+\cdots+a_{in}X_n+\cdots\in A[[X_1, \cdots, X_n]]$ for $1\leqq i\leqq n$. We set $\mathfrak{A}=(a_1, \cdots, a_n)$, the ideal of A generated by $a_1, \cdots, a_n$. Then there exists an A-automorphism φ of $A[[X_1, \cdots, X_n]]$ such that $\varphi(X_i)=Y_i$ for $1\leqq i\leqq n$ if and only if the following conditions hold:*

(1) *$\mathfrak{A}\subset\mathfrak{J}(A)$ and A is complete in the $\mathfrak{A}$-adic topology,*

(2) *the matrix*

$$\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n}\\
a_{21} & a_{22} & \cdots & a_{2n}\\
 & \cdots\cdots & & \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}$$

is invertible.

Proof. We assume that there exists an A-automorphism φ of $A[[X_1, \cdots, X_n]]$ satisfying $\varphi(X_i)=Y_i$ for $1\leqq i\leqq n$. Then it follows from Proposition 1 that $\mathfrak{A}\subset\mathfrak{J}(A)$ and A is complete in the $\mathfrak{A}$-adic topology. Let $\varphi^{-1}(X_i)=b_i+b_{i1}X_1+\cdots+b_{in}X_n+\cdots$ for $1\leqq i\leqq n$. Then we get

$$\begin{aligned}
X_i&=\varphi^{-1}(\varphi(X_i))\\
&=a_i+a_{i1}\varphi^{-1}(X_1)+\cdots+a_{in}\varphi^{-1}(X_n)+\cdots
\end{aligned}$$

by Proposition 1 applied to an isomorphism φ^{-1}. Comparing the coefficients of X's we have

$$\textstyle\sum_{k=1}^{n}a_{ik}b_{kj}\equiv\delta_{ij}\ (\mathrm{mod.}\ \mathfrak{J}(A))$$

