Osaka Mathematical Journal
Vol. 8 No. 1, June, 1956.

On Quotient Rings

By Yuzo UTUMI

An extension ring S of a ring T is called a left quotient ring of T if for any two elements $x \neq 0$ and y of S there exists an element a of T such that $ax \neq 0$ and ay belongs to T. Let R be a ring without total right zero divisors. Then R has always a unique maximal left quotient ring, and moreover the maximal left quotient ring of a total matrix ring of finite degree over R is a total matrix ring of the same degree over the maximal left quotient ring of R.

A left ideal $\mathfrak{l}$ of R is called an M-ideal if it contains every element x for which there exists a left ideal $\mathfrak{m}$ of R satisfying the condition that (1) $\mathfrak{m}x \subseteq \mathfrak{l}$ and (2) R is a left quotient ring of $\mathfrak{m}$. When S is a left quotient ring of R, M-ideals of R and those of S correspond one-one in a definite way. A left ideal $\mathfrak{l}$ of R is said to be complemented if there exists a left ideal $\mathfrak{l}'$ such that $\mathfrak{l}$ is a maximal one among left ideals which have zero intersection with $\mathfrak{l}'$. Every complemented left ideal is an M-ideal, but the converse is not true in general. In a ring without total right zero divisors, every M-ideal is complemented if and only if the ring has the zero left singular ideal. Another example of M-ideals is the annihilator left ideals. A sufficient condition for that every M-ideal of a ring with zero left singular ideal is an annihilator left ideal, is that the maximal left quotient ring coincides with the maximal right quotient ring.

Every semisimple I-ring has zero singular ideals and hence it has the left and the right maximal quotient rings. We discuss especially two types of semisimple I-rings, i.e., primitive rings with nonzero socle, and semisimple weakly reducible rings. Let P be a primitive ring with nonzero socle. Then the maximal left quotient ring of P is right completely primitive. Thus, the left and the right maximal quotient rings of P coincide if and only if P satisfies the minimum condition. Let W be a semisimple weakly reducible ring. The left and the right maximal quotient rings of W always coincide and is also semisimple weakly reducible. In particular, if W is plain then its maximal quotient ring is strongly regular. This implies that the (nilpotency) index of a total matrix ring of degree m over a semisimple I-ring of index n is mn.

The writer wishes to express his gratitude to Professor Goro Azumaya for his helpful suggestions.

1. For any subset A of a ring S and any family B of right operators of S the set of all the elements in S satisfying $xB \subseteq A$ is denoted by $(A/B)^S$. In particular, when B consists of the right multiplications of all elements in a subset C of S we write it as $(A/C)^S$.

(1.1) Let R be a subring of a ring S. We say that S is a (*left*) *quotient ring* of R if for any pair of elements $x \neq 0$ and y in S there exists an element a in R such that $ay \in R$ and $ax \neq 0$. Notation: $S \geq R$.

We may also define a similar concept by a slightly weaker condition: We write $S(\geq)R$ if any nonzero $x \in S$ there is an element $a \in R$ such that $0 \neq ax \in R$. Of course, $S \geq R$ implies $S(\geq)R$. But the following example shows that the converse is false. Let K be a field and S the ring $K[x]/(x^4)$. We denote the subring of S generated by $\bar{1}$, $\bar{x}^2$ and $\bar{x}^3$ as R. Then $S(\geq)R$, while no $\bar{a} \in R$ satisfies $\bar{a}\bar{x} \in R$ and $\bar{a}\bar{x}^3 \neq 0$ simultaneously.

Our main object is the quotient ring in the sense of (1.1).

(1.2) *Let $S \geq R$. The only homomorphism of S into itself which leaves R invariant is the identity mapping.*

If $x\theta \neq x$ for some $x \in S$, there would exist an element $a \in R$ such that $ax \in R$ and $a(x\theta - x) \neq 0$. But then $a(x\theta) = (a\theta)(x\theta) = (ax)\theta = ax$. This contradiction shows that $x\theta = x$ for every $x \in S$.

