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Osaka J. Math.
7 (1970), 443-449

NOTE ON A THEOREM DUE TO MILNOR

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(Received April 17, 1970)

1. Introduction

J. Milnor [1] has proved the following theorem: Let N be a closed topological manifold which is a mod 2 homology n-sphere, and T be a fixed point free involution on N. Then, for every continuous map $f:N\to N$ such that f_*: $H_n(N; \boldsymbol{Z}_2)\to H_n(N; \boldsymbol{Z}_2)$ is not trivial, there exists a point $y\in N$ such that $fT(y) = Tf(y)$.

In the present paper, we shall show that this result can be generalized as follows:

Theorem 1. *Let N and M be topological n-manifolds on each of which there is given a fixed point free involution T ($n\geqq 1$). Assume that N has the mod 2 homology of the n-sphere. Then, for every continuous map $f:N\to M$ such that f_*: $H_n(N; \boldsymbol{Z}_2)\to H_n(M; \boldsymbol{Z}_2)$ is not trivial, there exists a point $y\in N$ such that $fT(y) = Tf(y)$.*

Our method is different from Milnor [1], and we shall apply the method we used in [2] to prove a generalization of Borsuk-Ulam theorem.

REMARK. The theorem is regarded in some sense as a converse of Corollary 1 of the main theorem in [2].

Throughout this paper, all chain complexes and hence all homology and cohomology groups will be considered on $\boldsymbol{Z}_2$.

2. The chain map

Let Y be a Hausdorff space on which there is given a fixed point free involution T. Denote by π the cyclic group of order 2 generated by T. We shall denote by Y_π the orbit space of Y, and by p: $Y\to Y_\pi$ the projection. Consider the induced homomorphisms $T_\sharp:S(Y)\to S(Y)$ and $p_\sharp:S(Y)\to S(Y_\pi)$ of singular complexes. Then a chain map

$$\phi: S(Y_\pi)\to S(Y)$$

can be defined by

$$\phi(c) = \tilde{c}+T_\sharp(\tilde{c}), \quad p_\sharp(\tilde{c}) = c,$$

where $c \in S(Y_\pi)$, $\tilde{c} \in S(Y)$. Obviously ϕ is functorial with respect to equivariant continuous maps. Therefore ϕ induces homomorphisms

$$\phi_*: H_*(Y_\pi) \to H_*(Y), \quad \phi^*: H^*(Y) \to H^*(Y_\pi)$$

of homology and cohomology, which are functorial with respect to equivariant continuous maps.

As for the homomorphism $p^*: H^*(Y_\pi) \to H^*(Y)$ and the cap product, we have

Lemma 1. $\phi_*(\alpha \frown a) = p^*(\alpha) \frown \phi_*(a)$ *for* $\alpha \in H^*(Y_\pi)$, $a \in H_*(Y_\pi)$.

Proof. Let u be a singular cochain of Y_π, and c a singular chain of Y_π. Take a singular chain $\tilde{c}$ of Y such that $p_\sharp(\tilde{c}) = c$. Since

$$u \frown c = u \frown p_\sharp(\tilde{c}) = p_\sharp(p^\sharp u \frown c),$$

it follows that

$$\begin{aligned} \phi(u \frown c) &= p^\sharp u \frown \tilde{c} + T_\sharp(p^\sharp u \frown \tilde{c}) \\ &= p^\sharp u \frown \tilde{c} + T^\sharp p^\sharp u \frown T_\sharp \tilde{c} \\ &= p^\sharp u \frown \tilde{c} + p^\sharp u \frown T_\sharp \tilde{c} \\ &= p^\sharp u \frown (\tilde{c} + T_\sharp \tilde{c}) \\ &= p^\sharp u \frown \phi(c). \end{aligned}$$

This proves the desired lemma.

