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AN APPLICATION OF FUNCTIONAL HIGHER OPERATION

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Introduction

Let ${}^1M^n$ be the Moore space $M(n, Z_p)$ (i.e., a simply connected space with two non-vanishing homology groups $H_0({}^1M^n; Z)=Z$ and $H_n({}^1M^n; Z)=Z_p$), where p is an *odd* prime. Let ${}^1\pi_i$ be the stable homotopy group $\lim\,[{}^1M^{n+i}; {}^1M^n]$, and ${}^1\pi_*=\sum_i {}^1\pi_i$. Then, there are non-trivial elements $\alpha\in{}^1\pi_{2p-2}$ and $\beta_1\in{}^1\pi_{2p(p-1)-1}$[9].

Let ${}^2M^n$ be the mapping cone of α (i.e., ${}^2M^n={}^1M^n\cup_\alpha T{}^1M^{n+2p-2}$ for sufficiently large n), and ${}^2\pi_i$ be the stable homotopy group $\lim\,[{}^2M^{n+i}; {}^2M^n]$, ${}^2\pi_*=\sum_i 2\pi_i$. Corresponding to $\beta_1\in{}^1\pi_{2p(p-1)-1}$, we can define a non-trivial element $\beta\in{}^2\pi_{2p^2-2}$.

Then, our main theorem is

Theorem. *$\alpha^t\neq 0$ in ${}^1\pi_*$ and $\beta^t\neq 0$ in ${}^2\pi_*$ for all $t\geqq 1$.*

This paper is divided into three chapters. In the first chapter, we deal with the functionalization of Adams-Maunder higher cohomology operations [1], [3], and study some relations among them; in chapter 2, suitable chain complexes are constructed by means of the Milnor basis of the mod p Steenrod algebra [4]. In the last chapter, the main theorem is proved in a slightly general form using the results in preceding chapters, especially Proposition 4. 3.

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CHAPTER 1. FUNCTIONAL OPERATIONS

1. Preliminaries

In this paper, spaces are arcwise connected, based and having the homotopy type of a CW-complex. Maps take base point to base point and homotopies leave base point fixed. Base points are denoted by $*$. Groups are finitely generated and abelian. The additive group of integers is denoted by Z, and the additive group of integers modulo an *odd* prime p by Z_p. The closed interval $[0, 1]$ is denoted by I, $f \simeq g$ denotes that two maps f and g are homotopic, and $X \equiv Y$ means that two spaces X and Y are homotopy equivalent. A map and its homotopy class are often denoted by the same letter.

Most of cohomology groups are that of modulo p, so, unless otherwise stated, we shall denote $H^*(X)$ instead of $H^*(X; Z_p)$. The set of homotopy classess of maps $X \to Y$ is denoted by $[X; Y]$. A homomorphism of a set of homotopy classes into such a set is a correspondence such that it maps the class of the constant map into such a class and if both sets admit a group structure it is an (ordinary) homomorphism.

The (reduced) suspension of a space X is denoted by SX and the space of loops of X by ΩX. The mapping cylinder Y_f of a map f: $X \to Y$ is the space obtained from $X \times I \cup Y$ by identifying $(x, 1)$ with $f(x)$, $x \in X$. The mapping cone C_f of f is obtained from Y_f by identifying $(x, 0)$ with the base point $*$ for $x \in X$, and denoted often by $Y \cup_f TX$. The mapping track L_f of f is the space of maps $\lambda : I \to Y_f$ such that $\lambda(0) = *$ and $\lambda(1) \in X$, with the CO-topology.

For a map $f: X \to Y$ the map $Sf: SX \to SY$ is defined by $Sf(x, t) = (f(x), t)$, $x \in X$, $t \in I$, and the map $\Omega f: \Omega X \to \Omega Y$ is defined by $\Omega f(\lambda)(t) = f(\lambda(t))$, $\lambda \in \Omega X$, $t \in I$. There are homomorphisms $S_* : [X; Y] \to [SX; SY]$ and $\Omega_* : [X; Y] \to [\Omega X; \Omega Y]$ defined by $S_*(f) = Sf$ and $\Omega_*(f) = \Omega f$, respectively.

There is a canonical isomorphism $[SX; Y] \to [X; \Omega Y]$. Since the Eilenberg-MacLane space $K(\pi, n)$ is the space of loops of $K(\pi, n+1)$, the suspension homomorphism $s^* : H^{n+1}(SX; \pi) \to H^n(X; \pi)$ is an isomorphism for any coefficient group π and any integer $n > 0$.

It is well-known that if X is an $(n-1)$-connected space, then S_*: $\pi_i(X) \to \pi_{i+1}(SX)$ and $\Omega_* : H^i(X; \pi) \to H^{i-1}(\Omega X; \pi)$ are isomorphisms for $i < 2n-1$.

Since $\Omega K(\pi, n) = K(\pi, n-1)$, for $n > 2$, we may regard $\Omega^{-1} K(\pi, n-1)$ as $K(\pi, n)$. Let $f: K(\pi, n) \to K(\pi', m)$ be a map where $m < 2n-2$, then there is a map $f' : K(\pi, n+1) \to K(\pi', m+1)$ such that $\Omega f' \simeq f$. Let F and F' be the mapping tracks of f and f' respectively, then we may regard $\Omega^{-1} F$ as F' because $\Omega F' \equiv F$. Similarly, let $g: S^m \to S^n$ be a map where $m < 2n-2$, then there is a map $g' : S^{m-1} \to S^{n-1}$ such that $Sg' \simeq g$, and let M and M' be the mapping cones of g and g', then we may regard $S^{-1} M$ as M' because $SM' \equiv M$.

If we are only concerned with stable (cohomology and homotopy) elements, or spaces obtained from $K(\pi, n)$-spaces or spheres by stable elements, or maps into or from such a space, we say that we are "*in the stable range*".

Let A^* be the mod p Steenrod algebra where p is an *odd* prime. A chain complex is a sequence

$$\cdots \to C_r \xrightarrow{d_r} C_{r-1} \to \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

of finitely generated graded free A^*-modules C_i such that the component $(C_i)_q$ of degree q of C_i is zero for $i>q$, with A^*-maps d_i of degree 0 such that $d_{i-1}d_i=0$ for $i\geqq 2$.

Let $K=\times_i K(Z_p, n_i)$ be a (finite) cartesian product of Eilenberg-MacLane spaces, and let $n_i>n$ for some positive integer n, then by Künneth theorem, we have $H^j(K)=\sum_i H^j(Z_p, n_i\,;\,Z_p)$ for $j<2n-2$, i.e. in the stable range. Let $u\in H^*(X)$ be an element such that $u=\sum_i u_i$, $u_i\in H^{n_i}(X)$, and there be a positive integer n such that $n<n_i<2n-2$, then there is a map $\varphi: X\to K=\times_i K(Z_p, n_i)$ such that $\sum_i \varphi^*(\iota_i)=u$, (i.e. $\varphi^*(\iota_i)=u_i$). We shall often denote u by φ^*. Thus, for given a homomorphism $\eta: H^*(K)\to H^*(X)$, in the stable range, there is a map $\varphi: X\to K$ such that $\varphi^*=\eta: H^*(K)\to H^*(X)$.

Finally, the following lemma is easily proved.

Lemma 1.1. *Let $f: X\to Y$, $g: U\to X$ be two maps such that $fg\simeq 0$. Then, there are maps $\bar{g}: U\to L_f$ and $f': L_g\to \Omega Y$ such that $i_f\bar{g}=g$, $f'\tau_g\simeq \Omega f$ and $\bar{g}i_g\simeq -\tau_f f'$, where $i_f: L_f\to X$, $i_g: L_g\to U$ are projections and τ_f: $\Omega Y\to L_f$, $\tau_g: \Omega X\to L_g$ are injections.*

2. Cohomology operations of higher kind

Following Adams [1] and Maunder [3], we shall define a pyramid of stable cohomology operations $\{\Phi^{s,t}\}$ associated with a certain chain complex

$$\cdots \to C_r \xrightarrow{d_r} C_{r-1} \to \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0\,. \tag{2.1}$$

We shall say that a chain complex is *r-admissible* if we can construct a realization up to the r-th stage, that is, a sequence of spaces and maps

$$\begin{array}{ccccccccccc} & & B_r & & & & B_3 & & B_2 & & B_1 \\ & & \uparrow f_r & & & & \uparrow f_3 & & \uparrow f_2 & & \uparrow f_1 \\ F_r & \xrightarrow{i_r} & F_{r-1} & \to & \cdots & \to & F_2 & \xrightarrow{i_2} & F_1 & \xrightarrow{i_1} & F_0 \end{array} .$$

such that $F_j \xrightarrow{i_j} F_{j-1} \xrightarrow{f_j} B_j$ and so $\Omega B_j \xrightarrow{\tau_j} F_j \xrightarrow{i_j} F_{j-1}$ are fiberings for $j=1, \cdots, r$, and there are isomorphisms $\alpha_0 : C_0 \to H^*(F_0)$ and $\alpha_j : C_j \to H^*(B_j)$, $j=1, \cdots, r$, such that $f_1^*\alpha_1=\alpha_0 d_1$ and $\tau_{j-1}^* f_j^* \alpha_j=\alpha_{j-1}d_j$ for $j=2, \cdots, r$.

For any chain complex, we can construct a fibering $F_1 \to F_0 \to B_1$ as follows: Let $c_{0,1}$ be the generators of C_0 of degree q_i, then put $F_0=\underset{i}{\times} K(Z_p, m+q_i)$ where $\times$ denotes the cartesian product and m is a sufficiently large integer, and let $\alpha_0 : (C_0)_q \to H^{m+q}(F_0)$ be the canonical isomorphism. Let $c_{1,j}$ be the generators of C_1 of degree q'_j, then put $B_1=\underset{j}{\times} K(Z_p, m+q'_j)$ and let $\alpha_1 : (C_1)_q \to H^{m+q}(B_1)$ be the canonical isomorphism. A map $f_1 : F_0 \to B_1$ is defined by $f_1^* = \alpha_0 d_1 \alpha_1^{-1}$. We may regard f_1 as a fiber map and let F_1 be its fiber $i_1 : F_1 \to F_0$ be the injection.

Thus any chain complex is 1-*admissible.*

Next, let $C_j=A^*[c_{j,k}]$ where $c_{j,k}$ is of degree q_k, $j \geqq 2$, then we define $B_j=\underset{k}{\times} K(Z_p, m+q_k-j+1)$ and $\alpha_j : (C_j)_q \to H^{m+q-j+1}(B_j)$ to be the canonical isomorphism. Then, we have

Proposition 2.1. *Let a chain complex* (2.1) *be* $(r-1)$-*admissible, and if we have* $f_{r-1}^*\alpha_{r-1}d_r=0$ *for* $f_{r-1}^* : H^*(B_{r-1}) \to H^*(F_{r-2})$. *Then, the chain complex* (2.1) *is r-admissible.*

Proof. Since we are concerned only with the elements in the stable range, we have the following exact sequence

$$\cdots \to [F_{r-1}\,;\, X] \xrightarrow{\tau_{r-1}^\#} [\Omega B_{r-1}\,;\, X] \xrightarrow{(\Omega f_{r-1})^\#} [\Omega F_{r-2}\,;\, X] \to \cdots.$$

Since B_r is a cartesian product of Eilenberg-MacLane spaces, there is a map $h : \Omega B_{r-1} \to B_r$ such that $\alpha_{r-1}^{-1} h^* \alpha_r = d_r$. By the assumption, we have $f_{r-1}^*\alpha_{r-1}d_r=0$, so we have $f_{r-1}^* h^* \alpha_r=0$ and hence $(\Omega f_{r-1})^\# h=0$. So that, there is a map $f_r : F_{r-1} \to B_r$ such that $\tau_{r-1}^\# f_r = h$. This implies that $\alpha_{r-1}^{-1} f_r^* \tau_{r-1}^* \alpha_r = d_r$. We may regard f_r as a fiber map, and let F_r be its fiber, $i_r : F_r \to F_{r-1}$ be the injection. q.e.d.

REMARK. Since $d_1 d_2=0$ implies that $f_1^*\alpha_1 d_2=0$, any chain complex is 2-*admissible.*

Let a chain complex (2.1) be r-admissible, and let

$$C_s \to C_{s-1} \to \cdots \to C_{t+1} \to C_t, \tag{2.2}$$

$0 \leqq t < s \leqq r$, be a part of (2.1). Then we can construct a realization of (2.2), that is, a sequence of spaces and maps

$$\begin{array}{ccccccccc} & & B_s & & & & B_{t+2} & & B_{t+1} \\ & & \uparrow f_{t,s} & & & & \uparrow f_{t,t+2} & & \uparrow f_{t,t+1} \\ G_{t,s} & \xrightarrow{i_{t,s}} & G_{t,s-1} & \to & \cdots & \to & G_{t,t+1} & \xrightarrow{i_{t,t+1}} & G_{t,t} \end{array}$$

such that $G_{t,j} \xrightarrow{i_{t,j}} G_{t,j-1} \xrightarrow{f_{t,j}} B_j$ and $\Omega B_j \xrightarrow{\tau_{t,j}} G_{t,j} \xrightarrow{i_{t,j}} G_{t,j-1}$ are fiberings for $j=t+1, \cdots, s$ where $G_{t,t}=\Omega B_t$ and there are maps $\Delta_{t,j}$: $G_{t,j} \rightarrow F_j$ for $j=t+1, \cdots, s$, satisfying that

$$f_{t,j}^{*}=\Delta_{t,j-1}^{*} f_j^{*},\ \Delta_{t,j}^{*} i_j^{*}=i_{t,j}^{*} \Delta_{t,j-1}^{*} \text{ and } \tau_j^{*}=\tau_{t,j}^{*} \Delta_{t,j}^{*}, \text{ where } \Delta_{t,t}=\tau_t.$$

In fact, put $G_{t,t}=\Omega B_t$, $f_{t,t+1}=f_{t+1}\tau_t$, and let $\Omega B_{t+1} \xrightarrow{\tau_{t,t+1}} G_{t,t+1} \xrightarrow{i_{t,t+1}} \Omega B_t$ be the fibering induced from $\Omega B_{t+1} \xrightarrow{\tau_{t+1}} F_{t+1} \xrightarrow{i_{t+1}} F_t$ by $\tau_t=\Delta_{t,t}$. Then there is a natural map $\Delta_{t,t+1}: G_{t,t+1} \rightarrow F_{t+1}$ such that $i_{t+1}\Delta_{t,t+1}=\Delta_{t,t} i_{t,t+1}$ and $\tau_{t+1}=\Delta_{t,t+1}\tau_{t,t+1}$, and $G_{t,t+1} \xrightarrow{i_{t,t+1}} G_{t,t} \xrightarrow{f_{t,t+1}} B_{t+1}$ is also a fibering.