(1.3) *Let $S \geq R$. An element x belongs to the center of S if it is commutative with every element in R.*

Assume $xy \neq yx$. Then $ay \in R$ and $a(xy - yx) \neq 0$ for some $a \in R$. $axy = xay = ayx$. This is a contradiction.

(1.4) *Let $S \geq R$. For any finite number of elements $x_1 \neq 0, x_2, \cdots, x_n$ in S there exists an element $a \in R$ such that $ax_1, ax_2, \cdots, ax_n \in R$ and $ax_1 \neq 0$.*

The assertion is evidently true if $n = 1$. Let $n > 1$. We assume that $bx_1, bx_2, \cdots, bx_{n-1} \in R$, $bx_1 \neq 0$ for some $b \in R$. Since $S \geq R$ there is $c \in R$ such that $cbx_n \in R$ and $cbx_1 \neq 0$. Therefore $cb, cbx_1, \cdots, cbx_n \in R$ and $cbx_1 \neq 0$.

(1.5) *Let $S \supseteq R \supseteq T$. Then $S \geq R \geq T$ if and only if $S \geq T$.*

The "if" part is clear from the definition. To prove the "only if" part let $S \ni x (\neq 0)$, y. Then ax, $ay \in R$ and $ax \neq 0$ for some $a \in R$. Hence ca, $cay \in T$ and $cax \neq 0$ for some $c \in T$. This implies $S \geq T$.

We denote by $S^{\blacktriangle}$ the set of all left ideals $\mathfrak{l}$ satisfying $S \geq \mathfrak{l}$.

(1.6) *Let $\mathfrak{l}$ be a left ideal of S. Then $\mathfrak{l}\in S^{\blacktriangle}$ if (and only if) for any elements $x\neq 0$ and y in S there exists an element a in S such that $ay \ni \mathfrak{l}$ and $ax\neq 0$.*

In fact, it follows from the assumption that there is moreover an element $b\in S$ such that $ba\in\mathfrak{l}$ and $bax\neq 0$. Since $(ba)y=b(ay)\in\mathfrak{l}$ we have $S^{\blacktriangle}\ni\mathfrak{l}$.

(1.7) *If $S\geqq R$ and $\mathfrak{m}\in S^{\blacktriangle}$, then $S\geqq R\frown\mathfrak{m}$, $R\mathfrak{m}$.*

Let $S\ni x(\neq 0)$, y. Then $ax\neq 0$, $ay\in\mathfrak{m}$ for some $a\in\mathfrak{m}$. Hence ba, $bay\in R$ and $bax\neq 0$ for some $b\in R$. We see that ba, $bay\in(R\frown\mathfrak{m})\frown R\mathfrak{m}$. Therefore $S\geqq R\frown\mathfrak{m}$ and $S\geqq R\mathfrak{m}$.

(1.8) *Let $S\geqq R$ and let $\mathfrak{m}_x\in S^{\blacktriangle}$ be preassigned to each $x\in R$. Then $\Sigma_{x\in R}\mathfrak{m}_x x\in S^{\blacktriangle}$.*

Let $S\ni x(\neq 0)$, y. Then $ay\in R$ and $ax\neq 0$ for some $a\in R$. We set $\mathfrak{m}=\mathfrak{m}_a\frown\mathfrak{m}_{ay}$. By (1.7), $\mathfrak{m}\in S^{\blacktriangle}$. Hence $bax\neq 0$ for some $b\in\mathfrak{m}$ and then $ba\in\mathfrak{m}_a a$, $bay\in\mathfrak{m}_{ay}ay$.

(1.9) *Let $S^{\blacktriangle}\ni R$, T. If θ is an S-left homomorphism of R into S then $(T/\theta)^R\in S^{\blacktriangle}$.*

Let $S\ni x(\neq 0)$, y. Then $ay\in R$, $ax\neq 0$ for some $a\in R$. Moreover, $b(a\theta)$, $b((ay)\theta)\in T$ and $bax\neq 0$ for some $b\in T$. Hence ba, $bay\in(T/\theta)^R$. Thus $(T/\theta)^R\in S^{\blacktriangle}$.