where δ_{ij} denotes the Kronecker's symbol, because the coefficients of X's in

$\varphi^{-1}(X_1)^{i1}\cdots\varphi^{-1}(X_n)^{in}(i_1+\cdots+i_n\geqq 2)$ belong to the ideal $(b_1, \cdots, b_n)\subset\mathfrak{J}(A)$. Then $\det(a_{ij})\det(b_{ij})\equiv 1$ (mod. $\mathfrak{J}(A)$) and hence $\det(a_{ij})$ is a unit of A as asserted. Conversely we assume that the conditions (1) and (2) are satisfied. Since A is complete in the $\mathfrak{A}$-adic topology, $A[[X_1, \cdots, X_n]]$ is complete in its $(\mathfrak{A}[[X_1, \cdots, X_n]]+(X_1, \cdots, X_n))$-adic topology and hence $\sum a_{i_1\cdots i_n}Y_1^{i1}\cdots Y_n^{in}$ is a well defined power series in $A[[X_1, \cdots, X_n]]$. If we set $\varphi(\sum a_{i_1\cdots i_n}X_1^{i1}\cdots X_n^{in}) =\sum a_{i_1\cdots i_n}Y_1^{i1}\cdots Y_n^{in}$, then we see that φ is an A-endomorphism of $A[[X_1, \cdots, X_n]]$ satisfying $\varphi(X_i)=Y_i$ for $1\leqq i\leqq n$. In fact we shall show that φ is an automorphism. Let us consider an A-endomorphism τ of $A[[X_1, \cdots, X_n]]$ defined by $\tau(X_i)=X_i-a_i$ for $1\leqq i\leqq n$. It is immediate to see that τ is an automorphism and hence we have only to show that $\varphi\tau$ is an automorphism in order to complete our proof. Since $\varphi\tau(X_i)=a_{i1}X_1+\cdots+a_{in}X_n+\cdots$ for $1\leqq i\leqq n$, it is sufficient to prove assertion under the additional assumption: $a_i=0$ for $1\leqq i\leqq n$. The matrix (a_{ij}) being invertible, we can resolve $X_i= b_{i_1}Y_1+\cdots+b_{in}Y_n+f_i(X_1, \cdots, X_n)$ for $1\leqq i\leqq n$ conversely, where the non-zero terms of $f_i(X_1, \cdots, X_n)$ are of degree$\geqq 2$ in $X_1, \cdots, X_n$. Now we have $f_i(X_i, \cdots, X_n)=f_i(b_{11}Y_1+\cdots+b_{1n}Y_n+f_1(X_1, \cdots, X_n), \cdots, b_{n1}Y_1+\cdots+b_{nn}Y_n+f_n(X_1, \cdots, X_n))=\sum_{j,k}c_{jk}^{(i)}Y_jY_k+g_i(X_1, \cdots, X_n)$. Here the non-zero terms of $g_i(X_1, \cdots, X_n)$ are of degree $\geqq 3$ in $X_1, \cdots, X_n$. We repeat this procedure and eventually we can write $X_i=\sum b_{i_1\cdots i_n}Y_1^{i1}\cdots Y_n^{in}$. Since $a_i=0$ for $1\leqq i\leqq n$, we must have $b_{0\cdots 0}=0$. Then it is easy to see that φ is a surjection. Next we shall prove that φ is an injection. To the contrary, let us suppose that there is a non-zero power series $f(X_1, \cdots, X_n)\in A[[X_1, \cdots, X_n]]$ satisfying $\varphi(f(X_1, \cdots, X_n))=f(Y_1, \cdots, Y_n) =0$. Let k be the degree of first non-zero terms in $f(X_1, \cdots, X_n)$. Since $a_i =0$ for $1\leqq i\leqq n$, we have $f(0, \cdots, 0)=0$ and hence $k>0$. As is $f(Y_1, \cdots, Y_n) =0$, we get $\sum_{i1+\cdots+in=k} a_{i_1\cdots i_n}(a_{11}X_1+\cdots+a_{1n}X_n)^{i1}\cdots(a_{n1}X_1+\cdots+a_{nn}X_n)^{in}=0$, with some $a_{i1\cdots in}\neq 0$. Now the matrix (a_{ij}) is invertible by our assumption and therefore we have $A[X_1, \cdots, X_n]=A[a_{11}X_1+\cdots+a_{1n}X_n, \cdots, a_{n1}X_1+\cdots+a_{nn}X_n]$. This implies that $a_{11}X_1+\cdots+a_{1n}X_n, \cdots, a_{n1}X_1+\cdots+a_{nn}X_n$ are algebraically independent over A by the proof of (1.1) in [1]. Thus we obtain a contradiction and our proof is complete.

2. A condition that a complete local ring is isomorphic to a formal power series ring

Let A be a ring. A higher derivation on A is an infinite sequence of endomorphisms $D=\{\delta_0, \delta_1, \delta_2, \cdots\}$ of the underlying additive group of A satisfying the conditions: (1) δ_0=the identity mapping of A and (2) $\delta_n(ab)=\sum_{i+j=n}\delta_i(a)\delta_j(b)$ for any $a, b\in A$ and n.

Lemma 1. *Let A be a ring and let $D=\{\delta_0, \delta_1, \delta_2, \cdots\}$ be an infinite se-*

quence of mappings of A into itself. Then the following conditions are equivalent:

(1) *D is a higher derivation on A.*

(2) *The mapping φ: $a \to \delta_0(a)+\delta_1(a)t+\delta_2(a)t^2+\cdots$ is a ring homomorphism of A into $A[[t]]$ such that $\pi\varphi(a)=a$ for every $a \in A$ where π is the homormophism: $\sum_i a_i t^i \to a_0$.*

Proof. The equivalence between (1) and (2) is nothing but a reformulation of the definition.

Lemma 2. *Let A be a ring and let $\mathfrak{A}$ be an ideal of A such that $\bigcap_{m=1}^{\infty} \mathfrak{A}^m =(0)$. Suppose that A is complete in the $\mathfrak{A}$-adic topology and let $D=\{\delta_0, \delta_1, \delta_2, \cdots\}$ be a higher derivation on A. We assume that there exists an element $u \in \mathfrak{A}$ such that $\delta_1(u)=1$ and $\delta_i(u)=0$ for $i \geqq 2$. Then A contains a subring A_0 having the following properties:* (1) *u is analytically independent over A_0 and* (2) *A is the power series ring $A_0[[u]]$.*