Let, inductively, $\Omega B_j \xrightarrow{\tau_{t,j}} G_{t,j} \xrightarrow{i_{t,j}} G_{t,j-1}$, $j>t$, be the fibering induced from $\Omega B_j \xrightarrow{\tau_j} F_j \xrightarrow{i_j} F_{j-1}$ by a map $\Delta_{t,j-1}: G_{t,j-1} \rightarrow F_j$, then there is a natural map $\Delta_{t,j}: G_{t,j} \rightarrow F_j$ such that $i_j \Delta_{t,j}=\Delta_{t,j-1} i_{t,j}$ and $\tau_j=\Delta_{t,j}\tau_{t,j}$, and $G_{t,j} \xrightarrow{i_{t,j}} G_{t,j-1} \xrightarrow{f_{t,j}} B_j$ is also a fibering where $f_{t,j}=f_j \Delta_{t,j-1}$.

$$\begin{tikzcd}
 & B_s & & B_{t+2} & B_{t+1} & \\
F_s \arrow[r, "i_s"] & F_{s-1} \arrow[u, "f_s"'] \arrow[r] & \cdots \arrow[r] & F_{t+1} \arrow[u, "f_{t+2}"'] \arrow[r, "i_{t+1}"] & F_t \arrow[u, "f_{t+1}"'] \arrow[r] & \cdots \\
G_{t,s} \arrow[r, "i_{t,s}"] & G_{t,s-1} \arrow[u, "\Delta_{t,s-1}"'] \arrow[r] & \cdots \arrow[r] & G_{t,t+1} \arrow[u, "\Delta_{t,t+1}"'] \arrow[r, "i_{t,t+1}"] & G_{t,t} \arrow[u, "\Delta_{t,t}=\tau_t"'] &
\end{tikzcd}$$

Similarly, if $s>t$, there are maps $\Delta_{s,j}^{t}: G_{s,j} \rightarrow G_{t,j}$, for $j>s$, such that $\Delta_{s,j}=\Delta_{t,j}\Delta_{s,j}^{t}$ and

$$f_{s,j}^{*} = \Delta_{s,j-1}^{*} f_{t,j}^{*},\ \Delta_{s,j}^{t*} i_{t,j}^{*} = i_{s,j}^{*} \Delta_{s,j-1}^{t*},\ \tau_{t,j}^{*} = \tau_{s,j}^{*} \Delta_{s,j}^{t*}$$

for $j>s$, and the fibering $\Omega B_j \rightarrow G_{s,j} \rightarrow G_{s,j-1}$ is regarded as to be induced from $\Omega B_j \rightarrow G_{t,j} \rightarrow G_{t,j-1}$ by $\Delta_{s,j-1}^{t}$ where $\Delta_{s,s}^{t}=\tau_{t,s}$.

$$\begin{tikzcd}
 & B_r & & B_{s+2} & B_{s+1} & \\
F_r \arrow[r, "i_r"] & F_{r-1} \arrow[u, "f_r"'] \arrow[r] & \cdots \arrow[r] & F_{s+1} \arrow[u, "f_{s+2}"'] \arrow[r, "i_{s+1}"] & F_s \arrow[u, "f_{s+1}"'] \arrow[r] & \cdots \\
G_{t,r} \arrow[r, "i_{t,r}"] & G_{t,r-1} \arrow[u, "\Delta_{t,r-1}"'] \arrow[r] & \cdots \arrow[r] & G_{t,s+1} \arrow[u, "\Delta_{t,s+1}"'] \arrow[r, "i_{t,s+1}"] & G_{t,s} \arrow[u, "\Delta_{t,s}"'] \arrow[r] & \cdots \\
G_{s,r} \arrow[r, "i_{s,r}"] & G_{s,r-1} \arrow[u, "\Delta_{s,r-1}^{t}"'] \arrow[r] & \cdots \arrow[r] & G_{s,s+1} \arrow[u, "\Delta_{s,s+1}^{t}"'] \arrow[r, "i_{s,s+1}"] & G_{s,s} \arrow[u, "\Delta_{s,s}^{t}=\tau_{t,s}"'] &
\end{tikzcd}$$

For the simplicity, if it is necessary, we regard F_j (resp. τ_j, f_j, i_j, etc.) as $G_{0,j}$ (resp. $\tau_{0,j}$, $f_{0,j}$, $i_{0,j}$, etc.).

For given an element $u\in H^*(X)$ which is represented by a map $\varphi: X\to\Omega B_t$, (or $\varphi: X\to F_0$), we define

$$\Phi^{t+1,t}(u) = \varphi^* f^*_{t,t+1},$$

i.e., an element in $H^*(X)$ which is represented by $f_{t,t+1}\varphi$.

If $\Phi^{t+1,t}(u)=0$, there is a map $\varphi': X\to G_{t,t+2}$ such that $\varphi'^* i_{t,t+1}=\varphi^*$. We define

$$\Phi^{t+2,t}(u) = \{\varphi'^* f^*_{t,t+2}\},$$

for all such maps φ'.

Inductively, if $0\in\Phi^{s-1,t}(u)$, then there is a map $\varphi_0^{(s-t-2)}: X\to G_{t,s-2}$ such that

$$\varphi_0^{(s-t-2)*} i^*_{t,s-1}\cdots i^*_{t,t+1}=\varphi^* \text{ and } \varphi_0^{(s-t-2)*} f^*_{t,s-1}=0.$$

So that there is a map $\varphi^{(s-t-1)}: X\to G_{t,s-1}$ such that $\varphi^{(s-t-1)*} i^*_{t,s-1}=\varphi_0^{(s-t-2)*}$, and hence $\varphi^{(s-t-1)*} i^*_{t,s-1}\cdots i^*_{t,t+1}=\varphi^*$. We define

$$\Phi^{s,t}(u) = \{\varphi^{(s-t-1)*} f^*_{t,s}\},$$

for all such maps $\varphi^{(s-t-1)}$.

Then, we have

Proposition 2.2. (Cf. [3; Theorem 2.4.2]) *For given an r-admissible chain complex*

$$C_r \to C_{r-1} \to \cdots \to C_2 \to C_1 \to C_0,$$

there is a pyramid of stable cohomology operations $\{\Phi^{s,t}\}$, $r\geqq s>t\geqq 0$. *They satisfy that*

1) $\Phi^{s,t}$ *is defined for any element* $u\in H^*(X)$ *which is represented by a map* $\varphi: X\to\Omega B_t$, *provided that* $\Phi^{l,t}(u)\ni 0$ *for* $s>l>t$.

2) $\Phi^{s,t}(u)$ *is a coset of elements of* $H^*(X)$ *modulo Im* $\Phi^{s,t+1}$, *i.e., for any two elements* w, $w'\in\Phi^{s,t}(u)$, *there is an element* $v\in H^*(X)$ *which is represented by a map* φ: $X\to\Omega B_{t+1}$ *such that* $w-w'\in\Phi^{s,t+1}(v)$.

3) *For given a map* $g: Y\to X$ *and any element* $u\in H^*(X)$ *for which* $\Phi^{s,t}$ *is defined, we have* $g^*\Phi^{s,t}(u)\subset\Phi^{s,t}(g^*(u))$.

4) $s^*\Phi^{s,t}(u)=-\Phi^{s,t}(s^*(u))$ *for the suspension isomorphism* s^*: $H^*(SX)\to H^*(X)$, *if* $\Phi^{s,t}(u)$ *is defined.*

5) *Let* $\varepsilon: C_t\to H^*(X)$ *be an* A^**-map defined by* $\varepsilon=\varphi^*\alpha_t$ *for a map* $\varphi: X\to\Omega B_t$ *representing* $u\in H^*(X)$, *and* $\eta: C_s\to H^*(X)$ *be an* A^**-map defined by* $\eta=\psi^*\alpha_s$ *for a map* $\psi: X\to B_s$ *representing an element in* $\Phi^{s,t}(u)$. *If* ε *is of degree* m, *then* η *is of degree* $m-(s-t)+1$.

The proof is carried out similarly to that of [3; Theorem 2.4.2], so it is omitted.

A operation $\Phi^{s,t}$ is called an operation of the $(s-t)$-th kind. These operations $\Phi^{s,t}$ of the $(s-t)$-th kind are determined uniquely up to an operation of the $(s-t-1)$-th kind [3; Theorem 2.4.3].

Note that we have $\Phi^{s+1,s}\Phi^{s,t}(u)=0$ (mod zero) whenever $\Phi^{s,t}(u)$ is defined.

3. Functional operations of higher kind

In [5], stable functional cohomology operations were defined by the method of universal examples.

Now, we shall define stable functional cohomology operations of higher kind by making use of the above stable cohomology operations of higher kind.

Let

$$C_s \to C_{s-1} \to \cdots \to C_{t+1} \to C_t, \qquad r \geqq s > t \geqq 0,$$

be a part of an r-admissible chain complex with a realization

$$\begin{tikzcd} B_s & & & B_{t+2} & B_{t+1} \\ G_{t,s} \arrow[r] & G_{t,s-1} \arrow[u] \arrow[r] & \cdots \arrow[r] & G_{t,t+1} \arrow[u] \arrow[r] & G_{t,t} \arrow[u] \end{tikzcd}$$

Let $g: Y\to X$ be a map and $u\in H^*(X)$ be an element such that $\Phi^{t+1,t}(u)=0$ and $g^*(u)=0$. Then there is a map $\varphi: X\to \Omega B_t$ representing u and satisfying that $\varphi^* f^*_{t,t+1}=0$ and $g^*\varphi^*=0$. Hence we have a map φ': $X\to G_{t,t+1}$ such that $\varphi'^* i^*_{t,t+1}=\varphi^*$. But, since $g^*\varphi'^* i^*_{t,t+1}=g^*\varphi^*=0$, there is a map $\psi': Y\to \Omega B_{t+1}$ such that $g^*\varphi'^*=\psi'^*\tau^*_{t,t+1}$.

$$\begin{tikzcd} & & Y \arrow[r, "g"] \arrow[lld, "\psi'"'] & X \arrow[ld, "\varphi'"'] \arrow[d, "\varphi"] & \\ \Omega B_{t+1} \arrow[rr, "\tau_{t,t+1}"'] & & G_{t,t+1} \arrow[r, "i_{t,t+1}"'] & G_{t,t} \arrow[r, "{t,t+1}"'] & B_{t+1} \end{tikzcd}$$

We define

$$\Phi_g^{t+1,t}(u) = \{\psi'^*\},$$

for all such maps ψ'.

If u satisfies that $\Phi^{t+2,t}(u)\ni 0$ and $\Phi_g^{t+1,t}(u)\ni 0$, then for some maps φ_0', $\varphi_1': X\to G_{t,t+1}$, satisfying $\varphi_0'^* i^*_{t,t+1}=\varphi_1'^* i^*_{t,t+1}=\varphi^*$, we have $\varphi_0'^* f^*_{t,t+2}=0$ and $g^*\varphi_1'^*=0$. If there is a map $\varphi_0': X\to G_{t,t+1}$ such that

$$\varphi_0'^* i^*_{t,t+1} = \varphi^*, \ \varphi_0'^* f^*_{t,t+2} = 0 \text{ and } g^*\varphi_0'^* = 0. \tag{3.1}$$

Then there is a map $\psi'': Y\to \Omega B_{t+2}$ such that $\psi''^*\tau^*_{t,t+2}=g^*\varphi''^*$ for a map $\varphi'': X\to G_{t,t+2}$ satisfying $\varphi''^* i^*_{t,t+2}=\varphi_0'^*$. We define

$$\Phi_g^{t+2,t}(u) = \{\psi''^*\}$$

for all such maps ψ''.

If, inductively, $\Phi^{s,t}(u)$ and $\Phi_g^{s-1,t}(u)$ are defined, and $0\in\Phi^{s,t}(u)$, $0\in\Phi_g^{s-1,t}(u)$, and moreover there is a map $\varphi_0^{(s-t-1)}: X\to G_{t,s-1}$ such that

$$\begin{aligned}&\varphi_0^{(s-t-1)*}i_{t,s-1}^*\cdots i_{t,t+1}^* = \varphi^*,\\ &\varphi_0^{(s-t-1)*}f_{t,s}^* = 0 \text{ and } g^*\varphi_0^{(s-t-1)*} = 0\,.\end{aligned} \tag{3.1$'$}$$

Then we can find a map $\psi^{(s-t)}: Y\to\Omega B_s$ such that $\psi^{(s-t)*}\tau_{t,s}^* = g^*\varphi^{(s-t)*}$ where $\varphi^{(s-t)}: X\to G_{t,s}$ is a map satisfying that $\varphi^{(s-t)*}i_{t,s}^* = \varphi_0^{(s-t-1)*}$.

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[d, "\psi^{(s-t)}"'] & X \arrow[d, "\varphi^{(s-t)}"] \arrow[dr, "\varphi^{(s-t-1)}"] & & \\
\Omega G_{t,s-1} \arrow[r, "\Omega f_{t,s}"'] & \Omega B_s \arrow[r, "\tau_{t,s}"'] & G_{t,s} \arrow[r, "i_{t,s}"'] & G_{t,s-1} \arrow[r, "f_{t,s}"'] & B_s
\end{tikzcd}$$

We define

$$\Phi_g^{s,t}(u) = \{\psi^{(s-t)*}\}$$

for all such maps $\psi^{(s-t)}$.