This proof shows also the following

(1.10) *Let $S\geqq R$. If θ is an R-left homomorphism of R into S, then $(R/\theta)^R\in R^{\blacktriangle}$.*

(1.11) *Let $\bar{S}$ be a ring. The following conditions are equivalent*:

(1) *There exists a ring T such that $S\geqq T$ or $T\geqq S$.*

(2) *$S\geqq S$.*

(3) *S has no total right zero divisors, that is, $Sx=0$ implies $x=0$.*

This is evident from the definition and (1.5).

By virtue of the above lemmas the R. E. Johnson's method[1] for constructing the extended centralizer is verbatim applicable to our case.

Construction of $\bar{S}$. Let S be a ring such that $S\geqq S$. Then $S^{\blacktriangle}$ is non-vaid. We denote by $\mathfrak{A}_S$ the set of all S-left homomorphisms each of which is defined on a left ideal in $S^{\blacktriangle}$ and has values in S. The definition domain of $\theta\in\mathfrak{A}_S$ is denoted as M_θ. When $M_\theta=M_{\theta'}$, we define the addition by $x(\theta+\theta')=x\theta+x\theta'$. When $M_\theta\theta\subseteq M_{\theta'}$, the multiplication is defined by $x(\theta\theta')=(x\theta)\theta'$. For θ, $\theta'\in\mathfrak{A}_S$ if there exists $\mathfrak{l}\in S^{\blacktriangle}$ such that $\mathfrak{l}\subseteq M_\theta\frown M_{\theta'}$ and θ, $\bar{\theta}'$ coincide on $\mathfrak{l}$, we say that θ and θ' are

1) See [8].

equivalent. Then this relation is reflexive, symmetric and transitive. We denote the equivalence class containing θ as $\bar{\theta}$ and the set of all the classes as $\bar{S}$. By (1.7), (1.9) it is easy to see that $\bar{S}$ forms a ring in a natural way. For any $x \in S$ the right multiplication x_r belongs to $\mathfrak{A}_S$. We identify x with $\bar{x}_r$ and regard $\bar{S}$ as an extension ring of S.

(1.12) *If* $x \in M_\theta$, *then* $x\theta = x\bar{\theta}$.

This follows easily from that $y(x\theta) = (yx)\theta$ for every $y \in S$.

(1.13) $\bar{S} \geq S$.

In fact, let $\bar{\theta}, \bar{\varphi} \in S$ and $\bar{\varphi} \neq 0$. By (1.7), $M_\theta \cap M_\varphi \in S^\blacktriangle$. Hence $a\varphi \neq 0$ for some $a \in M_\theta \cap M_\varphi$. Then $a\bar{\theta} = a\theta \in S$ and $a\bar{\varphi} = a\varphi \neq 0$ by (1.12). This implies $\bar{S} \geq S$.

Theorem 1. *Let $T \geq S$. Then T is isomorphic, over S, to $\bar{S}$ if and only if T satisfies either the following condition* (1) *or* (2).

In this case, we say that T is the (*left*) *maximal quotient ring* of S.

CONDITION (1). *For any $\theta \in \mathfrak{A}_S$ there are $x \in T$ and $\mathfrak{m} \in S^\blacktriangle$ such that $\mathfrak{m} \subseteq M_\theta$ and $y\theta = yx$ for every $y \in \mathfrak{m}$.*

CONDITION (2). *If $R \geq S$, then there exists an isomorphism, over S, of R into T.*