Proof. The mapping σ: $A \to A$, given by $\sigma(a)=\sum_{i=0}^{\infty}(-1)^i\delta_i(a)u^i$ is a ring homomorphism. We put $\mathrm{Im}(\sigma)=A_0$. A_0 is a subring of A. From the definition of σ it follows that $a=\sigma(a)+\delta_1(a)u-\delta_2(a)u^2+\cdots$ for $a \in A$. Similarly we see $\delta_1(a)=\sigma(\delta_1(a))+\delta_1^2(a)u-\delta_2\delta_1(a)u^2+\cdots$ and therefore we can write $a=\sigma(a)+\sigma(\delta_1(a))u+(\delta_1^2(a)-\delta_2(a))u^2+(-\delta_2\delta_1(a)+\delta_3(a))u^3+\cdots$. Proceeding in this way we have $a=\sum_{i=0}^{\infty}a_i u^i$ with $a_i \in A_0$. Next we shall prove that u is analytically independent over A_0. Since $\delta_1(u)=1$ and $\delta_i(u)=0$ for $i \geqq 2$, we get $u \in \mathrm{Ker}(\sigma)$. For $a \in A_0$ there exists $b \in A$ such that $a=\sigma(b)=b-\delta_1(b)u+\delta_2(b)u^2-\cdots$. Thus it follows that $a=b-uc$ for some $c \in A$. If $a \in \mathrm{Ker}(\sigma) \cap A_0$, we obtain $b=a+uc \in \mathrm{Ker}(\sigma)$ and hence $a=\sigma(b)=0$. Let us suppose that $\sum_{i=0}^{\infty}a_i u^i=0$ with $a_i \in A_0$. Since $a_0=-(\sum_{i=1}^{\infty}a_i u^{i-1})u$ and $u \in \mathrm{Ker}(\sigma)$, we have $a_0 \in \mathrm{Ker}(\sigma) \cap A_0=(0)$. By induction it will be shown that all $a_i=0$. If we assume $a_i=0$ for $0 \leqq i \leqq n$, we get $0=a_{n+1}u^{n+1}+a_{n+2}u^{n+2}+\cdots$. Then we have $0=\delta_{n+1}(a_{n+1}u^{n+1}+a_{n+2}u^{n+2}+\cdots)=a_{n+1}+ub$ for some $b \in A$ and therefore $a_{n+1} \in \mathrm{Ker}(\sigma) \cap A_0=(0)$ as desired. Hence A is the power series ring $A_0[[u]]$.

An ideal $\mathfrak{A}$ of a ring A is said to be differential if we have $\delta_1(\mathfrak{A}) \subset \mathfrak{A}$ for every higher derivation $\{\delta_0, \delta_1, \delta_2, \cdots\}$ on A.

Theorem 3. *A complete local ring A is isomorphic to a formal power series ring $A_0[[X]]$ if and only if the maximal ideal $\mathfrak{M}$ of A is not differential.*

Proof. We assume that A is isomorphic to a formal power series ring $A_0[[X]]$. Then A_0 is a complete local ring. Let $\mathfrak{M}_0$ be the maximal ideal of A_0. It is well-known that the maximal ideal of $A_0[[X]]$ is $\mathfrak{M}_0[[X]]+(X)$. We consider a mapping δ_n of $A_0[[X]]$ into itself defined by $\delta_n(\sum_{i=0}^{\infty}a_i X^i)=\sum_{i=0}^{\infty}\binom{i}{n}a_i X^{i-n}$ where $\binom{i}{n}=0$ for $i<n$. It is easy to see that $\{\delta_0, \delta_1, \delta_2, \cdots\}$ is a