Then, easily we have

Proposition 3.1. 1) *$\Phi_g^{s,t}(u)$ is defined for any element $u\in H^*(X)$ which is represented by a map $\varphi: X\to\Omega B_t$ provided that $\Phi^{s,t}(u)$ and $\Phi_g^{s-1,t}(u)$ are defined and contain* 0, *and there is a map $\varphi_0^{(s-t-1)}: X\to G_{t,s-1}$ satisfaing* (3.1)$'$.

2) *$\Phi_g^{s,t}(u)$ is a coset of elements of $H^*(Y)$ modulo $g^*H^*(X)+\Phi^{s,t}H^*(Y)$ (or more precisely, $g^*[X;\ \Omega B_s]+(\Omega f_{t,s})_*[Y;\ \Omega G_{t,s-1}]$).*

3) *Let $\varepsilon: C_t\to H^*(X)$ be an A^*-map defined by $\varepsilon=\varphi^*\alpha_t$, and $\eta: C_s\to H^*(Y)$ an A^*-map defined by $\eta=\psi^*\alpha_s$ for a map $\psi: Y\to\Omega B_s$ representing an element in $\Phi_g^{s,t}(u)$. If ε is of degree m, then η is of degree $m-(s-t)$.*

REMARK. If $\Phi^{s,t}(u)\ni 0$, $\Phi_g^{s-1,t}(u)\ni 0$ and at least one of them is reduced to zero (*mod zero*), there is a map $\varphi_0^{(s-t-1)}$ satisfying (3.1)$'$.

By definition, if $g\simeq h$ then we have $\Phi_g^{s,t}(u)=\Phi_h^{s,t}(u)$ whenever one of them is defined, and if $g\simeq 0$ then for any operation $\Phi^{s,t}$, $\Phi_g^{s,t}(u)$ is defined and $\Phi_g^{s,t}(u)=0$ (*mod zero*) provided that $\Phi^{s,t}(u)$ is defined and $\Phi^{s,t}(u)\ni 0$.

Let $h: U\to Y$ be a map, and θ be an operation of the first kind, then it is easily verified that

Proposition 3.2. (i) *$h^*\Phi_g^{s,t}(u)\subset\Phi_{gh}^{s,t}(u)$ if $\Phi_g^{s,t}(u)$ is defined.*

(ii) *$\Phi_h^{s,t}(g^*(u))\supset\Phi_{gh}^{s,t}(u)$ if $\Phi_{gh}^{s,t}(u)$ is defined.*

Proposition 3. 3. (i) *$\theta(\Phi_g^{s,t}(u))\subset(\theta\Phi^{s,t})_g(u)$ if $\Phi_g^{s,t}(u)$ is defined.*
(ii) *$\theta_g(\Phi^{s,t}(u))\supset(\theta\Phi^{s,t})_g(u)$ if $(\theta\Phi^{s,t})_g(u)$ is defined.*

4. Some relations among functional operations

Peterson and Stein [5] proved two formulas in connection with relations of stable functional operations of the first kind.

We shall begin with to give a generalizations of these formulas.

Proposition 4. 1. *$\Phi^{s+1,s}\Phi_g^{s,t}(u)\equiv g^*\Phi^{s+1,t}(u)$ modulo $\mathrm{Im}\, g^*\Phi^{s+1,t+1}$ (i.e. $g^*f_{t+1,s+1*}[X;\, G_{t+1,s}]$), whenever $\Phi_g^{s,t}(u)$ is defined.*

Proof. Let $\varphi: X\to\Omega B_t$ be a map representing $u\in H^*(X)$ for which $\Phi_g^{s,t}(u)$ is defined, then there are maps $\varphi^{(s-t)}: X\to G_{t,s}$ and $\psi^{(s-t)}: Y\to\Omega B_s$ such that $\varphi^{(s-t)*}i_{t,s}^*\cdots i_{t,t+1}^*=\varphi^*$ and $\psi^{(s-t)*}\tau_{t,s}^*=g^*\varphi^{(s-t)*}$. By definition, $\Phi_g^{s,t}(u)$ is the set of elements $\psi^{(s-t)*}$ for all such maps $\psi^{(s-t)}$, so that $\Phi^{s+1,s}\Phi_g^{s,t}(u)$ is the set of elements $\psi^{(s-t)*}f_{s,s+1}^*=\psi^{(s-t)*}\tau_{t,s}^*f_{t,s+1}^*=g^*\varphi^{(s-t)*}f_{t,s+1}^*$.

$$\begin{tikzcd}
 & Y \arrow[rr, "g"] \arrow[d, "\psi^{(s-t)}"] & & X \arrow[d, "\varphi^{(s-t)}"] & \\
\Omega G_{t,s-1} \arrow[r, "\Omega f_{t,s}"] & \Omega B_s \arrow[r, "\tau_{t+1,s}"] & G_{t+1,s} \arrow[r, "\Delta_{t+1,s}^t"] & G_{t,s} \arrow[r, "f_{t,s+1}"] & B_{s+1}
\end{tikzcd}$$

On the other hand, since $\Phi_g^{s,t}(u)$ is defined, we have $\Phi^{s,t}(u)\ni 0$, hence $\Phi^{s+1,t}(u)$ is defined and is the set of elements $\varphi^{(s-t)*}f_{t,s+1}^*$. So that, $\Phi^{s+1,s}\Phi_g^{s,t}(u)$ and $g^*\Phi^{s+1,t}(u)$ have a common element.

But, we have

$$\begin{aligned}
&\textit{the indeterminacy of } \Phi^{s+1,s}\Phi_g^{s,t}(u)\\
&\quad = f_{s,s+1*}(g^*[X;\,\Omega B_s]+(\Omega f_{t,s})_*[Y;\,\Omega G_{t,s-1}])\\
&\quad = f_{s,s+1*}g^*[X;\,\Omega B_s]\\
&\quad \subset f_{t+1,s+1*}g^*[X;\,G_{t+1,s}]\\
&\quad = \textit{the indeterminacy of } g^*\Phi^{s+1,t}(u).
\end{aligned}$$

q.e.d.

Proposition 4. 2. *$\Phi_g^{s+1,s}\Phi^{s,t}(u)\equiv-\Phi^{s+1,t}(g^*(u))$ modulo $\mathrm{Im}\, g^*+\mathrm{Im}\, \Phi^{s+1,t+1}$ (i.e. $g^*[X;\,\Omega B_{s+1}]+(\Omega f_{t+1,s+1})_*[Y;\,G_{t+1,s}]$), provided that $\Phi^{s,t}(u)$ is defined and $g^*\Phi^{s,t}(u)\ni 0$.*

Proof. Let $\varphi: X\to\Omega B_t$ be a map representing u. Then, there is a map $\varphi^{(s-t-1)}: X\to G_{t,s-1}$ such that $\varphi^{(s-t-1)*}i_{t,s-1}^*\cdots i_{t,t+1}^*=\varphi^*$. Since $g^*\Phi^{s,t}(u)\ni 0$, for some $\varphi^{(s-t-1)}$, we have $g^*\varphi^{(s-t-1)*}f_{t,s}^*=0$, so there is a map $\chi: Y\to G_{t,s}$ satisfying that $\chi^*i_{t,s}^*=g^*\varphi^{(s-t-1)*}$.

Since $\Omega^{-1}f_{s,s+1}\cdot f_{t,s}\simeq 0$, there are maps $\bar f_{t,s}: G_{t,s-1}\to\Omega^{-1}G_{s,s+1}$ and

$f'_{s,s+1}: G_{t,s} \to B_{s+1}$, by Lemma 1. 1, such that

$$\text{(4. 1)} \qquad \begin{aligned} &\Omega^{-1} i_{s,s+1} \cdot \bar{f}_{t,s} \simeq f_{t,s}, \quad \bar{f}_{t,s} \cdot i_{t,s} \simeq -\Omega^{-1}\tau_{s,s+1} \cdot f'_{s,s+1}, \\ &f'_{s,s+1} \cdot \tau_{t,s} \simeq f_{s,s+1}, \end{aligned}$$

so that the equality $\tau^*_{t,s} f'^*_{s,s+1} = f^*_{s,s+1}$ implies that $f'^*_{s,s+1} = f^*_{t,s+1} + i^*_{t,s}\lambda^*$ for some map $\lambda : G_{t,s-1} \to B_{s+1}$.

Hence, there is a map $\rho : Y \to B_{s+1}$ such that $\rho^*(\Omega^{-1}\tau_{s,s+1})^* = g^*\varphi^{(s-t-1)*}\bar{f}^*_{t,s}$ because $g^*\varphi^{(s-t-1)*}\bar{f}^*_{t,s}(\Omega^{-1}i_{s,s+1})^* = 0$, and we have $\Phi^{s+1,s}_g \Phi^{s,t}(u) = \{\rho^*\}$ for all such maps ρ.

$$\begin{tikzcd}
& Y \arrow[r, "g"] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\chi"] \arrow[dd, bend right, "\rho"'] & X \arrow[d, "\varphi^{(s-t-1)}"] & & \\
\Omega B_s \arrow[r, "\tau_{t,s}"] \arrow[d, "\Delta^{t+1}_{s,s}"'] & G_{t,s} \arrow[r, "i_{t,s}"] \arrow[d, "f_{t,s+1}"] & G_{t,s-1} \arrow[r, "f_{t,s}"] \arrow[d, "\bar{f}_{t,s}"] & B_s \arrow[r, "\Omega^{-1}f_{s,s+1}"] & \Omega^{-1}B_{s+1} \\
G_{t+1,s} \arrow[r, "f_{t+1,s+1}"'] \arrow[ur, "\Delta^{t}_{t+1,s}"] & B_{s+1} \arrow[r, "\Omega^{-1}\tau_{s,s+1}"'] & \Omega^{-1}G_{s,s+1} \arrow[ur, "\Omega^{-1}i_{s,s+1}"'] & &
\end{tikzcd}$$

On the other hand, since $0 \in \Phi^{s,t}(g^*(u))$ (because $g^*\Phi^{s,t}(u) \subset \Phi^{s,t}(g^*(u))$), there is a map $\sigma : Y \to G_{t,s}$ such that $\sigma^* i^*_{t,s} \cdots i^*_{t,t+1} = g^*\varphi^*$, and $\Phi^{s+1,t}(g^*(u))$ is the set of elements $\sigma^* f^*_{t,s+1}$ for all such σ.

But, by its definition, we have $\chi^* = \sigma^* + \mu^*\Delta^{t*}_{t+1,s+1}$ for some map $\mu : Y \to G_{t+1,s}$.

Since

$$\begin{aligned} \rho^*(\Omega^{-1}\tau_{s,s+1})^* &= g^*\varphi^{(s-t-1)*}\bar{f}^*_{t,s} = \chi^* i^*_{t,s}\bar{f}^*_{t,s} \\ &= -\chi^* f'^*_{s,s+1}(\Omega^{-1}\tau_{s,s+1})^*, \qquad \text{by (4. 1)}, \end{aligned}$$

we have $\rho^* + \chi^* f'^*_{s,s+1} = \nu^* f^*_{s,s+1}$ for some map $\nu : Y \to \Omega B_s$.

Thus, we conclude that

$$\begin{aligned} \rho^* + \sigma^* f^*_{t,s+1} &= \nu^* f^*_{s,s+1} - \mu^* f^*_{t+1,s+1} - g^*\varphi^{(s-t-1)*}\lambda^* \\ &\in g^*[X;\ B_{s+1}] + f_{t+1,s+1*}[Y;\ G_{t+1,s}]. \qquad \text{q.e.d.} \end{aligned}$$

The following proposition is useful in the later arguments.

Proposition 4. 3. *Let $h : U \to Y$ be a map, then we have*

$$\Phi^{s+1,s}_h \Phi^{s,t}_g(u) \equiv \Phi^{s+1,t}_{gh}(u) \qquad \textit{modulo } \mathrm{Im}\, h^* + \mathrm{Im}\, \Phi^{s+1,t}$$

(*i.e.*, $h^*[Y;\ \Omega B_{s+1}] + (\Omega f_{t,s+1})_*[U;\ \Omega G_{t,s}]$), *provided that $\Phi^{s,t}_g(u)$ is defined $h^*\Phi^{s,t}_g(u) = 0$ (mod zero) and $\Phi^{s+1,t}(u) \ni 0$.*

Proof. Let $\varphi : X \to \Omega B_t$ be a map representing u. Then, we have a map $\varphi^{(s-t)} : X \to G_{t,s}$ such that $g^*\varphi^{(s-t)*} i^*_{t,s} = 0$, and $\varphi^{(s-t)*} f^*_{t,s+1} = 0$. We have, therefore, a map $\psi^{(s-t)} : Y \to \Omega B_s$ such that

(4. 2) $$\psi^{(s-t)*}\tau^*_{t,s} = g^*\varphi^{(s-t)*} \text{ and } h^*\psi^{(s-t)*} = 0\,,$$

and hence we have

$$\psi^{(s-t)*}f^*_{s,s+1} = \psi^{(s-t)*}\tau^*_{t,s}f^*_{t,s+1} = g^*\varphi^{(s-t)*}f^*_{t,s+1} = 0\,.$$

So that there is a map $\chi : Y \to G_{s,s+1}$ such that

(4. 3) $$\chi^* i^*_{s,s+1} = \psi^{(s-t)*}\,,$$

and hence

$$h^*\chi^* i^*_{s,s+1} = h^*\psi^{(s-t)*} = 0\,. \qquad \text{by (4. 2)}$$

This implies that we have a map $\rho : U \to \Omega B_{s+1}$ such that

(4. 4) $$\rho^*\tau^*_{s,s+1} = h^*\chi^*\,.$$

By definition, $\Phi^{s+1,s}_h\Phi^{s,t}_g(u) = \{\rho^*\}$ for all such maps ρ.