Proof. To see the "only if" part it is sufficient to prove that $\bar{S}$ satisfies these conditions. (1) is evident from (1.12). Let $R \geq S$. By (1.10), $(S/x)^S \in S^\blacktriangle$ for every $x \in R$. Hence the right multiplication θ_x of x on $(S/x)^S$ belongs to $\mathfrak{A}_S$. Associating each $x \in R$ with $\bar{\theta}_x \in \bar{S}$ we obtain an isomorphism, over S, of R into $\bar{S}$. Therefore $\bar{S}$ satisfies (2). If R satisfies the condition (1), this isomorphism is onto. This proves the first half of the "if" part of Theorem. Finally, let T satisfy (2). Then, since $\bar{S} \geq S$ by (1.13), $\bar{S}$ is isomorphic, over S, into T. On the other hand, since $T \geq S$ and $\bar{S}$ satisfies the condition (2), T is isomorphic, over S, into $\bar{S}$. Then product of these isomorphisms is the identity mapping of $\bar{S}$ by (1.2). It follows from this that $\bar{S}$ and T are isomorphic over S. This completes the proof.

The following (1.14)–(1.17) are easily proved by Theorem 1 and we omit their proofs.

(1.14) *If $T \geq S$, then $\bar{T} \simeq \bar{S}$ over S.*

(1.15) $\bar{\bar{S}} = \bar{S}$.

(1.16) *If $T \geq S$ and $T = \bar{T}$, then $T \simeq \bar{S}$ over S.*

(1. 17) *Every automorphism of* S *can be extended uniquely to that of* $\bar{S}$.

2. (2. 1) *Let* $\{S_\alpha\}$ *be a family of rings with the property* $S_\alpha \geq S_\alpha$ *for every* α. *Then* $\Sigma_{\oplus}^{c} \bar{S}_\alpha$ *is the maximal quotient ring of* $\Sigma_{\oplus} S_\alpha$, *where* $\Sigma_{\oplus}^{c}$ *denotes the complete direct sum, while* $\Sigma_{\oplus}$ *the* (*restricted*) *direct sum.*

(1) First we note that if $\Sigma_{\oplus}^{c} T_\alpha \geq R$ and $T_\alpha \geq T_\alpha$ then $T_\alpha \geq R \frown T_\alpha$. In fact, let $T_\alpha \ni x\,(\neq 0)$, y. By the assumption, $bx \neq 0$ for some $b \in T_\alpha$. Hence ab, $aby \in R$ and $abx \neq 0$ for some $a \in R$. Then ab, $aby \in R \frown T_\alpha$. Therefore $T_\alpha \geq R \frown T_\alpha$. (2) Let $T_\alpha \geq R_\alpha$ for every α. Then it is easy to see that $\Sigma_{\oplus}^{c} T_\alpha \geq \Sigma_{\oplus} R_\alpha$. (3) We set $P = \Sigma_{\oplus}^{c} \bar{S}_\alpha$. Let $\theta \in \mathfrak{A}_P$ and denote its restriction to $M_\theta \frown \bar{S}_\alpha$ as $\bar{\theta}_\alpha$. Then $\bar{\theta}_\alpha \in \mathfrak{A}_{\bar{S}_\alpha}$ since $M_\theta \frown \bar{S}_\alpha \in \bar{S}_\alpha^{\blacktriangle}$ by (1). By Theorem 1 there is $x_\alpha \in \bar{S}_\alpha$ such that $yx_\alpha = y\theta_\alpha$ for every $y \in M_\theta \frown \bar{S}_\alpha$. Hence $y \Sigma_{\oplus}^{c} x_\alpha = y\theta$ for every $y \in \Sigma\,(M_\theta \frown \bar{S}_\alpha)$. By (2), $\Sigma\,(M_\theta \frown \bar{S}_\alpha) \in P^{\blacktriangle}$. Therefore it follows from Theorem 1 that $P = \bar{P}$ because of $P \geq P$. By (2), $P \geq \Sigma\, S_\alpha$. Thus we see that $P \simeq \overline{\Sigma\, S_\alpha}$ over $\Sigma\, S_\alpha$ by (1. 16).