higher derivation on $A_0[[X]]$. Since $\delta_1(X)=1$, the ideal $\mathfrak{M}_0[[X]]+(X)$ is not differential and hence $\mathfrak{M}$ is so. Conversely we assume that the maximal ideal $\mathfrak{M}$ of A is not differential. Then exists a higher derivation $\{\delta_0, \delta_1, \delta_2, \cdots\}$ on A such that $\delta_1(u)$ is a unit of A for some $u\in\mathfrak{M}$. By Lemma 1 the mapping φ: $a\to\sum_{i=0}^{\infty}\delta_i(a)t^i$ is a ring homomorphism of A into the power series ring $A[[t]]$. We shall set $s=\delta_1(u)t+\delta_2(u)t^2+\cdots$. Since $\delta_1(u)$ is a unit of A, we can resolve $t=u_1s+u_2s^2+\cdots(u_i\in A)$ conversely, where $u_1=\delta_1(u)^{-1}$ is a unit of A. Obviously s is analytically independent over A and we have $A[[t]]=A[[s]]$. For $a\in A$ we shall define $d_n(a)\in A$ by the following identity:

$$\begin{aligned} &a+\delta_1(a)t+\delta_2(a)t^2+\cdots+\delta_n(a)t^n+\cdots \\ &= a+\delta_1(a)(u_1s+u_2s^2+\cdots)+\delta_2(a)(u_1s+u_2s^2+\cdots)^2+\cdots \\ &= a+d_1(a)s+d_2(a)s^2+\cdots+d_n(a)s^n+\cdots . \end{aligned}$$

Then the mapping ψ: $a\to a+d_1(a)s+d_2(a)s^2+\cdots$ is a ring homomorphism of A into $A[[s]]$. It follows from Lemma 1 that $\{d_0=1, d_1, d_2, \cdots\}$ is a higher derivation on A. Since $u+\delta_1(u)t+\delta_2(u)t^2+\cdots=u+s$, we have $d_1(u)=1$ and $d_i(u)=0$ for $i\geqq 2$. Hence by Lemma 2 we see that A is isomorphic to a formal power series ring $A_0[[X]]$.

3. Power invariant rings and strongly power invariant rings

Let A be a ring. We say that A is n-power invariant if whenever B is a ring and $A[[X_1, \cdots, X_n]]\cong B[[Y_1, \cdots, Y_n]]$, then we have $A\cong B$. A is said to be strongly n-power invariant if whenever B is a ring and $A[[X_1, \cdots, X_n]]\cong B[[Y_1, \cdots, Y_n]]$ under φ, then there exists a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ such that $\varphi(X_i)=\psi(Y_i)$ for $1\leqq i\leqq n$. We first observe that if A is strongly n-power invariant and $A[[X_1, \cdots, X_n]]\cong B[[Y_1, \cdots, Y_n]]$ under φ, there is a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ such that $\varphi(X_i)=\psi(Y_i)$ for $1\leqq i\leqq n$ and hence $\psi^{-1}\varphi$ is an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$ satisfying $\psi^{-1}\varphi(X_i)=Y_i$ for $1\leqq i\leqq n$. Hence we have

$$\begin{aligned} &A\cong A[[X_1, \cdots, X_n]]/(X_1, \cdots, X_n)\cong B[[Y_1, \cdots, Y_n]]/(Y_1, \cdots, Y_n) \\ &\cong B . \end{aligned}$$

Thus a strongly n-power invariant ring A is n-power invariant.

Theorem 4.* *A semisimple ring A (the Jacobson radical of $A=(0)$) is strongly n-power invariant for any n.*

Proof. Let B be a ring and let φ be an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$. By Proposition 1 we have

* This result is essentially due to M.J. O'Malley [4].

$$\varphi(X_i) = b_i + b_{i1}Y_1 + \cdots + b_{in}Y_n + \cdots (1 \leqq i \leqq n),$$
$$\varphi^{-1}(Y_i) = a_{i1}X_1 + \cdots + a_{in}X_n + \cdots (1 \leqq i \leqq n)$$

where $b_i \in \mathfrak{J}(B)$, $b_{ij} \in B$, $a_{ij} \in A$ and B is $(b_1, \cdots, b_n)$-adic complete. Let $\varphi(a_{ij}) = b_{ij}' + b_{ij1}Y_1 + \cdots + b_{ijn}Y_n + \cdots$ for $1 \leqq i, j \leqq n$, where $b_{ij}', b_{ijk} \in B$. Then by Propositon 1

$$\begin{aligned} Y_i &= \varphi(\varphi^{-1}(Y_i)) \\ &= \varphi(a_{i1})\varphi(X_1) + \cdots + \varphi(a_{in})\varphi(X_n) + \cdots \\ &= \textstyle\sum_{j=1}^n (b_{ij}' + \sum_{k=1}^n b_{ijk}Y_k + \cdots)(b_j + \sum_{k=1}^n b_{jk}Y_k + \cdots) + \cdots . \end{aligned}$$