On the other hand, since $\varphi^{(s-t)*}f^*_{t,s+1}=0$, there is a map $\varphi^{(s-t+1)}$: $X \to G_{t,s+1}$ such that

(4. 5) $$\varphi^{(s-t+1)*}i^*_{t,s+1} = \varphi^{(s-t)*}\,.$$

$$\begin{tikzcd}
 & U \arrow[rr, "h"] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\rho"] & & Y \arrow[rr, "g"] \arrow[dl, "\chi"'] \arrow[dd, "\psi^{(s-t)}"] & & X \arrow[dl, "\varphi^{(s-t+1)}"'] \arrow[dd, "\varphi^{(s-t)}"] & \\
\Omega G_{t,s} \arrow[r, "\Omega f_{t,s+1}"'] & \Omega B_{s+1} \arrow[r, "\tau_{s,s+1}"'] & G_{s,s+1} \arrow[rr, "\Delta^t_{s,s+1}"] \arrow[dr, "i_{s,s+1}"'] & & G_{t,s+1} \arrow[dr, "i_{t,s+1}"'] & & G_{t,s-1} \\
 & & & \Omega B_s \arrow[rr, "\tau_{t,s}"'] \arrow[ur, phantom, "(1)"] & & G_{t,s} \arrow[ur, "i_{t,s}"'] \arrow[r, "f_{t,s+1}"'] & B_{s+1}
\end{tikzcd}$$

Then, we have

$$h^*g^*\varphi^{(s-t+1)*}i^*_{t,s+1} \underset{(4.5)}{=} h^*g^*\varphi^{(s-t)*} \underset{(4.2)}{=} 0\,,$$

so that there is a map $\sigma : U \to \Omega B_{s+1}$ such that

(4. 7) $$\sigma^*\tau^*_{t,s+1} = h^*g^*\varphi^{(s-t+1)*}\,.$$

By definition, $\Phi^{s+1,t}_{gh}(u) = \{\sigma^*\}$ for all such maps σ.

But, since

$$\begin{aligned}
\chi^*\Delta^{t*}_{s,s+1}i^*_{t,s+1} &\underset{(1)}{=} \chi^* i^*_{s,s+1}\tau^*_{t,s} \underset{(4.3)}{=} \psi^{(s-t)*}\tau^*_{t,s} \\
&\underset{(4.2)}{=} g^*\varphi^{(s-t)*} \underset{(4.5)}{=} g^*\varphi^{(s-t+1)*}i^*_{t,s+1}\,,
\end{aligned}$$

we have $\chi^*\Delta^{t*}_{s,s+1} = g^*\varphi^{(s-t+1)*} + \lambda^*\tau^*_{t,s+1}$ for some map $\lambda : Y \to \Omega B_{s+1}$. Hence,

$$\begin{aligned}\rho^*\tau^*_{t,s+1} = \rho^*\tau^*_{s,s+1}\Delta^{t*}_{s,s+1} &\underset{(4.4)}{=} h^*\chi^*\Delta^{t*}_{s,s+1} \\ &= h^*g^*\varphi^{(s-t+1)*} + h^*\lambda^*\tau^*_{t,s+1} \\ &\underset{(4.6)}{=} \sigma^*\tau^*_{t,s+1} + h^*\lambda^*\tau^*_{t,s+1}\,.\end{aligned}$$

This implies that

$$\rho^* - \sigma^* - h^*\lambda^* = \mu^*(\Omega f_{t,s+1})^*$$

for some map $\mu : U \to \Omega G_{t,s}$.

Thus, we have

$$\begin{aligned}\rho^* - \sigma^* &= h^*\lambda^* + \mu^*(\Omega f_{t,s+1})^* \\ &\in h^*[Y;\ \Omega B_{s+1}] + (\Omega f_{t,s+1})_*[U;\ \Omega G_{t,s}]\,.\end{aligned}$$ q.e.d.

CHAPTER 2. CONSTRUCTION OF CHAIN COMPLEXES

5. The Steenrod algebra

Recall that p is an *odd prime*.

It is well-known [2] that the mod p Steenrod algebra A^* has a multiplicative basis $\Delta \in A^1$, $\mathcal{P}^{p^k} \in A^{2p^k(p-1)}$, $k=0,1,2,\cdots$, and they satisfy the Adem's relations.

On the other hand, Milnor [4] determined another basis, so called Milnor basis, as follows:

Theorem 5.1. [4; Theorem 4. a] *The elements $Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots\mathcal{P}^R$ form an additive basis for A^*, where $\varepsilon_0, \varepsilon_1, \cdots$ are zero or one, almost all zero, and $R=(r_1, r_2, \cdots)$ is an infiinite sequence of non-negative integers almost all zero.*

The Milnor basis Q_k and $\mathcal{P}^R$ satisfy the following relations:

$$\begin{aligned}&Q_jQ_k + Q_kQ_j = 0, \\ (5.1)\qquad &\mathcal{P}^RQ_k - Q_k\mathcal{P}^R = \sum_{j\geqq 1} Q_{k+j}\mathcal{P}^{R-S_j(p^k)}, \\ &\mathcal{P}^R\mathcal{P}^S = \sum_X b(X)\mathcal{P}^{T(X)},\end{aligned}$$

where $S_j(s)$ is the sequence consisting of zeros except for one positive integer s in the j-th place, and if $R=(r_1, r_2, \cdots)$ and $S=(s_1, s_2, \cdots)$, $R-S=(r_1-s_1, r_2-s_2, \cdots)$ if $r_i - s_i \geqq 0$ for all i, and $\mathcal{P}^{R-S}=0$ if at least one of $r_i-s_i<0$, and $T(X)=(t_1(X), t_2(X), \cdots)$, where $t_n(X)=\sum_{i+j=n} x_{ij}$ for a matrix $X=(x_{ij})$ consisting of non-negative integers x_{ij}, $i, j=0,1,2,\cdots$(x_{00} is omitted), almost all zero, such that

$$(5.2)\qquad \sum_{j\geqq 0} p^j x_{ij} = r_i,\ i = 1,2,\cdots;\ \sum_{i\geqq 0} x_{ij} = s_j,\ j = 1,2,\cdots,$$

and $b(X)=(\prod_n t_n(X)!)/(\prod_{i,j} x_{ij}!)$, and the sum extends over all matrices X satisfying (5.2). (See [4; Theorem 4b])

It is directly verified, by (5.1), that the elements $\mathcal{P}^R Q_0^{\varepsilon_0} Q_0^{\varepsilon_1}\cdots$ also form an additive basis for A^*.

Milnor also gave some relations between the Adem's basis and his basis:

$$(5.3) \qquad \begin{aligned} &Q_0 = \Delta,\ Q_{k+1} = [\mathcal{P}^{p^k}, Q_k], \\ &\mathcal{P}^{S_1(s)} = \mathcal{P}^s,\ \mathcal{P}^{S_k(1)} = [\mathcal{P}^{p^k}, \mathcal{P}^{S_{k-1}(1)}], \end{aligned}$$

where $[a, b]=ab-(-1)^{\deg a\cdot\deg b}ba$.

$\mathcal{P}^R$ is denoted simply by $\mathcal{P}^{r_1,r_2,\cdots r_m}$ if $R=(r_1, r_2, \cdots, r_m, 0, 0, \cdots)$.

For the simplicity, we shall often denote $Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots Q_n^{\varepsilon_n}\mathcal{P}^{r_1,r_2,\cdots r_m}$ by $Q(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n)\mathcal{P}(r_1, r_2, \cdots, r_m)$, and the sequence consisting of zeros of number k by O^k (i.e., $O^k=(\overbrace{0, \cdots, 0}^{k})$).

Since the degree of Q_k is $2p^k-1$ and that of $\mathcal{P}(r_1, \cdots, r_m)$ is $r_1(2p-2)+\cdots+r_m(2p^m-2)$, the degree $d(\alpha)$ of a monomial $\alpha=Q(\varepsilon_0, \cdots, \varepsilon_n)\cdot\mathcal{P}(r_1, \cdots, r_m)$ is

$$\begin{aligned} d(\alpha) &= \varepsilon_0+\varepsilon_1(2p-1)+\cdots+\varepsilon_n(2p^n-1)+r_1(2p-2)+\cdots+r_m(2p^m-2) \\ &= \varepsilon_0+\cdots+\varepsilon_n+2(p-1)[(\varepsilon_1+r_1)+\cdots+(\varepsilon_l+r_l)(p^{l-1}+\cdots+1)] \end{aligned}$$

where $l=\max(m, n)$ and $\varepsilon_i=0$ for $i>n$, $r_j=0$ for $j>m$.

We define the height $h(\alpha)$ of a monomial $\alpha=Q(\varepsilon_0, \cdots, \varepsilon_n)\mathcal{P}^R$ to be

$$h(\alpha) = \varepsilon_0+\cdots+\varepsilon_n.$$

Then, since p is odd, we have

Lemma 5.2. *$d(\alpha)\equiv h(\alpha)$ (mod 4) for any monomial $\alpha\in A^*$.*

For $i\geqq 0$, let $M_i=A^*Q_0+\cdots+A^*Q_i$ and $M_i'=Q_0A^*+\cdots+Q_iA^*$, then $M_i\subset M_{i+1}$ and $M_i'\subset M_{i+1}'$. Let $M_\infty=\bigcup_i M_i$ and $M_\infty'=\bigcup_i M_i'$, then M_∞ and M_∞' are submodules of A^* generated, respectively, by the elements $Q(\varepsilon_0, \varepsilon_1, \cdots)\mathcal{P}^R$ and $\mathcal{P}^R Q(\varepsilon_0, \varepsilon_1, \cdots)$ such that at least one of $\varepsilon_j\neq 0$. They are subalgebras (actually ideals) of A^*, and, by (5.1), $M_\infty=M_\infty'$.

For $i\geqq 0$, let L_i and L_i' be submodules of A^* generated by the elements $Q(O^{i-1}, \varepsilon_i, \varepsilon_{i+1}, \cdots)\mathcal{P}^R$ and $\mathcal{P}^R Q(O^{i-1}, \varepsilon_i, \varepsilon_{i+1}, \cdots)$ (i.e. $\varepsilon_0=\cdots=\varepsilon_{i-1}=0$). Then, $L_0=L_0'=A^*$, $L_i\supset L_{i+1}$, $L_i'\supset L_{i+1}'$, and L_i, L_i' are subalgebras of A^*. It follows from (5.1) that $L_i=L_i'$. Let $L_\infty=\bigcap_i L_i$ and $L_\infty'=\bigcap_i L_i'$, then they are submodules of A^* generated by the elements $\mathcal{P}^R$ (i.e. $\varepsilon_0=\varepsilon_1=\cdots=0$), and hence $L_\infty=L_\infty'$.

Lemma 5.3. $A^*=M_i\oplus L_{i+1}=M_\infty\oplus L_\infty$.

Proof. It is easily seen that $A^{*}=M_{i}+L'_{i+1}=M_{i}+L_{i+1}$ and $A^{*}=M_{\infty}+L_{\infty}$. But, by definition, $M_{i}\cap L_{i+1}=M_{i}\cap L'_{i+1}=\{0\}$ and $M_{\infty}\cap L_{\infty}=\{0\}$.

Lemma 5. 4. $L_{i}=L_{i+1}Q_{i}\oplus L_{i+1}$.

Proof. Since $L'_{i}=L_{i}$ and $L'_{i+1}=L_{i+1}$, this follows from $L'_{i}=L'_{i+1}Q_{i}\oplus L'_{i+1}$ which is easily verified. q.e.d.

Lemma 5. 5. $A^{*}=(\sum_{k=0}^{i}L_{k+1}Q_{k})\oplus L_{i+1}=(\sum_{k\geqq 0}L_{k+1}Q_{k})\oplus L_{\infty}$, *where* $\sum$ *denotes a direct sum.*

Proof. Since $A^{*}=L_{0}$ and, by Lemma 5. 4, $L_{i}=L_{i+1}Q_{i}\oplus L_{i+1}$, we have the first decomposition. Since $L_{i}\supset L_{i+1}$, $\lim L_{i}=\bigcap_{i}L_{i}=L_{\infty}$. So that the second decomposition is obtained. q.e.d.

Let $\eta_{i}: L_{i+1}\rightarrow A^{*}/M_{i}$ (resp. $\eta_{\infty}: L_{\infty}\rightarrow A^{*}/M_{\infty}$) be a homomorphism defined by the composition of the injection $L_{i+1}\rightarrow A^{*}$ (resp. $L_{\infty}\rightarrow A^{*}$) and the projection $A^{*}\rightarrow A^{*}/M_{i}$ (resp. $A^{*}\rightarrow A^{*}/M_{\infty}$). Then, as a direct consequence of Lemma 5. 3, we have

Lemma 5. 6. *η_{i} (resp. η_{∞}) is an L_{i+1}-(resp. L_{∞}-) isomorphism.*

Let $\tilde{L}_{i}$ and $\tilde{L}'_{i}$ be submodules of A^{*} generated by the elements $Q(O^{i-1}, \varepsilon_{i}, \varepsilon_{i+1}, \cdots)\mathcal{P}^{R}$ and $\mathcal{P}^{R}Q(O^{i-1}\varepsilon_{i}, \varepsilon_{i+1}, \cdots)$ with at least one nonzero ε_{j}, respectively (i.e., $\tilde{L}_{i}=L_{i}\cap M_{\infty}$, $\tilde{L}'_{i}=L'_{i}\cap M_{\infty}$). Then, $\tilde{L}_{0}=\tilde{L}'_{0}=M_{\infty}$, $\tilde{L}_{i}=\tilde{L}'_{i}$, and $\tilde{L}_{i}$ is a subalgebra of M_{∞}.

Similarly to the above Lemmas, we have

Lemma 5. 3′. $M_{\infty}=M_{i}\oplus\tilde{L}_{i+1}$.

Lemma 5. 4′. $\tilde{L}_{i}=L_{i+1}Q_{i}\oplus\tilde{L}_{i+1}$.