We have also

Lemma 2. *If Y is a closed topological n-manifold, then $\phi_*: H_n(Y_\pi) \to H_n(Y)$ sends the* (mod 2) *fundamental class of Y_π to that of Y.*

Proof. Let y be any point of Y. Then ϕ induces a homomorphism ϕ_*: $H_*(Y_\pi, Y_\pi - p(y)) \to H_*(Y, Y - \{y, T(y)\})$, and the following commutative diagram holds:

$$\begin{tikzcd} H_n(Y_\pi) \arrow[r, "j_{1*}"] \arrow[d, "\phi_*"] & H_n(Y_\pi, Y_\pi - p(y)) \arrow[d, "\phi_*"] \\ H_n(Y) \arrow[r, "j_{2*}"] \arrow[dr, "j_{3*}"'] & H_n(Y, Y - \{y, T(y)\}) \arrow[d, "j_{4*}"] \\ & H_n(Y, Y - y) \end{tikzcd}$$

where j_{i*} ($i = 1, 2, 3, 4$) are induced by the inclusions. If $w \in H_n(Y_\pi)$ is the fundamental class, then $j_{1*}(w)$ is the generator of $H_n(Y_\pi, Y_\pi - p(y))$. It is easily seen that $j_{4*} \circ \phi_*$ sends the generator of $H_n(Y_\pi, Y_\pi - p(y))$ to that of $H_n(Y, Y - y)$. Therefore $j_{3*} \phi_*(w)$ is the generator of $H_n(Y, Y - y)$. Consequently $\phi_*(w)$ is the fundamental class of $H_n(Y)$. This completes the proof of Lemma 2.

REMARK. ϕ is a kind of transfer map.

3. The element θ'

Let N and M be connected closed topological manifolds, on each of which there is given a fixed point free involution T. Consider the product manifolds $N\times M$ and $N\times M^2=N\times M\times M$ on which π acts without fixed point by

$$T(y, x) = (T(y), T(x)), \quad T(y, x, x') = (T(y), x', x)$$

($y\in N$, x, $x'\in M$). Let $N\underset{\pi}{\times} M$, $N\underset{\pi}{\times} M^2$ denote the orbit spaces; these are connected closed topological manifolds.

Define a continuous map $d_0'\colon N\times M\to N\times M^2$ by

$$d_0'(y, x) = (y, x, T(x))$$

($y\in N$, $x\in M$). Then d_0' induces a continuous map $d_0'\colon N\underset{\pi}{\times} M\to N\underset{\pi}{\times} M^2$, and hence a homomorphism $d_{0*}'\colon H_*(N\underset{\pi}{\times} M)\to H_*(N\underset{\pi}{\times} M^2)$. Let $\tau\in H_{m+n}(N\underset{\pi}{\times} M)$ denote the fundamental class of the manifold $N\underset{\pi}{\times} M$ and define

$$\theta_0'\in H^m(N\underset{\pi}{\times} M^2)$$

to be the element which is the Poincaré dual of $d_{0*}'(\tau)$, where $n=\dim N$, $m=\dim M$.

Assume now that $n\geqq m$ and N has the mod 2 homology of the sphere ($n\geqq 1$). Consider the space N^∞ constructed in §5 of [2]. Then it follows from Theorem 6 of [2] that there exists a unique element $\theta'\in H^m(N^\infty\underset{\pi}{\times} M^2)$ such that

$$i^*(\theta') = \theta_0'$$

for the homomorphism $i^*\colon H^m(N^\infty\underset{\pi}{\times} M^2)\to H^m(N\underset{\pi}{\times} M^2)$ induced by the inclusion.

With the notation in [2], we have

Theorem 2. $$\theta' = P(1, \bar{\mu})+\delta',$$
where $\bar{\mu}\in H^m(M)$ is the generator, and δ' is a linear combination of elements of the type $P(\alpha, \beta)$ with $\deg \alpha>0$, $\deg \beta>0$. (Compare Theorem 7 in [2].)

Proof. Consider the orbit space $N^\infty\underset{\pi}{\times} M$ of $N^\infty\times M$ on which π acts by $T(y, x)=(T(y), T(x))$, ($y\in N^\infty$, $x\in M$). Then the projection $N^\infty\times M\to M$ defines a fibration $q\colon N^\infty\underset{\pi}{\times} M\to M_\pi$ with fibre N^∞. Since $\widetilde{H}_*(N^\infty)=0$, it follows that $q_*\colon H_*(N^\infty\underset{\pi}{\times} M)\cong H_*(M_\pi)$ and in particular $H_{n+m}(N^\infty\underset{\pi}{\times} M)=0$.