Equating the coefficients of Y's we have

$$\textstyle\sum_{j=1}^n b_{ij}' b_{jk} \equiv \delta_{ik} (\mathrm{mod.}\ \mathfrak{J}(B))$$

because the coefficients of Y's in $\varphi(a)\varphi(X_1)^{i_1} \cdots \varphi(X_n)^{i_n} (a \in A,\ i_1 + \cdots + i_n \geqq 2)$ belong to $\mathfrak{J}(B)$. Then it is immediate to see that the matrix (b_{ij}) is invertible. Thus it follows from Theorem 2 that there exists a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leqq i \leqq n$.

Corollary. *An affine domain A over a field is strongly n-power invariant for any n.*

Proof. By Hilbert's Nullstellensatz we see that $\mathfrak{J}(A) = (0)$. Now our assertion follows from Theorem 4.

From now on we exclusively consider local rings which may not be noetherian (see [2], p. 13) and for such a ring A we denote the unique maximal ideal by $\mathfrak{M}(A)$.

Theorem 5. *Let A be a local ring which may not be noetherian and let φ be an isomorphism of $A[[X_1, \cdots, X_n]]$ onto $B[[Y_1, \cdots, Y_n]]$. Then we have the following facts:*

(1) *B is a local ring which may not be noetherian.*

(2) *There is either a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leqq i \leqq n$, or A(resp. B) contains a local ring A_0(resp. B_0) which may not be noetherian and an element $a \in \mathfrak{M}(A)$ (resp. $b \in \mathfrak{M}(B)$) such that a (resp. b) is analytically independent over A_0(resp. B_0) and $A = A_0[[a]]$ (resp. $B = B_0[[b]]$).*

Proof. (1) It is obvious by Proposition 1.

(2) By Proposition 1 we can express

$$\varphi(X_i) = b_i + b_{i1}Y_1 + \cdots + b_{in}Y_n + \cdots (1 \leqq i \leqq n),$$
$$\varphi^{-1}(Y_i) = a_i + a_{i1}X_1 + \cdots + a_{in}X_n + \cdots (1 \leqq i \leqq n)$$

where $a_i \in \mathfrak{M}(A)$ and $b_i \in \mathfrak{M}(B)$ for $1 \leqq i \leqq n$. Here A is $(a_1, \cdots, a_n)$-adic complete and B is $(b_1, \cdots, b_n)$-adic complete. Let

$$\varphi(a_i) = b_i' + b_{i1}'Y_1 + \cdots + b_{in}'Y_n + \cdots (1 \leqq i \leqq n),$$
$$\varphi(a_{ij}) = b_{ij}'' + b_{ij1}Y_1 + \cdots + b_{ijn}Y_n + \cdots (1 \leqq i, j \leqq n).$$

We see that b_i' is in $\mathfrak{M}(B)$, as is $a_i \in \mathfrak{M}(A)$. If the matrix

$$(\#) \quad \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots\cdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

is invertible, then it follows from Theorem 2 that there exists a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leqq i \leqq n$. To the contrary we assume that matrix (#) is not invertible. From Proposition 1 we have

$$\begin{aligned} Y_i &= \varphi(\varphi^{-1}(Y_i)) \\ &= \varphi(a_i) + \varphi(a_{i1})\varphi(X_1) + \cdots + \varphi(a_{in})\varphi(X_n) + \cdots \\ &= (b_i' + \textstyle\sum_{k=1}^n b_{ik}'Y_k + \cdots) + \sum_{j=1}^n (b_{ij}'' + \sum_{k=1}^n b_{ijk}Y_k + \cdots) \\ &\quad (b_j + \textstyle\sum_{k=1}^n b_{jk}Y_k + \cdots) + \cdots. \end{aligned}$$