Lemma 5. 5′. $M_{\infty}=(\sum_{k=0}^{i}L_{k+1}Q_{k})\oplus\tilde{L}_{i+1}=\sum_{k\geqq 0}L_{k+1}Q_{k}$. (direct sum)

Lemma 5. 6′. *The homomorphism $\tilde{\eta}_{i}:\tilde{L}_{i+1}\rightarrow M_{\infty}/M_{i}$ defined by the injection $\tilde{L}_{i+1}\rightarrow M_{\infty}$ and the projection $M_{\infty}\rightarrow M_{\infty}/M_{i}$ is an $\tilde{L}_{i+1}$-isomorphism.*

Let $M_{\infty}^{k}=M_{\infty}\cdot M_{\infty}^{k-1}$ and $\tilde{L}_{i}^{k}=\tilde{L}_{i}\cdot\tilde{L}_{i}^{k-1}$, $k\geqq 2$, then $M_{\infty}^{k}\subset M_{\infty}^{k-1}$, $\tilde{L}_{i}^{k}\subset\tilde{L}_{i}^{k-1}$, $\tilde{L}_{i}^{k}\subset M_{\infty}^{k}$, and we have

Lemma 5. 3″. $M_{\infty}^{k}=\tilde{L}_{i+1}^{k}\oplus(M_{\infty}^{k}\cap M_{i})$.

Lemma 5. 6″. *$\tilde{\eta}_{i,k}:\tilde{L}_{i+1}^{k}\rightarrow M_{\infty}^{k}/(M_{\infty}^{k}\cap M_{i})$ is an $\tilde{L}_{i+1}$-isomorphism.*

The following Lemmas are useful for later arguments.

Lemma 5. 7. *Let $\alpha\in M_{\infty}/M_{i}$ be an element such that $Q_{i}\alpha\equiv 0$ (mod M_{i}), then there is an element $\beta\in A^{*}$ such that $Q_{i}\beta\equiv\alpha$ (mod M_{i}). That is, the sequence $A^{*}\xrightarrow{Q_{i*}}M_{\infty}/M_{i}\xrightarrow{Q_{i*}}A^{*}/M_{i}$ is exact.*

Proof. We shall identify M_∞/M_i with $\tilde{L}_{i+1}$ by $\tilde{\eta}_i$. Any two monomials α, α' can be written in the form $\alpha=Q(O^i, \varepsilon_{i+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_n)$ $\alpha'=Q(O^i, \varepsilon'_{i+1}, \cdots, \varepsilon'_{i+n})\mathcal{P}(r'_1, \cdots, r'_n)$, without loss of generality, by adding zeros of ε_{i+j} (resp. ε'_{i+j}) or r_j (resp. r'_j), if necessary.

For the convenience we introduce an order among monomials in $\tilde{L}_{j+1}$: For two monomials $\alpha=Q(O^i, \varepsilon_{i+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_n)$ and $\alpha'=Q(O^i, \varepsilon'_{i+1}, \cdots, \varepsilon'_{i+n})\mathcal{P}(r'_1, \cdots, r'_n)$, we define $\alpha>\alpha'$ if there is an integer k, $1\leqq k\leqq n$, such that

$$\varepsilon_{i+j} = \varepsilon'_{i+j} \text{ and } r_j = r'_j \text{ for } 1\leqq j<k, \text{ and } \varepsilon_{i+k}>\varepsilon'_{i+k},$$
$$\text{or, } \varepsilon_{i+j} = \varepsilon'_{i+j} \text{ and } r_j = r'_j \text{ for } 1\leqq j<k,\ \varepsilon_{i+k} = \varepsilon'_{i+k} \text{ and } r_k>r'_k\,.$$

Let $\alpha=x_1\alpha_1+\cdots$ be an element of $\tilde{L}_{i+1}$ such that $Q_i\alpha\equiv 0 \pmod{M_i}$, where α_1 is the first (largest) monomial in the above order, and $x_1\neq 0$. Then $\alpha_1=Q(O^i, \varepsilon_{i+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_n)$ must satisfy the condition that there exists an integer k, $1\leqq k\leqq n$, such that

$$\varepsilon_{i+j} = 0,\ r_j<p^i \text{ for } 1\leqq j<k, \text{ and } \varepsilon_{i+k} = 1\,. \tag{5.4}$$

For, suppose that this condition were not satisfied, then there is an integer l, $1\leqq l\leqq n$, such that $\varepsilon_{i+j}=0$, $r_j<p^i$ for $1\leqq j<l$, $\varepsilon_{i+l}=0$ and $r_l\geqq p^i$, (The case where $\varepsilon_{i+j}=0$ and $r_j<p^i$ for all $j\leqq n$ is omitted since $\alpha_1\in M_\infty$.) Hence, by (1.5), we have

$$Q_i\alpha \equiv \pm x_1 Q(O^{i+l-1}, 1, \varepsilon_{i+l+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_l-p^i, \cdots, r_n)+\cdots$$

and the monomial $Q(O^{i+l-1}, 1, \varepsilon_{i+l+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_l-p^i, \cdots, r_n)$ is larger than any other monomials in $Q_i\alpha$. So that it is not cancelled. This contradicts to the assumption that $Q_i\alpha\equiv 0$.

While, the monomial

$$\beta_1 = Q(O^{i+k}, \varepsilon_{i+k+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_k+p^i, \cdots, r_n)$$

satisfies that $Q_i\beta_1\equiv -\alpha_1+$smaller terms $\pmod{M_i}$ and hence, if we put $\alpha'=\alpha+Q_i(x_1\beta_1)$, we have $Q_i\alpha'\equiv 0 \pmod{M_i}$ and α' consists of monomials smaller than α_1. By repeating such a process, we conclude that there is an element $\beta=-x_1\beta_1+\cdots\in A^*$ such that $Q_i\beta\equiv\alpha \pmod{M_i}$. q.e.d.

The subalgebra M^k_∞, $k\geqq 1$, is identified with a submodule of A^* generated by the elements $Q(\varepsilon_0, \varepsilon_1, \cdots)\mathcal{P}^R$ with at least k non-zero ε_j's. Then we have

Lemma 5.8. *Let $\alpha\in M^2_\infty/(M^2_\infty\cap M_i)$ be an element such that $Q_i\alpha\equiv 0$ (mod M_i) and $Q_{i-1}\alpha\equiv 0$ (mod M_i), then there is an element $\beta\in A^*$ such that $Q_i\beta\equiv 0$ (mod M_i) and $Q_{i-1}\beta\equiv\alpha$ (mod M_i). That is, the sequence*

$$\text{Ker } Q_i \xrightarrow{Q_{i-1*}} \text{Ker } Q_i\cap[M^2_\infty/(M^2_\infty\cap M_i)] \xrightarrow{Q_{i-1*}} A^*/M_i \text{ is exact.}$$

Proof. We shall identify $M_\infty^2/(M_\infty^2 \cap M_i)$ with $\tilde{L}_{i+1}^2$ by $\tilde{\eta}_{i,2}$. Let $\alpha = x_1\alpha_1 +$smaller terms be an element of $\tilde{L}_{i+1}^2$ such that $Q_i\alpha \equiv 0 \pmod{M_i}$ and $Q_{i-1}\alpha \equiv 0 \pmod{M_i}$. Then, we conclude, by a similar argument as in the proof of the above Lemma, that α_1 must satisfy a condition that there is an integer k, $1 \leqq k \leqq n$, such that

$$(5.5) \qquad \varepsilon_{i+j} = 0 \text{ for } j<k,\ r_1<p^i,\ r_j<p^{i-1} \text{ for } 1<j<k, \text{ and } \varepsilon_{i+k}=1.$$

Since $\alpha \in \tilde{L}_{i+1}^2 \subset M_\infty^2$, α_1 contains at least one non-zero ε_{i+l} other than ε_{i+k}, and by (5.5), $l>k$.

Put

$$\gamma_1 = Q(O^{i+k}, \varepsilon_{i+k+1}, \cdots, \varepsilon_{i+l-1}, 0, \varepsilon_{i+l+1}, \cdots, \varepsilon_{i+n})\mathcal{P}(r_1, \cdots, r_{k+1}+p^{i-1}, \cdots, r_l+p^i, \cdots, r_n)$$

then it satisfies that $Q_iQ_{i-1}\gamma_1 \equiv (-1)^{l-k}\alpha_1 +$smaller terms. So that $\alpha' = \alpha - Q_iQ_{i-1}((-1)^{l+k}x_1\gamma_1)$ satisfies that $Q_i\alpha' \equiv 0$ and $Q_{i-1}\alpha' \equiv 0$, and consisting of monomials smaller than α_1. Thus, after a finite number of steps, we have an element $\gamma = (-1)^{l-k+1}x_1\gamma_1 + \cdots$ such that $Q_iQ_{i-1}\,\gamma \equiv \alpha \pmod{M_i}$. Put $\beta = -Q_i\gamma$, then we have the required element β. q.e.d.

REMARK. By a similar argument as in the proofs of the above Lemmas, we can conclude that for any $k \geqq 0$, the sequence

$$K_{k-1} \xrightarrow{Q_{i-k*}} K_{k-1} \cap [M_\infty^{k+1}/(M_\infty^{k+1} \cap M_i)] \xrightarrow{Q_{i-k*}} A^*/M_i$$

is exact, where $K_{k-1} = \operatorname{Ker} Q_i \cap \cdots \cap \operatorname{Ker} Q_{i-k+1}$, $K_{-1} = A^*$.

6. Construction of chain complexes

Now, we shall construct admissible chain complexes

$$(6.1)_i \qquad \cdots \to {}^iC_r \xrightarrow{{}^id_r} {}^iC_{r-1} \to \cdots \to {}^iC_2 \xrightarrow{{}^id_2} {}^iC_1 \xrightarrow{{}^id_1} {}^iC_0$$

for $i \geqq 0$, which are used in the later arguments.

Let ${}^iC_0$ be a free A^*-module generated by one generator c of degree 0, and ${}^iC_r$ be a free A^*-module generated by the generators $c_{j_1,\cdots,j_r}$, $0 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_r \leqq i$, of degree $2(p^{j_1}+\cdots+p^{j_r})-r$, for $r \geqq 1$. For the convenience, we shall denote $c_{j_1,\cdots,j_r}$ by $[j_1, \cdots, j_r]$ and c by $[\ ]$. Then, an A^*-map ${}^id_r : {}^iC_r \to {}^iC_{r-1}$ is defined as follows: Let $j_1 = \cdots = j_{s_1-1} < j_{s_1} = \cdots = j_{s_2-1} < j_{s_2} = \cdots < \cdots < \cdots = j_{s_k-1} < j_{s_k} = \cdots = j_r$,

$$(6.2) \qquad {}^id_r[j_1, \cdots, j_r] = \sum_{\lambda=0}^{k} Q_{j_{s_\lambda}}[j_1, \cdots, j_{s_\lambda-1}, j_{s_\lambda+1}, \cdots, j_r]$$

Where $j_{s_0} = j_1$. In particular, ${}^id_1[j] = Q_j[\ ]$. The A^*-map ${}^id_r$ is of degree 0 for any $r>0$, and it is easily checked that ${}^id_{r-1}{}^id_r=0$ for $r\geqq 2$. Thus we have obtained chain complexes $(6.1)_i$ for $i\geqq 0$.

Lemma 6.1. *The chain complex* $(6.1)_i$ *is exact* (*i.e.* $\operatorname{Ker}{}^id_r=\operatorname{Im}{}^id_{r+1}$) *for all* $i\geqq 0$ *and* $r\geqq 1$.

Proof. By Lemma 5.5, any element $\alpha\in A^*$ can be written uniquely in the form $\alpha=\alpha'+\sum\alpha_kQ_k$, for $\alpha'\in L_\infty$ and $\alpha_k\in L_{k+1}$.

We define a chain homotopy ${}^is_r:{}^iC_r\to{}^iC_{r+1}$, $r\geqq 0$, by

$$\left.\begin{aligned}&{}^is_r(\alpha Q_k[j_1,\cdots,j_r])=\alpha[k,j_1,\cdots,j_r] \text{ if } k\leqq j_1,\ \alpha\in L_{k+1},\\ &{}^is_r(\beta[j_1,\cdots,j_r])=0 \quad \text{ if } \beta\in L_{j_1+1},\end{aligned}\right\}\quad \text{for } r\geqq 1;$$
$$\begin{aligned}&{}^is_0(\alpha Q_k[\ \ ])=\alpha[k] \quad \text{ if } k\leqq i,\ \alpha\in L_{k+1},\\ &{}^is_0(\beta[\ \ ])=0 \quad \text{ if } \beta\in L_{i+1}.\end{aligned}$$

Although ${}^is_r$ is not an A^*-map but an L_{i+1}-map, by a direct calculation, we have ${}^is_{r-1}{}^id_r+{}^id_{r+1}{}^is_r=\text{identity}$ as an A^*-map. This implies the exactness of $(6.1)_i$. q.e.d.

Let ${}^iF_0=K(Z_p,m)$ for a sufficiently large integer m, and ${}^iB_r=\underset{0\leqq j_1\leqq\cdots\leqq j_r\leqq i}{\times} K(Z_p, m+2(p^{j_1}+\cdots+p^{j_r}-(r-1))-1)$, for $r\geqq 1$. The canonical isomorphisms $\alpha_0:{}^iC_0\to H^*({}^iF_0)$ and $\alpha_r:{}^iC_r\to H^*({}^iB_r)$, $r\geqq 1$, are given by

$$\alpha_0[\ \ ]=\iota,$$
$$\alpha_r[j_1,\cdots,j_r]=\iota_{j_1,\cdots,j_r},$$

where $\iota\in H^m({}^iF_0)$, $\iota_{j_1,\cdots,j_r}\in H^{m+k}({}^iB_r)$, $k=2(p^{j_1}+\cdots+p^{j_r}-(r-1))-1$, are the fundamental classes.

Let ${}^if_1:{}^iF_0\to{}^iB_1$ be a map such that ${}^if_1^*\iota_j=Q_j\iota$, and ${}^iF_1$ be the fiber of the fiber map ${}^if_1$. Then, we have

Lemma 6.2. *In the stable range,* $H^*({}^iF_1)$ *is generated by elements* $a_0\in H^m({}^iF_1)$ *and* $a_{j_1,j_2}\in H^{m+k}({}^iF_1)$, $k=2(p^{j_1}+p^{j_2}-1)-1$, *for* $0\leqq j_1\leqq j_2\leqq i$, *with the fundamental relations*

$$(6.3)_1\qquad\begin{aligned}&Q_0a_0=Q_1a_0=\cdots=Q_ia_0=0\,;\ Q_ja_{j,j}=0 \text{ for } 0\leqq j\leqq i,\\ &Q_{j_1}a_{j_1,j_2}+Q_{j_2}a_{j_1,j_1}=0,\ Q_{j_1}a_{j_2,j_2}+Q_{j_2}a_{j_1,j_2}=0,\ 0\leqq j_1<j_2\leqq i,\\ &Q_{j_1}a_{j_2,j_3}+Q_{j_2}a_{j_1,j_3}+Q_{j_3}a_{j_1,j_2}=0 \text{ for } 0\leqq j_1<j_2<j_3\leqq i.\end{aligned}$$

This Lemma will be proved in the next section.