For the continuous map $d'\colon N^\infty\underset{\pi}{\times} M\to N^\infty\underset{\pi}{\times} M^2$ defined similarly to d_0', the following commutative diagram holds:

$$\begin{array}{ccc} H_{n+m}(N \underset{\pi}{\times} M) & \xrightarrow{d'_{0*}} & H_{n+m}(N \underset{\pi}{\times} M^2) \\ \downarrow i_* & & \downarrow i_* \\ H_{n+m}(N^\infty \underset{\pi}{\times} M) & \xrightarrow{d'_*} & H_{n+m}(N^\infty \underset{\pi}{\times} M^2), \end{array}$$

where i are the inclusions. Hence we have

$$i_* d'_{0*}(\tau) = d'_* i_*(\tau) = 0.$$

For the generator λ of $H_{n+2m}(N \underset{\pi}{\times} M^2)$, we have

$$d'_{0*}(\tau) = \theta'_0 \frown \lambda.$$

Therefore it follows that

$$0 = i_*(\theta'_0 \frown \lambda) = i_*(i^*(\theta') \frown \lambda) = \theta' \frown i_*(\lambda).$$

Let $\mu \in H_m(M)$ be the generator, and $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be a basis of the module $H^*(M)$ such that $\alpha_1 = 1$ and $\alpha_l = \bar{\mu}$. Then, with the notations in [2], we have

$$i_*(\lambda) = P_n(\mu)$$

(see §6 of [2]), and

$$\theta' = \sum_{i,j} g_{ij} P_i(\alpha_j) + \sum_{j<k} h_{jk} P(\alpha_j, \alpha_k)$$

where g_{ij}, $h_{jk} \in Z_2$ (see Theorem 4 of [2]). Thus it follows from Theorem 4 of [2] that

$$0 = \theta' \frown P_n(\mu) = \sum_{i,j} g_{ij} P_{n-i}(\alpha_j \frown \mu)$$

and hence $g_{ij} = 0$.

It remains now to prove that $h_{1l} = 1$. To do this, we consider the following diagram:

$$\begin{array}{ccccc} H_{n+m}(N \underset{\pi}{\times} M) & \xrightarrow{d'_{0*}} & H_{n+m}(N \underset{\pi}{\times} M^2) & \xleftarrow{\frown \lambda} & H^m(N \underset{\pi}{\times} M^2) \\ \downarrow \phi_* & & \downarrow \phi_* & & \downarrow p^* \\ H_{n+m}(N \times M) & \xrightarrow{d'_{0*}} & H_{n+m}(N \times M^2) & \xleftarrow{\frown \phi_*(\lambda)} & H^m(N \times M^2), \end{array}$$

where $p: N \times M^2 \to N \underset{\pi}{\times} M^2$ is the projection. It follows from Lemma 1 that the diagram is commutative, and from Lemma 2 that

$$\phi_*(\tau) = \nu \times \mu, \quad \phi_*(\lambda) = \nu \times \mu \times \mu$$

where $\nu \in H_n(N)$ is the generator. Therefore we have

$$d'_{0*}(\nu\times\mu)=d'_{0*}\phi_*(\tau)=p^*(\theta'_0)\frown\phi_*(\lambda)$$
$$=p^*i^*(\theta')\frown(\nu\times\mu\times\mu).$$

Consider next the commutative diagram

$$\begin{array}{ccc} H^m(N^\infty\underset{\pi}{\times}M^2) & \xrightarrow{i^*} & H^m(N\underset{\pi}{\times}M^2) \\ \downarrow{p^*} & & \downarrow{p^*} \\ H^m(N^\infty\times M^2) & \xrightarrow{i^*} & H^m(N\times M^2), \end{array}$$

where p are the projections and i are the inclusions. Since it is obvious that

$$p^*(P(\alpha_j,\ \alpha_k))=1\times\alpha_j\times\alpha_k+1\times\alpha_k\times\alpha_j,$$

we have

$$d'_{0*}(\nu\times\mu)=i^*p^*(\theta')\frown(\nu\times\mu\times\mu)$$
$$=i^*p^*(\sum_{j<k}h_{jk}P(\alpha_j,\ \alpha_k))\frown(\nu\times\mu\times\mu)$$
$$=\nu\times(\sum_{j<k}h_{jk}(a_j\times a_k+a_k\times a_j)),$$

where $a_i=\alpha_i\frown\mu$.