As a corollary of (2. 1),

(2. 2) If $\bar{S} = \mathfrak{a} \oplus \mathfrak{a}'$ where $\mathfrak{a}$ and $\mathfrak{a}'$ are two-sided ideals of $\bar{S}$, then $\mathfrak{a}$ *is the maximal quotient ring of* $\mathfrak{a} \frown S$.

From (1) of the proof of (2. 1) we get $\mathfrak{a} \geq \mathfrak{a} \frown S$. Now $S = \bar{\bar{S}} = \bar{\mathfrak{a}} \oplus \bar{\mathfrak{a}}'$. Hence $\mathfrak{a} = \bar{\mathfrak{a}}$. Owing to (1. 16) this implies $\mathfrak{a} \simeq \overline{\mathfrak{a} \frown S}$ over $\mathfrak{a} \frown S$.

We use the notation R_n for the total matrix ring of degree n over a ring R.

(2. 3) *If* $S \geq S$, *then* $(\bar{S})_n$ *is the maximal quotient ring of* S_n.

First, we assume that S has a unit element. (1) $S \geq T$ implies $S_n \geq T_n$. In fact, let $S_n \ni A_k = \Sigma\, a_{ij}^{(k)} e_{ij}$ for $k = 0, 1$ and let $a_{pq}^{(0)} \neq 0$. Then there is $a \in T$ such that $aa_{pq}^{(0)} \neq 0$ and $aa_{p\mu}^{(1)} \in T$ $(\mu = 1, \cdots, m)$. Hence ae_{pp}, $ae_{pp}A_1 \in T_n$ and $ae_{pp}A_0 \neq 0$. This shows $S_n \geq T_n$. (2) If $(S_n)^{\blacktriangle} \ni R$, then $\mathfrak{m}_n \leq R$ for some $\mathfrak{m} \in S^{\blacktriangle}$. In fact, we denote by $\mathfrak{m}_k$ the set of all the elements of S each of which is a coefficient of a matrix in $R \frown S_n e_{kk}$. This is evidently a left ideal of S. Let $S \ni x\ (\neq 0)$, y. By the assumption there is a matrix $A = \Sigma\, a_{ij} e_{ij} \in R$ such that Ae_{kk}, $A(ye_{kk}) \in R$ and $A(xe_{kk}) \neq 0$. Hence $a_{ik}x \neq 0$ for some i. Since a_{ik}, $a_{ik}y \in \mathfrak{m}_k$, this implies $\mathfrak{m}_k \in S^{\blacktriangle}$. By (1. 7), $\mathfrak{m} = \bigwedge \mathfrak{m}_k \in S^{\blacktriangle}$. For any element $y \in \mathfrak{m}$ there exists a matrix $D \in R \frown S_n e_{kk}$ whose $(1, k)$-coefficient is y. $ye_{jk} = e_{jl}D \in R$. Therefore $\mathfrak{m}_n \leq R$. (3) Let $\theta \in \mathfrak{A}_{S_n}$. By (2), $\mathfrak{m}_n \leq M$ for some $\mathfrak{m} \in S^{\blacktriangle}$. For any $x \in \mathfrak{m}$ we denote $(xe_{1k})\theta = e_{11}(xe_{1k})\theta$ as $\Sigma_j\,(x\theta_{kj})\,e_{1j}$. Then θ_{kj} are S-left homomorphisms of $\mathfrak{m}$ into S so that they belong to $\mathfrak{A}_S$. Hence there are $a_{kj} \in \bar{S}$ such that $x\theta_{kj} = xa_{kj}$ for every $x \in \mathfrak{m}$. Therefore, for every $\Sigma\, x_{ik}e_{ik} \in \mathfrak{m}_n$, $(\Sigma\, x_{ik}e_{ik})\,\theta = \Sigma_{ik}\, e_{i1}\,(x_{ik}e_{1k})\,\theta = \Sigma_{ikj}\, e_{i1}\,(x_{ik}\theta_{kj})\,e_{1j}$ $= \Sigma_{ij}\,(\Sigma_k\, x_{ik}a_{kj})\,e_{ij} = (\Sigma\, x_{ik}e_{ik})(\Sigma\, a_{ik}e_{ik})$. This shows that $(\bar{S})_n \sim \overline{(S_n)}$ over S_n