Inductively, we assume that $H^*({}^iF_{r-1})$, $r\geqq 2$, is generated by the elements $a_0\in H^m({}^iF_{r-1})$ and $a_{j_1,\cdots,j_r}\in H^{m+k}({}^iF_{r-1})$, $k=2(p^{j_1}+\cdots+p^{j_r}-(r-1))-1$, for $0\leqq j_1\leqq\cdots\leqq j_r\leqq i$, with the fundamental relations

$$(6.3)_{r-1}\qquad Q_0a_0=\cdots=Q_ia_0=0, \text{ and } \rho(j_1,\cdots,j_r,j_{r+1})=0$$

for $0\leqq j_1\leqq\cdots\leqq j_r\leqq j_{r+1}\leqq i$, where $\rho(j_1,\cdots,j_r,j_{r+1})$ is defined as follows: Let $j_1=\cdots=j_{s_1-1}<j_{s_1}=\cdots<\cdots<\cdots=j_{s_k-1}<j_{s_k}=\cdots=j_{r+1}$, then

(6. 4) $$\rho(j_1, \cdots, j_r, j_{r+1}) = \sum_{\lambda=0}^{k} Q[j_{s_\lambda}]\langle j_1, \cdots, j_{s_\lambda -1}, j_{s_\lambda +1}, \cdots, j_{r+1}\rangle,$$

where $j_{s_0}=j_1$, $Q[j]=Q_j$, and $\langle j_1, \cdots, j_s\rangle$ stands for $a_{j_1,\cdots,j_s}$.

Let ${}^if_r : {}^iF_{r-1}\to {}^iB_r$ be a map such that ${}^if_r^*(\iota_{j_1,\cdots,j_r})=a_{j_1,\cdots,j_r}$. Then, we have ${}^if_r^*\alpha_r{}^id_{r+1}=0$ because $\rho(j_1, \cdots, j_{r+1})=0$. Let ${}^iF_r$ be the fiber of the fiber map ${}^if_r$. Then we have

Lemma 6. 3. *If $i=0,1,2$, or $r<p^4+p^3-2$, $H^*({}^iF_r)$ is generated, in the stable range, by elements $a_0\in H^m({}^iF_r)$ and $a_{j_1,\cdots,j_{r+1}}\in H^{m+k}({}^iF_r)$, $k=2(p^{j_1}+\cdots+p^{j_{r+1}}-r)-1$, for $0\leqq j_1\leqq\cdots\leqq j_{r+1}\leqq i$, with the fundamental relations*

(6. 3)$_r$ $$Q_0a_0 = \cdots = Q_ia_0 = 0, \text{ and } \rho(j_1, \cdots, j_{r+1}, j_{r+2}) = 0$$

for $0\leqq j_1\leqq\cdots\leqq j_{r+1}\leqq j_{r+2}\leqq i$.

This Lemma will be proved in the section 8.

Thus, we have

Theorem 6. 4. *The chain complex (6.1)$_i$ is r-admissible, for all $r\geqq 1$ if $i\leqq 2$, and for $r<p^4+p^3-2$ if $i\geqq 3$.*

Therefore we can speak of the pyramids of stable cohomology operations $\{{}^i\Phi^{s,t}\}$ associated with the chain complex (6.1)$_i$.

7. Proof of Lemma 6. 2

For the convenience, we shall denote $a_{j_1,\cdots,j_s}$ by $\langle j_1, \cdots, j_s\rangle$, and $\iota_{j_1,\cdots,j_s}$ by $\iota[j_1, \cdots, j_s]$, if it is necessary.

From the stable cohomology exact sequence of the fibering ${}^iF_1\to {}^iF_0\to {}^iB_1$, we have exact sequences

$$H^{m+k}({}^iB_1) \xrightarrow{{}^if_1^*} A^k[\iota] \xrightarrow{{}^ii_1^*} H^{m+k}({}^iF_1) \xrightarrow{{}^i\tau^*} H^{m+k+1}({}^iB_1) \xrightarrow{{}^if_1^*} A^{k+1}[\iota],$$

for $k=0,1,2,\cdots$, because $H^{m+k}({}^iF_0)\approx A^k[\iota]$.

For $k=0$, we have an element $a_0={}^ii_1^*(\iota)$. Since ${}^if_1^*(\iota_j)=Q_j\iota$, we have $Q_ja_0=0$ for $j=0,1,\cdots,i$.

Since ${}^if_1^*\alpha_1=\alpha_0{}^id_1$, we have ${}^if_1^*(\alpha_1{}^id_2[j_1, j_2])=\alpha_0({}^id_1{}^id_2[j_1, j_2])=0$. Hence, we have elements

$$\langle j_1, j_2\rangle = {}^i\tau_1^{*-1}(\alpha_1{}^id_2[j_1, j_2]) \qquad \text{for } 0\leqq j_1\leqq j_2\leqq i\,,$$

which are in $H^{m+k}({}^iF_1)$, $k=2(p^{j_1}+p^{j_2}-1)-1$, because the degree of $[j_1, j_2]$ is $2(p^{j_1}+p^{j_2}-1)$ and ${}^i\tau_1^*$ increases the degree by one.

For $k=6p^j-4$, $0\leqq j\leqq i$, $Q_j\langle j, j\rangle\in {}^ii_1^*(A^k[\iota])$ since ${}^i\tau_1^*(Q_j\langle j, j\rangle)=Q_jQ_j\iota_j=0$. For $k=2(2p^{j_1}+p^{j_2}-2)$, $0\leqq j_1<j_2\leqq i$, we have $Q_{j_1}\langle j_1, j_2\rangle+Q_{j_2}\langle j_1, j_1\rangle\in {}^ii_1^*(A^k[\iota])$, because

$$\begin{aligned}{}^i\tau_1^*(Q_{j_1}\langle j_1, j_2\rangle + Q_{j_2}\langle j_1, j_1\rangle) &= Q_{j_1}(Q_{j_1}\iota[j_2] + Q_{j_2}\iota[j_1]) \\ &\quad + Q_{j_2}Q_{j_1}\iota[j_1] = 0 .\end{aligned}$$

Similarly, for $k=2(p^{j_1}+2p^{j_2}-2)$, $0\leqq j_1<j_2\leqq i$, we have $Q_{j_1}\langle j_2, j_2\rangle + Q_{j_2}\langle j_1, j_2\rangle \in {}^ii_1^*(A^k[\iota])$, and for $k=2(p^{j_1}+p^{j_2}+p^{j_3}-2)$, $0\leqq j_1<j_2<j_3\leqq i$, $Q_{j_1}\langle j_2, j_3\rangle + Q_{j_2}\langle j_1, j_3\rangle + Q_{j_3}\langle j_1, j_2\rangle \in {}^ii_1^*(A^k[\iota])$.

On the other hand, for each case, we have $k\equiv 2 \pmod 4$. Hence, by Lemma 5.2, $h(\alpha)\not\equiv 0 \pmod 4$ so that $h(\alpha)\neq 0$ for any monomial $\alpha\in A^k$. But, by Lemma 5.5, any element $\alpha\in A^*$ can be written in the form $\alpha' + \sum \alpha_j Q_j$, $\alpha'\in L_\infty$, $\alpha_j\in L_{j+1}$. While, since $h(\alpha)\neq 0$ and $k<2p^{i+1}-1$, we have $\alpha = \sum_{j=0}^{i}\alpha_j Q_j$. Therefore, we have ${}^ii_1^*(A^k[\iota])=0$ because ${}^ii_1^*(\iota)=a_0$ and $Q_j a_0=0$ for $0\leqq j\leqq i$.

Thus, we have $\rho(j_1, j_2, j_3)=0$ for $0\leqq j_1\leqq j_2\leqq j_3\leqq i$.

Conversely, let u be any element in $H^*({}^iF_1)$, then $\alpha_0{}^if_1^*{}^i\tau_1^*u = {}^id_1\alpha_1{}^i\tau_1^*u=0$, so that $\alpha_1{}^i\tau_1^*u\in \operatorname{Ker}{}^id_1$. But, by Lemma 6.1, the chain complex $(6.1)_i$ is exact so

$$\alpha_1{}^i\tau_1^*u = {}^id_2(\sum \beta(j_1, j_2)[j_1, j_2]) = \sum \beta(j_1, j_2){}^id_2[j_1, j_2]$$

for some $\beta(j_1, j_2)\in A^*$. This implies that $u = \sum \beta(j_1, j_2)\langle j_1, j_2\rangle + \beta_0 a_0$, for some $\beta(j_1, j_2)$, $\beta_0\in A^*$.

Let $\sum \beta(j_1, j_2)\langle j_1, j_2\rangle + \beta_0 a_0=0$ for some $\beta(j_1, j_2)$, $\beta_0\in A^*$. Then, ${}^i\tau_1^*(\sum \beta(j_1, j_2)\langle j_1, j_2\rangle + \beta_0 a_0) = \sum \beta(j_1, j_2)\alpha_1{}^id_2[j_1, j_2]=0$. So that we have ${}^id_2(\sum \beta(j_1, j_2)[j_1, j_2])=0$, Again, by Lemma 6.1, we have $\sum\beta(j_1, j_2)[j_1, j_2] = \sum\gamma(j_1, j_2, j_3){}^id_3[j_j, j_2, j_3]$ for some $\gamma(j_1, j_2, j_3)\in A^*$. So we conclude that $\sum \beta(j_1, j_2)\langle j_1, j_2\rangle = \sum \gamma(j_1, j_2, j_3)\rho(j_1, j_2, j_3)$, and hence $\beta_0 = \sum_{j=0}^{i}\gamma_j Q_j$ for some $\gamma_j\in A^*$.

This completes the proof of Lemma 6.2.

8. Proof of Lemma 6.3

Let the chain complex $(6.1)_i$ be $(r-1)$-admissible, and $H^*({}^iF_{r-1})$ is generated by a_0 and $\langle j_1, \cdots, j_r\rangle$, $0\leqq j_1\leqq\cdots\leqq j_r\leqq i$, with the fundamental relations $(6.3)_{r-1}$. We define, then, a map ${}^if_r : {}^iF_{r-1}\to {}^iB_r$ by ${}^if_r^*(\iota[j_1, \cdots, j_r]) = \langle j_1, \cdots, j_r\rangle$, for the fundamental classes $\iota[j_1, \cdots, j_r]\in H^*({}^iB_r)$. Since ${}^iB_r$ is a cartesian product of Eilenberg-MacLane spaces, the map ${}^if_r$ is well-defined. Let ${}^iF_r$ be the fiber of the fiber map ${}^if_r$, and let $i\leqq 2$ or $r<p^4+p^3-2$.

Let E_s^* be the subalgebra of $H^*({}^iF_s)$ generated by a_0, then

Lemma 8.1. *${}^ii_r^*\,|\,E_{r-1}^*$ is isomorphic and $E_r^* = \operatorname{Im}{}^ii_r^*$.*

Proof. It follows immediately from the definition that $(\operatorname{Im}{}^if_r^*)\cap E_r^*$

$=\{0\}$, so that ${}^i i_r^* | E_r^*$ is isomorphic by the exactness of the sequence

$$\cdots \to H^*({}^iB_r) \xrightarrow{{}^if_r^*} H^*({}^iF_{r-1}) \xrightarrow{{}^ii_r^*} H^*({}^iF_r) \xrightarrow{{}^i\tau_r^*} H^*({}^iB_r) \xrightarrow{{}^if_r^*} H^*({}^iF_r) \to \cdots$$

While, since ${}^if_r^*(\iota[j_1, \cdots, j_r]) = \langle j_1, \cdots, j_r\rangle$ and $H^*({}^iF_{r-1})$ is generated by the elements a_0 and $\langle j_1, \cdots, j_r\rangle$, we have $\mathrm{Im}\, {}^ii_r^* = E_r^*$. q.e.d.

Next, easily we have $\rho(j_1, \cdots, j_{r+1}) = {}^if_r^*(\alpha_r {}^id_{r+1}[j_1, \cdots, j_{r+1}])$ in $H^*({}^iF_{r-1})$. But, by inductive assumption, we have $\rho(j_1, \cdots, j_{r+1})=0$, hence there are elements $\langle j_1, \cdots, j_{r+1}\rangle = {}^i\tau_r^{*-1}(\alpha_r {}^id_{r+1}[j_1, \cdots, j_{r+1}])$ in $H^*({}^iF_r)$. The degree of $\langle j_1, \cdots, j_{r+1}\rangle$ is $2(p^{j_1}+ \cdots + p^{j_{r+1}}-r)-1$, because that of $[j_1, \cdots, j_{r+1}]$ is $2(p^{j_1}+\cdots+p^{j_{r+1}}-r)$ and ${}^id_{r+1}$ is of degree 0.

Let $\rho(j_1, \cdots, j_{r+1}, j_{r+2})$, $0 \leqq j \leqq \cdots \leqq j_{r+1} \leqq j_{r+2} \leqq i$, be the elements defined as in (6.4), then the degree of $\rho(j_1, \cdots, j_{r+2})$ is $2(p^{j_1}+\cdots+p^{j_{r+2}}-r)-2 \equiv 2 \pmod 4$, and ${}^i\tau_r^*(\rho(j_1, \cdots, j_{r+2})) = \alpha_r {}^i(d_{r+1} {}^id_{r+2}[j_1, \cdots, j_{r+2}])=0$. Hence, $\rho(j_1, \cdots, j_{r+2}) \in \mathrm{Im}\, {}^ii_r^* = E_r^*$. On the other hand, by a simple calculation, we have

Lemma 8.2. *If* $j_1 = \cdots = j_{s_1-1} < j_{s_1} = \cdots < \cdots < \cdots = j_{s_k-1} < j_{s_k} = \cdots = j_{r+3}$,

$$\sum_{\lambda=0}^{k} Q[j_{s_\lambda}]\rho(j_1, \cdots, j_{s_\lambda-1}, j_{s_\lambda+1}, \cdots, j_{r+3}) = 0\,,$$

without assuming $\rho(j_1, \cdots, j_{r+2})=0$, *where* $j_{s_0}=j_1$ *and* $Q[j]=Q_j$.