Comparing the coefficients of Y's we get

$$\textstyle\sum_{j=1}^n b_{ij}''b_{jk} + b_{ik}' \equiv \delta_{ik} \ (\mathrm{mod.}\ \mathfrak{M}(B))$$

because the coefficients of Y's in $\varphi(a)\varphi(X_1)^{i_1} \cdots \varphi(X_n)^{i_n} (a \in A,\ i_1 + \cdots + i_n \geqq 2)$ belong to $\mathfrak{M}(B)$. Thus we have

$$\begin{pmatrix} b_{11}'' & b_{12}'' & \cdots & b_{1n}'' \\ b_{11}'' & b_{12}'' & \cdots & b_{2n}'' \\ & \cdots\cdots & & \\ b_{n1}'' & b_{n2}'' & \cdots & b_{nn}'' \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots\cdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \equiv \begin{pmatrix} 1-b_{11}' & -b_{11}' & \cdots & -b_{1n}' \\ -b_{21}' & 1-b_{22}' & \cdots & -b_{2n}' \\ & \cdots\cdots & & \\ -b_{n1}' & -b_{n2}' & \cdots & 1-b_{nn}' \end{pmatrix}$$

(mod. $\mathfrak{M}(B)$). By our assumption the matrix (#) is not invertible and so det $(b_{ij}) \in \mathfrak{M}(B)$. Since $\det(\delta_{ij} - b_{ij}') \equiv \det(b_{ij}'')\det(b_{ij})$ (mod. $\mathfrak{M}(B)$), we must have $\det(\delta_{ij} - b_{ij}') \in \mathfrak{M}(B)$ where δ_{ij} is the Kronecker's symbol. Hence there exists *a* $b_{ij} \notin \mathfrak{M}(B)$. Then it is easy to see that the matrix

$$\begin{array}{c} \\ \\ \\ j \supset \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ b_{i1}' & \cdots & b_{ij}' & \cdots & b_{in}' & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is invertible. Now we shall show that B is (b_i')-adic complete. Let $\{c_k\}$ be a Cauchy sequence in B under the (b_i')-adic topology. Then there is a subsequence $\{d_k\}$ of $\{c_k\}$ such that $d_k=\sum_{j=0}^{k} r_j b_i'^j$ for each k, where $r_j \in B$. Let $f_j=\varphi^{-1}(r_j)$ and we set $f=\sum_{j=0}^{\infty} a_i^j f_j$ which is a well defined power series in $A[[X_1, \cdots, X_n]]$, because A is $(a_1, \cdots, a_n)$-adic complete. Then $\varphi(f)=\sum_{j=0}^{\infty} \varphi(a_i)^j r_j=\sum_{j=0}^{\infty} r_j b_i'^j+g$ in $B^*[[Y_1, \cdots, Y_n]]$, where B^* denotes the (b_i')-adic completion of B and g has no constant term. Since $\varphi(f)\in B[[Y_1, \cdots, Y_n]]$, we see that $\sum_{j=0}^{\infty} r_j b_i'^j \in B$, that is, $\{d_k\}$ converges in B and hence $\{c_k\}$ converges in B. Together with $b_i' \in \mathfrak{M}(B)$, it follows from Theorem 2 that there exists a B-automorphism σ of $B[[Y_1, \cdots, Y_n]]$ such that $\sigma(Y_j)=\varphi(a_i)$ and $\sigma(Y_k)=Y_k$ for $k \neq j$, that is, $\varphi(a_i)$ is analytically independent over $B[[Y_1, \cdots, Y_{j-1}, Y_{j+1}, \cdots, Y_n]]$ and $B[[Y_1, \cdots, Y_n]]=B[[Y_1, \cdots, Y_{j-1}, \varphi(a_i), Y_{j+1}, \cdots, Y_n]]$. We consider the following sequence of ring homomorphisms:

$$A \xrightarrow{\iota} A[X_1, \cdots, X_n]] \overset{\varphi}{\rightrightarrows} B[[Y_1, \cdots, Y_n]] = B[[Y_1, \cdots, Y_{j-1},$$
$$\varphi(a_i), Y_{j+1}, \cdots, Y_n]] \xrightarrow{\tau} B[[Y_1, \cdots, Y_{j-1}, \varphi(a_i), Y_{j+1}, \cdots, Y_n]]$$
$$[[t]] \overset{\tilde{\varphi}^{-1}}{\rightrightarrows} A[[X_1, \cdots, X_n]]\,[[t]] \xrightarrow{\nu} A[[t]]$$