Let $\Delta_*: H_*(M)\to H_*(M\times M)$ denote the homomorphism induced by the diagonal map. Then we have

$$d'_{0*}(\nu\times\mu)=\nu\times(1\times T)_*\,\Delta_*(\mu).$$

Therefore it holds that

$$(1\times T)_*\,\Delta_*(\mu)=\sum_{j<k}h_{jk}(a_j\times a_k+a_k\times a_j).$$

Thus it follows that

$$1=\langle\bar{\mu},\ \mu\rangle=\langle 1\times\bar{\mu},\ \Delta_*(\mu)\rangle$$
$$=\langle 1\times T^*(\bar{\mu}),\ \sum_{j<k}h_{jk}(a_j\times a_k+a_k\times a_j)\rangle$$
$$=\sum_{j<k}h_{jk}(\langle 1\times\bar{\mu},\ a_j\times a_k\rangle+\langle 1\times\bar{\mu},\ a_k\times a_j\rangle)$$
$$=h_{1l}.$$

This completes the proof of Theorem 2.

4. Proof of Theorem 1

In what follows we shall prove Theorem 1.

We note first that M may be assumed to be a connected closed topological manifold.

Consider continuous maps $s: N\to N\times M^2$ and $k: N\to N^\infty\times N^2$ defined by

$$s(y) = (y, f(y), fT(y)),$$
$$k(y) = (y, y, T(y)), \ (y \in N).$$

Then, as in the proof of Lemma 4 in [2], we have by Theorem 2

$$\begin{aligned} s^*(\theta'_0) &= s^*i^*(\theta') = k^*(1\times f^2)^*(\theta') \\ &= k^*(1\times f^2)^*(P(1, \bar{\mu})+\delta') \end{aligned}$$

in $H^n(N_\pi)$. From this and the hypothesis, it follows that

$$s^*(\theta'_0) = k^*P(1, \bar{\nu}),$$

where $\bar{\nu} \in H^n(N)$ is the generator.

We have a commutative diagram

$$\begin{tikzcd} & H^n(N^3) \arrow[dr, "\Delta_*"] & \\ H^n(N^\infty \times N^2) \arrow[ur, "(1\times 1\times T)^*"] \arrow[rr, "k^*"] \arrow[d, "\phi^*"] & & H^n(N) \arrow[d, "\phi^*"] \\ H^n(N^\infty \underset{\pi}{\times} N^2) \arrow[rr, "k^*"] & & H^n(N_\pi), \end{tikzcd}$$

where $N^3 = N\times N\times N$ and $\Delta: N \to N^3$ is the diagonal map. It is easily seen that

$$P(1, \bar{\nu}) = \phi^*(1\times 1\times \bar{\nu}).$$

Therefore we have

$$\begin{aligned} k^*P(1, \bar{\nu}) &= k^*\phi^*(1\times 1\times \bar{\nu}) \\ &= \phi^*\Delta^*(1\times 1\times T^*(\bar{\nu})) \\ &= \phi^*\Delta^*(1\times 1\times \bar{\nu}) \\ &= \phi^*(\bar{\nu}), \end{aligned}$$

which proves

$$s^*(\theta'_0) \neq 0.$$

Put

$$A'(f) = \{y \in N \mid fT(y) = Tf(y)\}$$

and

$$B'(f) = \text{Image of } A'(f) \text{ under the projection } N \to N_\pi.$$

Then we have the following commutative diagram which is similar to the diagram in the proof of Lemma 5 in [2]:

$$\begin{tikzcd} H_{2n}(d_0'(N \underset{\pi}{\times} M)) \arrow[r, "j_*"] \arrow[d, "\backslash \gamma_2'"] & H_{2n}(N \underset{\pi}{\times} M^2) \arrow[d, "\backslash \gamma_1"] \\ H^n(N \underset{\pi}{\times} M^2, N \underset{\pi}{\times} M^2 - d_0'(N \underset{\pi}{\times} M)) \arrow[r, "j^*"] \arrow[d, "s^*"] & H^n(N \underset{\pi}{\times} M^2) \arrow[d, "s^*"] \\ H^n(N_\pi, N_\pi - B'(f)) \arrow[r, "j^*"] & H^n(N_\pi). \end{tikzcd}$$

Therefore $s^*(\theta_0') \neq 0$ implies $H^n(N_\pi, N_\pi - B'(f)) \neq 0$, which shows $B'(f) \neq \phi$. Thus $A'(f) \neq \phi$ and the proof completes.

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References

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