Now, since $\rho(i, \cdots, i) = Q_i \langle i, \cdots, i\rangle \in E_r^{m+k}$, $k \not\equiv 0 \pmod 4$, and $Q_j a_0 = 0$ for $j \leqq i$, there is an element $\alpha \in \tilde{L}_{j+1}$ such that $\rho(i, \cdots, i) = \alpha a_0$. Then, we have $Q_i \alpha a_0 = 0$ because $Q_i \rho(i, \cdots, i)=0$, and this implies that $Q_i\alpha \equiv 0 \pmod{M_i}$. But, by Lemma 5.7, there is an element $\beta \in A^*$ such that $Q_i\beta \equiv \alpha \pmod{M_i}$. Thus, if we replace $\langle i, \cdots, i\rangle$ by $\langle i, \cdots, i\rangle - \beta a_0$, then we have $Q_i\langle i, \cdots, i\rangle = 0$ and still ${}^i\tau_r^*(\langle i, \cdots, i\rangle) = \alpha_r {}^id_{r+1}[i, \cdots, i]$.

For $(r+2)$-tuples $(j_1, \cdots, j_{r+2})$ and $(j_1', \cdots, j_{r+2}')$ with $0 \leqq j_1 \leqq \cdots \leqq j_{r+2} \leqq i$, $0 \leqq j_1' \leqq \cdots \leqq j_{r+2}' \leqq i$, we define that $(j_1', \cdots, j_{r+2}') > (j_1, \cdots, j_{r+2})$ if there is an integer s, $1 \leqq s \leqq r+2$, such that $j_k' = j_k$ for $s < k \leqq r+2$ and $j_s' > j_s$.

If $\rho(j_1', \cdots, j_{r+2}')=0$ for any $(j_1', \cdots, j_{r+2}') > (j_1, \cdots, j_{r+2})$, then $Q_l\rho(j_1, \cdots, j_{r+2})=0$ for any $l \geqq j_{r+2}$. For, by Lemma 8.2, $Q_l\rho(j_1, \cdots, j_{r+2}) = -\sum_{\lambda} Q[j_{s_\lambda}]\rho(j_1, \cdots, j_{s_\lambda-1}, j_{s_\lambda+1}, \cdots, j_{r+2}, l)$, for any $l \geqq j_{r+2}$, and the terms in the right hand side $>(j_1, \cdots, j_{r+2})$, so they vanish.

Assume, inductively, that $\rho(j_1', \cdots, j_{r+1}', i)=0$ for any $(j_1', \cdots, j_{r+1}', i)$, $>(j_1, \cdots, j_{r+1}, i)$, then we have $Q_i\rho(j_1, \cdots, j_{r+1}, i)=0$. But, we may put $\rho(j_1, \cdots, j_{r+1}, i) = \alpha a_0$ for some $\alpha \in \tilde{L}_{i+1}$, and we have $Q_i\alpha \equiv 0 \pmod{M_i}$. Again, by Lemma 5.7, there is an element $\beta \in A^*$ such that $Q_i\beta \equiv \alpha \pmod{M_i}$. Replace $\langle j_1, \cdots, j_{r+1}\rangle$ by $\langle j_1, \cdots, j_{r+1}\rangle - \beta a_0$, then we have $\rho(j_1, \cdots, j_{r+1}, i)=0$ and still ${}^i\tau_r^*(\langle j_1, \cdots, j_{r+1}\rangle) = \alpha_r {}^id_{r+1}[j_1, \cdots, j_{r+1}]$.

Thus, we have $\rho(j_1, \cdots, j_{r+2})=0$ provided that $j_{r+2}=i$.

If $i=1$, $\rho(j_1, \cdots, j_{r+2})$ without $j_{r+2}=i$ is only $\rho(0, \cdots, 0)=Q_0\langle 0, \cdots, 0\rangle \in E_r^{m+2}=0$. Hence, $\rho(j_1, \cdots, j_{r+2})=0$ for any $(j_1, \cdots, j_{r+2})$. (The fact that $\rho(0, \cdots, 0)=0$ shows that the admissibility of $(6.1)_0$.)

If $i\geqq 2$, $\rho(j_1, \cdots, j_{r+2})$ without $j_{r+2}=i$ and of the maximal degree is $\rho(i-1, \cdots, i-1)=Q_{i-1}\langle i-1, \cdots, i-1\rangle$. We may put $\rho(i-1, \cdots, i-1)=\alpha a_0$ for some $\alpha\in\tilde{L}_{i+1}^2$. Since $Q_i\rho(i-1, \cdots, i-1)=Q_{i-1}\rho(i-1, \cdots, i-1)=0$, we have $Q_i\alpha\equiv 0 \pmod{M_i}$ and $Q_{i-1}\alpha\equiv 0 \pmod{M_i}$. Hence, by Lemma 5.8, we can find an element $\beta\in A^*$ such that $Q_i\beta\equiv 0 \pmod{M_i}$ and $Q_{i-1}\beta\equiv\alpha \pmod{M_i}$. Replace $\langle i-1, \cdots, i-1\rangle$ by $\langle i-1, \cdots, i-1\rangle-\beta a_0$, then we have $\rho(i-1, \cdots, i-1)=0$ and still $\rho(i-1, \cdots, i-1, i)=0$ and ${}^i\tau_r^*(\langle i-1, \cdots, i-1\rangle) =\alpha_r{}^id_{r+1}[i-1, \cdots, i-1]$.

Similarly, we can reduce $\rho(j_1, \cdots, j_{r+1}, i-1)$ to zero without altering $\rho(j_1, \cdots, j_{r+1}, i)$ and ${}^i\tau_r^*(\langle j_1, \cdots, j_{r+1}\rangle)$.

If $i=2$, $\rho(j_1, \cdots, j_{r+2})$ with $j_{r+2}<i-1$ is only $\rho(0, \cdots, 0)=0$.

If $i\geqq 3$, $\rho(j_1, \cdots, j_{r+2})$ with $j_{r+2}<i-1$ and of the maximal degree is $\rho(i-2, \cdots, i-2)$ and its degree is $2((r+2)p^{i-2}-(r+1))$, and hence any $\rho(j_1, \cdots, j_{r+2})$ with $j_{r+2}<i-1$ has degree not greater than $2((r+2)p^{i-2}-(r+1))$. On the other hand, we may put $\rho(j_1, \cdots, j_{r+2})=\alpha a_0$ for some $\alpha\in\tilde{L}_{i+1}^2$, and since $r<p^4+p^3-2$,

$$d(\alpha)\leqq 2((r+2)p^{i-2}-(r+1))<2(p^{i+2}+p^{i+1}-1) = d(Q_{i+1}Q_{i+2})\,.$$

Hence, we conclude that $\alpha\equiv 0 \pmod{M_i}$.

Thus, if $i\leqq 2$ or $r<p^4+p^3-2$, we have $\rho(j_1, \cdots, j_{r+2})=0$ for any $(j_1, \cdots, j_{r+2})$.

Similarly to the proof of Lemma 6.2, it is easily verified that any element in $H^*({}^iF_r)$ can be written in the form

$$\sum\beta(j_1, \cdots, j_{r+1})\langle j_1, \cdots, j_{r+1}\rangle+\beta_0 a_0\,,$$

and that all relations in $H^*({}^iF_r)$ are generated by $(6.3)_r$.

This completes the proof of Lemma 6.3.

CHAPTER 3. NON-TRIVIALITY OF STABLE HOMOTOPY ELEMENTS

9. Some stable homotopy elements

Let S, ${}^1\boldsymbol{M}$ and ${}^2\boldsymbol{M}$ be S–spectrum [10] such that $S=\{S^m|m\geqq 1\}$, ${}^1\boldsymbol{M}=\{{}^1M^m=S^m\bigcup_{p\iota}e^{m+1}|m\geqq 2\}$ and ${}^2\boldsymbol{M}=\{{}^2M^m={}^1M^m\bigcup_{\alpha}T({}^1M^{m+2p-2})|m\geqq 2p-1\}$, respectively, where α is the stable homotopy element defined in [9] which corresponds to the element $\alpha_1\in\pi_{m+2p-3}(S^m)$ of the stable homotopy group of sphere [6].

The (stable) mod p (where p is an odd prime) cohomology structures of ${}^1M^N$ and ${}^2M^N$ (where N is a sufficiently large integer) are as follows:

$$H^*({}^1M^N) = \{e^N,\ e^{N+1} \mid \Delta e^N = e^{N+1}\},$$

$$H^*({}^2M^N) = \{e^N,\ e^{N+1},\ e^{N+2p-1},\ e^{N+2p} \mid \Delta e^N = e^{N+1},\ \Delta e^{N+2p-1} = e^{N+2p},$$
$$\mathcal{P}^1 e^{N+1} = (-1)^{N+1} e^{N+2p-1}\}\ .$$

Let $G_k = \lim\,[S^{m+k}, S^m]$, ${}^1\pi_k = \lim\,[{}^1M^{m+k}, {}^1M^m]$ and ${}^2\pi_k = \lim\,[{}^2M^{m+k}, {}^2M^m]$, and $G_* = \sum G_k$, ${}^1\pi_* = \sum {}^1\pi_k$, ${}^2\pi^* = \sum {}^2\pi_k$. Then we have the following exact sequences

$$\cdots \to G_k \xrightarrow{(p\iota)_*} G_k \xrightarrow{j_*} \bar{G}_k \xrightarrow{k_*} G_{k-1} \xrightarrow{(\rho\iota)_*} G_{k-1} \to \cdots \tag{9.1}$$
$$\cdots \to \bar{G}_{k+1} \xrightarrow{(p\iota)^*} \bar{G}_{k+1} \xrightarrow{k^*} {}^1\pi_k \xrightarrow{j^*} \bar{G}_k \xrightarrow{(p\iota)^*} \bar{G}_k \to \cdots$$

$$\cdots \to {}^1\pi_{k-2p+2} \xrightarrow{\alpha_*} {}^1\pi_k \xrightarrow{j'_*} {}^1\bar{\pi}_k \xrightarrow{k'_*} {}^1\pi_{k-2p+1} \xrightarrow{\alpha_*} {}^1\pi_{k-1} \to \cdots \tag{9.2}$$
$$\cdots \to {}^1\bar{\pi}_{k+1} \xrightarrow{\alpha^*} {}^1\bar{\pi}_{k+2p-1} \xrightarrow{k'^*} {}^2\pi_k \xrightarrow{j'^*} {}^1\bar{\pi}_k \xrightarrow{\alpha^*} {}^1\bar{\pi}_{k+2p-2} \to \cdots$$

where $\bar{G}_k = \lim\,[S^{m+k}, {}^1M^m]$, ${}^1\bar{\pi}_k = \lim\,[{}^1M^{m+k}, {}^2M^m]$, and $j : S^m \to {}^1M^m$, $j' : {}^1M^m \to {}^2M^m$ are injections and $k : {}^1M^m \to S^{m+1}$, $k' : {}^2M^m \to {}^1M^{m+2p-1}$ are shrinking maps.

Since $p\iota \circ \alpha_1 = \alpha_1 \circ p\iota = 0$ in G_* and $\mathcal{P}^1_{\alpha_1} e^N = (-1)^N e^{N+2p-3}$ for the generators $e^N \in H^N(S^N)$ and $e^{N+2p-3} \in H^{N+2p-3}(S^{N+2p-3})$ [6], we have a nontrivial element $\alpha = j^{*-1} k_*^{-1}(\alpha_1) \in {}^1\pi_{2p-2}$ such that

$$\mathcal{P}^1_{\alpha} e^{N+1} = (-1)^{N+1} e^{N+2p-2} \tag{9.3}$$

for the generators $e^{N+1} \in H^{N+1}({}^1M^N)$ and $e^{N+2p-1} \in H^{N+2p-1}({}^1M^{N+2p-2})$. Also, since $\alpha \circ \beta_1 = \beta_1 \circ \alpha = 0$ in ${}^1\pi_*$ and $\mathcal{P}^p_{\beta_1} e^{N+1} = (-1)^{N+1} e^{N+2p(p-1)}$ for the generators $e^{N+1} \in H^{N+1}({}^1M^N)$ and $e^{N+2p(p-1)} \in H^{N+2p(p-1)}({}^1M^{N+2p(p-1)-1})$ [6], [9], we have a non-trivial element $\beta = j'^{*-1} k'^{-1}_*(\beta_1) \in {}^2\pi_{2p^2-2}$ such that

$$\mathcal{P}^p_{\beta} e^{N+2p} = (-1)^N e^{N+2p^2-1} \tag{9.4}$$

for the generators $e^{N+2p} \in H^{N+2p}({}^2M^N)$ and $e^{N+2p^2-1} \in H^{N+2p^2-1}({}^2M^{N+2p^2-2})$.