where $\iota(a)=a$ for $a\in A$, φ is the given isomorphism, $\tau(\varphi(a_i))=\varphi(a_i)+t$, $\tau(Y_k)=Y_k+t$ for $k\neq j$, $\tilde{\varphi}^{-1}$ is the ismomorphism induced by φ^{-1}, and $\nu(X_i)=0$ for $1\leqq i\leqq n$. We set ρ the composite of these homomorphisms. Then ρ is a ring homomorphism of A into $A[[t]]$ such that $\pi\rho(a)=a$ where π is the homomorphism: $\sum_i a_i t^i \to a_0$. Thus we can express $\rho(a)=a+\delta_1(a)t+\delta_2(a)t^2+\cdots$. Thence $\{1, \delta_1, \delta_2, \cdots\}$ is a higher derivation on A by Lemma 1. Since $\rho(a_i)=a_i+t$, we have $\delta_1(a_i)=1$, $\delta_j(a_i)=0$ for $j\geqq 2$ and by Lemma 2 we see that A contains a subring A_0 satisfying the properties: a_i is analytically independent over A_0 and $A=A_0[[a_i]]$. It is obvious that A_0 is a local ring which may not be noetherian. On the other hand

$$\begin{aligned} X_l &= \varphi^{-1}(\varphi(X_l)) \\ &= \varphi^{-1}(b_l)+\varphi^{-1}(b_{l1})\varphi^{-1}(Y_1)+\cdots+\varphi^{-1}(b_{ln})\varphi^{-1}(Y_n)+\cdots. \end{aligned}$$

We set

$$\begin{aligned} \varphi^{-1}(b_l) &= a_l'+a_{l1}'X_1+\cdots+a_{ln}'X_n+\cdots(1\leqq l\leqq n), \\ \varphi^{-1}(b_{lm}) &= a_{lm}''+a_{lm1}X_1+\cdots+a_{lmn}X_n+\cdots(1\leqq l, m\leqq n). \end{aligned}$$

Here a_l' is in $\mathfrak{M}(A)$, as is $b_l\in\mathfrak{M}(B)$. Thus

$$\begin{aligned} X_l = {} & (a_l'+\textstyle\sum_{k=1}^{n} a_{lk}'X_k+\cdots)+\sum_{m=1}^{n}(a_{lm}''+\sum_{k=1}^{n} a_{lmk}X_k+\cdots) \\ & (a_m+\textstyle\sum_{k=1}^{n} a_{mk}X_k+\cdots)+\cdots. \end{aligned}$$

Comparing the coefficients of X's we get

$$\sum_{m=1}^{n} a_{lm}'' a_{mk} + a_{lk}' \equiv \delta_{lk} \ (\mathrm{mod.}\ \mathfrak{M}(A)) .$$

In the matrix notation

$$(a_{ij}'')(a_{ij}) \equiv (\delta_{ij} - a_{ij}') \ (\mathrm{mod.}\ \mathfrak{M}(A)) .$$

Now we have $\det(\varphi^{-1}(b_{lm})) \equiv \det(a_{lm}'')$ (mod. $(X_1, \cdots, X_n)$). Since $\det(\varphi^{-1}(b_{lm})) = \varphi^{-1}(\det(b_{lm}))$ and $\det(b_{lm}) \in \mathfrak{M}(B)$ by our assumption, it is immediate to see that $\det(a_{lm}'') \in \mathfrak{M}(A)$. Thus the same argument as above implies that some $a_{lm}' \notin \mathfrak{M}(A)$ and we have $A[[X_1, \cdots, X_n]] = A[[X_1, \cdots, X_{m-1}, \varphi^{-1}(b_l), X_{m+1}, \cdots, X_n]]$. Then we see that B contains a subring B_0 satisfying the properties: b_l is analytically independent over B_0 and $B = B_0[[b_l]]$. Obviously B_0 is a local ring which may not be noteherian and our proof is now complete.