10. A non-vanishing theorem

We shall say that an r-admissible chain complex

$$C_r \to \cdots \to C_1 \to C_0 \tag{10.1}$$

with a realization

$$\begin{array}{ccccccccc} B_r & & & & & & B_2 & & B_1 \\ \uparrow & & & & & & \uparrow & & \uparrow \\ F_r & \to & F_{r-1} & \to & \cdots & \to & F_1 & \to & F_0 \end{array}$$

is *canonical* if there are injections $j_t : \Omega^{-kt} F_0 \to \Omega B_t$, $\bar{j}_t : \Omega^{-kt} B_1 \to B_{t+1}$ for

$1 \leqq t \leqq r-1$, and a fixed integer $k>0$ such that

$$\begin{array}{ccc} \Omega^{-kt}F_0 & \xrightarrow{\Omega^{-kt}f_1} & \Omega^{-kt}B_1 \\ j_t \downarrow & f_{t,t+1} & \downarrow \bar{j}_t \\ \Omega B_t & \longrightarrow & B_{t+1} \end{array}$$

is commutative. Then, the following diagram is commutative up to a homotopy

$$(10.2)_t \quad \begin{array}{ccccccc} \Omega^{-kt+1}B_1 & \xrightarrow{\Omega^{-kt}\tau_1} & \Omega^{-kt}F_1 & \xrightarrow{\Omega^{-kt}i_1} & \Omega^{-kt}F_0 & \xrightarrow{\Omega^{-kt}f_1} & \Omega^{-kt}B_1 \\ \Omega\bar{j}_t \downarrow & \tau_{t,t+1} & \downarrow \bar{j}_t & i_{t,t+1} & \downarrow j_t & f_{t,t+1} & \downarrow \bar{j}_t \\ \Omega B_{t+1} & \longrightarrow & G_{t,t+1} & \longrightarrow & \Omega B_t & \longrightarrow & B_{t+1}. \end{array}$$

An S-spectrum $\boldsymbol{M}=\{M_N\}$ is said to be *of the type l* for an integer $l \geqq 0$, if $H^{N+l}(M_N) \neq 0$ and $H^i(M_N)=0$ for $i<N$, $i>N+l$.

Then, we have the following non-vanishing theorem for the iterated powers of a certain stable homotopy element.

Therem 10. 1. *Let $\boldsymbol{M}$ be an S-spectrum of the type l, and $\alpha \in \pi_k(\boldsymbol{M}, \boldsymbol{M})$ (i.e. $\lim [M_{N+k}, M_N]$), $k>l$, be an element such that*

$$\Phi_\alpha^{1,0} e^m = x e^{m+k} \qquad (\textit{mod zero})$$

for the elements $e^m \in H^m(M_N)$ and $e^{m+k} \in H^{m+k}(M_{N+k})$, $m=N+l$, corresponding to the same element $e \in H^*(\boldsymbol{M})$ (i.e., $s^{*k}(e^{m+k})=e^m$ for the suspension isomorphism $s^*: H^*(M_{N+1}) \to H^*(M_N)$), and $x \neq 0$, where $\Phi^{1,0}$ is a stable cohomology operation associated with an r-admissible canonical chain complex (10. 1). *Then we have $\alpha^t \neq 0$ for $t \leqq r$.*

Proof. There is a map $\varphi: M_N \to F_0$ representing e^m, i.e., $\varphi^*(\iota)=e^m$ for the fundamental class $\iota \in H^*(F_0)$. Since, by the assumption, $\Phi_\alpha^{1,0}e^m = xe^{m+k}$, a map $\psi: M_{N+k} \to \Omega B_1$ representing $\Phi_\alpha^{1,0}e^m$ can be factored into $j_1 \circ \psi'$ by a map $\psi': M_{N+k} \to \Omega^{-k}F_0$ which is homotopic to $\Omega^{-k}\varphi$, and the injection $j_1: \Omega^{-k}F_0 \to \Omega B_1$. Hence, by the commutativity of $(10.2)_1$ we conclude that $\Phi_\alpha^{2,1}e^{m+k} = (\Omega\bar{j}_1)^*\Phi_\alpha^{1,0}e^{m+k} = xe^{m+2k}$.

On the other hand, since $e^{m+k} = s^{*-k}(e^m)$ and $H^i(M_N)=0$ for $i>m$, $\alpha^*\Phi_\alpha^{1,0}e^m = x\alpha^*e^{m+k}=0$ and $\Phi^{2,0}e^m=0$. Hence, $\Phi_{\alpha^2}^{2,0}e^m$ is defined and, by Proposition 4. 3, we have

$$\Phi_{\alpha^2}^{2,0}e^m \equiv \Phi_\alpha^{2,1}\Phi_\alpha^{1,0}e^m = x\Phi_\alpha^{2,1}e^{m+k} = x^2 e^{m+2k} \neq 0$$

mod $\alpha^*[M_{N+k}, B_2] + (\Omega f_2)_*[M_{N+2k}, F_2]$. But, since $\alpha \in \pi_k(\boldsymbol{M}, \boldsymbol{M})$ and $\boldsymbol{M}$ is of the type l for $l<k$, we have $\alpha^*[M_{N+k}, B_2]=0$ and $(\Omega f_2)_*[M_{N+2k}, F_2]=0$. Therefore we have $\alpha^2 \neq 0$.

Assume, inductively, that $\Phi^{t-1,0}_{\alpha^{t-1}}e^m = x^{t-1}e^{m+(t-1)k}$ (mod zero), for the elements $e^m \in H^m(M_N)$ and $e^{m+(t-1)k} \in H^{m+(t-1)k}(M_{N+(t-1)k})$ corresponding to the same element $e \in H^*(\boldsymbol{M})$. Then, a map representing $\Phi^{t-1,0}_{\alpha^{t-1}}e^m$ is factored into $j_{t-1}\cdot\psi'_{t-1}$ by a map $\psi'_{t-1}: M_{N+(t-1)k} \to \Omega^{-(t-1)k}F_0$ which is homotopic to $\Omega^{-(t-1)k}\varphi$, and the injection $j_{t-1}: \Omega^{-(t-1)k}F_0 \to \Omega B_{t-1}$. Hence, we have $\Phi^{t,t-1}_{\alpha}e^{m+(t-1)k} = (\Omega\bar{j}_{t-1})^*\Phi^{1,0}_{\alpha}e^{m+(t-1)k} = xe^{m+tk}$, by the commutativity of $(10.2)_{t-1}$.

On the other hand, similarly to the above argument, we have $\alpha^*\Phi^{t-1,0}_{\alpha^{t-1}}e^m = 0$ and $\Phi^{t,0}e^m = 0$. Hence $\Phi^{t,0}_{\alpha^t}e^m$ is defined and, by Proposition 4.3, we have

$$\Phi^{t,0}_{\alpha^t}e^m \equiv \Phi^{t,t-1}_{\alpha}\Phi^{t-1,0}_{\alpha^{t-1}}e^m = x^{t-1}\Phi^{t,t-1}_{\alpha}e^{m+(t-1)k} = x^t e^{m+tk} \neq 0$$

mod $\alpha^*[M_{N+(t-1)k}, \Omega B_t] + (\Omega f_t)_*[M_{N+tk}, F_t]$. But, since $\boldsymbol{M}$ is of the type l, and $l<k$, we have $\alpha^*[M_{N+(t-1)k}, \Omega B_t] = 0$ and $(\Omega f_t)_*[M_{N+tk}, F_t] = 0$.

Thus, we have $\alpha^t \neq 0$ for $t \leqq r$. q.e.d.

REMARK. The assumption that the chain complex (10.1) is canonical and that $\boldsymbol{M}$ is of type l are not essential, but they simplify the proof.

11. Non-triviality of α^t and β^t

It follows immediately from the definition that

Lemma 11.1. *The chain complex* $(6.1)_i$ *is canonical for all* $i \geqq 0$.

As a direct consequences of Theorem 10.1, we have the following non-triviality theorems for α^t and β^t.

Theorem 11.2. *For all* $t \geqq 1$, $\alpha^t \neq 0$ *in* ${}^1\pi_*$, *where* $\alpha \in {}^1\pi_{2p-2}$ *is the element defined by* (9.3).

Proof. The S-spectrum ${}^1\boldsymbol{M}$ is of type 1 and $\alpha \in \pi_k({}^1\boldsymbol{M}, {}^1\boldsymbol{M})$ for $k = 2p-2 > 1$. While, since ${}^1f_1^*(\iota_j) = Q_j\iota$, $j=0,1$, by (9.3) and Proporition 3.3, we have

$$\begin{aligned} {}^1\Phi^{1,0}_{\alpha}e^m &= Q_{0,\alpha}e^m + Q_{1,\alpha}e^m = \Delta_\alpha e^m + (\mathcal{P}^1\Delta - \Delta\mathcal{P}^1)_\alpha e^m \\ &\equiv \mathcal{P}^1_\alpha\Delta e^m - \Delta\mathcal{P}^1_\alpha e^m = (-1)^{m+1}e^{m+k} \end{aligned}$$

for $m = N+k+1$, mod $(Q_0+Q_1)H^m({}^1M^{N+k}) + \mathcal{P}^1H^{m+1}({}^1M^{N+k}) + \Delta\mathcal{P}^1H^m({}^1M^{N+k}) + \alpha^*H^{m+k}({}^1M^N) = 0$. (The fact that $(\theta+\theta')_\alpha(u) \equiv \theta_\alpha(u) + \theta'_\alpha(u)$ *mod* $\operatorname{Im}\theta + \operatorname{Im}\theta' + \operatorname{Im}\alpha^*$ for operations of the first kind θ, θ' is easily verified).

Hence, the condition of Theorem 10.1 is satisfied by the chain complex $(6.1)_1$ and the element α. Thus, we have $\alpha^t \neq 0$ for all $t \geqq 1$. q.e.d.

Theorem 11.3. *For all* $t \geqq 1$, $\beta^t \neq 0$ *in* ${}^2\pi_*$, *where* $\beta \in {}^2\pi_{2p^2-2}$ *is the* *element defined by* (9.4).

Proof. The S-spectrum ${}^2\boldsymbol{M}$ is of type $2p$ and $\beta\in\pi_k({}^2\boldsymbol{M},{}^2\boldsymbol{M})$ for $k=2p^2-2>2p$. Since ${}^2f_1^*(\iota_j)=Q_j\iota$, $j=0,1,2$, similarly to the proof of the above Theorem, we have

$$
\begin{aligned}
&{}^2\Phi_\beta^{1,0}e^m = (-1)^m e^{m+k} \qquad \text{for} \quad m=N+2p\,,\\
&\bmod\,(Q_0+Q_1+Q_2)H^m({}^2M^{N+k})+\mathcal{P}^pH^{m+2p-1}({}^2M^{N+k})+\mathcal{P}^1\Delta\mathcal{P}^pH^m({}^2M^{N+k})\\
&\qquad +\Delta\mathcal{P}^2\mathcal{P}^pH^m({}^2M^{N+k})+\beta^*H^{m+k}({}^2M^N) = 0\,.
\end{aligned}
$$

Thus, the condition of Theorem 10.1 is fulfilled by the chain complex $(6.1)_2$ and the element β. Hence, we have $\beta^t\neq 0$ for all $t\geqq 1$, in ${}^2\pi_*$.
q.e.d.

Finally, we have the following direct consequences of Theorems 11.2 and 11.3.

Let $\delta\in{}^1\pi_{-1}$ be the elements such that $\delta^*e_1^N=e_2^N$ for the generators $e_1^N\in H^N({}^1M^N)$ and $e_2^N\in H^N({}^1M^{N-1})$ [9]. In [9], we proved that $2\alpha\delta\alpha=\alpha^2\delta+\delta\alpha^2$ and this implies that $\alpha^{rp}\delta=\delta\alpha^{rp}$. Then,

Proposition 11.4. *For all* $t\geqq 1$, $\delta\alpha^t\neq 0$ *and* $\alpha^t\delta\neq 0$ *in* ${}^1\pi_*$.

Proof. By Theorem 11.2 and Proposition 3.2, we have

$${}^1\Phi_{\delta\alpha^t}^{t,0}e_1^{N+1} = {}^1\Phi_{\alpha^t}^{t,0}\delta^*e_1^{N+1} = {}^1\Phi_{\alpha^t}^{t,0}e_2^{N+1} = (-1)^{tN}e^{N+tk+1}\neq 0$$

for the generators $e_1^{N+1}\in H^{N+1}({}^1M^{N+1})$, $e_2^{N+1}\in H^{N+1}({}^1M^N)$ and $e^{N+tk+1}\in H^{N+tk+1}({}^1M^{N+tk})$. Hence, we have $\delta\alpha^t\neq 0$ for all $t\geqq 1$. If $\alpha^t\delta=0$ for some $t>1$, then we have $0=\alpha^{rp-t}\alpha^t\delta=\delta\alpha^{rp}\neq 0$ for r such that $rp>t$. This is a contradiction. q.e.d.

REMARK. By making use of the result of Toda [7], [8], we can conclude that $\alpha^t\delta\alpha\delta\neq 0$ for all $t\geqq 1$ [9]. But, we can not prove this fact using our method only.

Let $\bar\delta\in{}^2\pi_{1-2p}$ be the element such that $\bar\delta^*e_1^{N+i}=e_2^{N+i}$, $i=0,1$, for the generators $e_1^{N+i}\in H^{N+i}({}^2M^N)$ and $e_2^{N+i}\in H^{N+i}({}^2M^{N-2p+1})$. Then,

Lemma 11.5. $2\beta\bar\delta\beta=\beta^2\bar\delta+\bar\delta\beta^2$, *if* $p\geqq 5$.

Proof. By the structure of ${}^1\pi_*$ [9] and the exactness of (9.2), ${}^2\pi_{4p^2-2p-3}=\{\beta^2\bar\delta\}+\{\bar\delta\beta^2\}$, if $p\geqq 5$. So that the proof is carried out similarly to that of Proposition 5.1 of [9] using the Adem's relation $2\mathcal{P}^p\mathcal{P}^1\mathcal{P}^p=\mathcal{P}^p\mathcal{P}^p\mathcal{P}^1+\mathcal{P}^1\mathcal{P}^p\mathcal{P}^p$ instead of $2\mathcal{P}^1\Delta\mathcal{P}^1=\mathcal{P}^1\mathcal{P}^1\Delta+\Delta\mathcal{P}^1\mathcal{P}^1$. q.e.d.

By the above Lemma, easily we have $\beta^{rp}\bar\delta=\bar\delta\beta^{rp}$. So that similarly to Proposition 11.4, we have

Proposition 11.6. *For all* $t\geqq 1$, $\bar\delta\beta^t\neq 0$ *and* $\beta^t\bar\delta\neq 0$ *in* ${}^2\pi_*$, *if* $p\geqq 5$.

REMARK. It seems true that $\beta^t\bar{\delta}\beta\bar{\delta}\neq 0$ for all $t\geqq 1$, but we have no idea to prove it.

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