Theorem 6. *Let A be a local ring which may not be noetherian. Then we have only one of the followings:*

(1) *A is strongly n-power invariant for any n.*

(2) *A is isomorphic to a formal power series ring $A_0[[X]]$.*

Proof. We assume that A is not strongly n-power invariant for some n. Then we have a ring B and an isomorphism φ: $A[[X_1, \cdots, X_n]] \cong B[[Y_1, \cdots, Y_n]]$ such that there is never a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ satisfying $\varphi(X_i) = \psi(Y_i)$ for $1 \leqq i \leqq n$. Now Theorem 5 implies that A must be isomorphic to a power series ring $A_0[[X]]$. Conversely it is easy to see that a power series ring $A_0[[X]]$ is not strongly n-power invariant for any n.

Thus a local ring which may not be noetherian can simply be called to be stronly power invariant without reference to the number n of variables.

Corollary 1. *An artinian local ring is strongly power invariant.*

Proof. An artinian local ring A is not isomorphic to a power series ring $A_0[[X]]$ and hence A is strongly power invariant.

Corollary 2. *Let P be a point on an irreducible affine algebraic curve over an algebraically closed field k and let A be the local ring of P. Then the following conditions are equivalent:*

(1) *P is a singular point.*

(2) *The completion $\hat{A}$ is strongly power invariant.*

Proof. Let us suppose that P is non-singular. Then it is obvious that $\hat{A}$ is isomorphic to the power series ring $k[[X]]$ and hence by Theorem 6 $\hat{A}$ is not strongly power invariant. Conversely we assume that $\hat{A}$ is not strongly power invariant. Then it follows from Theorem 6 that $\hat{A}$ is isomorphic to a formal power series ring $A_0[[X]]$. Since $\hat{A}$ is reduced and dim $\hat{A}=1$, A_0 is reduced and dim $A_0=0$. Now it is immediate to show that $A_0\simeq k$ and therefore $\hat{A}\simeq k[[X]]$. Hence P is non-singular.

Corollary 3. *Let V be an irreducible affine varety over a field of characteristic zero and let A be the local ring of a component of the singular locus of V. Then the completion $\hat{A}$ is strongly power invariant.*

Proof. If $\hat{A}$ is not strongly power invariant, $\hat{A}$ is isomorphic to a formal power series ring $A_0[[X]]$. Then we can obtain a contradiction by the same argument as that of Theorem 5 in [5].

Theorem 7. *Let A be a complete local ring. Then A is strongly power invariant if and only if the maximal ideal $\mathfrak{M}(A)$ of A is differential.*

Proof. The assertion follows from Theorem 3 and Theorem 6 immediately.

Theorem 8**) *A noetherian local ring is n-power invariant for any n.*

Proof. Let A be a noetherian local ring. We shall prove our assertion by induction on Krull dimension of A. If dim $A=0$, then A is strongly power invariant by Corollary 1 of Theorem 6 and hence A is n-power invariant for any n according to the remark preceding to Theorem 4. Let us suppose dim $A>0$. Let B be a ring and let $A[[X_1, \cdots, X]]\simeq B[[Y_1, \cdots, Y_n]]$ under φ. If there exists a B-automorphism ψ of $B[[Y_1, \cdots, Y_n]]$ such that $\varphi(X_i)=\psi(Y_i)$ for $1\leqq i\leqq n$, then $A\simeq B$ by the remark preceding to Theorem 4. Unless such an automorphism exists, it follows from Theorem 5 that A(resp. B) is a power series ring $A_0[[a]]$ (resp. $B_0[[b]]$). Here A_0 and B_0 are local rings. Thus we have an isomorphism $A_0[[a, X_1, \cdots, X_n]]\simeq B_0[[b, Y_1, \cdots, Y_n]]$. Since dim A_0 $<$ dim A, our induction hypothesis means that A_0 is n-power invariant for any n. Hence we have $A_0\simeq B_0$ and $A=A_0[[a]]\simeq B_0[[b]]=B$, as desired.

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** After this paper is completed, the author has observed that E. Hamann obtained the result: a quasi-local ring is n-power invariant for any n, in her paper "On Power Invariance", to appear.

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