since $(\bar{S})_n \geq S_n$ by (1). For S without unit element we denote by S' the subring of $\bar{S}$ generated by S and the unit element of $\bar{S}$. Then $S \simeq S'$ over S by (1.14). Moreover, $\overline{(S_n')} \simeq \overline{(S_n)}$ over S_n since $S_n' \geq S_n$ by (1). From these facts it follows that $(\bar{S})_n \simeq \overline{(S_n)}$ over S_n as required.

3. In this section we shall consider some correspondence between ideals of a ring and those of its quotient ring.

Let $R \leq S$ and $\mathfrak{l}$ be an R-left submodule of S. We denote by $\Delta_R^S \mathfrak{l}$ the set of all elements $x \in S$ satisfying $(\mathfrak{l}/x)^R \in R^\blacktriangle$.

(3.1) *$\Delta_R^S \mathfrak{l}$ is a left ideal of S containing $\mathfrak{l}$.*

For, let $\Delta_R^S \mathfrak{l} \ni x$ and $S \ni y$. Since $(\mathfrak{l}/x)^R \in R^\blacktriangle$ we see that $((\mathfrak{l}/x)^R/y)^R \in R^\blacktriangle$ by (1.10). Now $((\mathfrak{l}/x)^R/y)^R yx \subseteq (\mathfrak{l}/x)^R x \subseteq \mathfrak{l}$. Hence $(\mathfrak{l}/yx)^R \in R^\blacktriangle$, or $yx \in \Delta_R^S \mathfrak{l}$.

(3.2) $\Delta_R^S (\mathfrak{l} \cap \mathfrak{l}') = \Delta_R^S \mathfrak{l} \cap \Delta_R^S \mathfrak{l}'$.

This is easy to verify by (1.5), (1.7).

(3.3) $\Delta_R^S (\mathfrak{l} \cap R) \supseteq \mathfrak{l}$.

If $x \in \mathfrak{l}$, then $(R/x)^R \in R^\blacktriangle$ by (1.10). This means $x \in \Delta_R^S (\mathfrak{l} \cap R)$ since $(R/x)^R = (R \cap \mathfrak{l}/x)^R$.

(3.4) $\Delta_R^S (\mathfrak{l}x) \supseteq (\Delta_R^S \mathfrak{l}) x$ *for every* $x \in S$.

(3.5) *Let $R \leq S \leq T$ and $\mathfrak{l}$ be an S-left submodule of T. Then $\Delta_S^T \mathfrak{l} = \Delta_R^T \mathfrak{l}$.*

(3.6) *Let $R \leq S \leq T$. If $\mathfrak{l}$ is an R-left submodule of S then $\Delta_S^T (\Delta_R^S \mathfrak{l}) = \Delta_R^T \mathfrak{l}$.*

Since $\Delta_R^S \mathfrak{l}$ is a left ideal of S containing $\mathfrak{l}$, we see that $\Delta_S^T (\Delta_R^S \mathfrak{l}) = \Delta_R^T (\Delta_R^S \mathfrak{l}) \supseteq \Delta_R^T \mathfrak{l}$ by (3.2), (3.5). On the other hand, let $x \in \Delta_S^T (\Delta_R^S \mathfrak{l})$ or $(\Delta_R^S \mathfrak{l}/x)^R \in R^\blacktriangle$. If $y \in (\Delta_R^S \mathfrak{l}/x)^R$, then $yx \in \Delta_R^S \mathfrak{l}$; hence $(\mathfrak{l}/yx)^R \in R^\blacktriangle$. It follows from (1.8) that $\Sigma\, (\mathfrak{l}/yx)^R y \in R^\blacktriangle$, where Σ denotes the sum for all $y \in (\Delta_R^S \mathfrak{l}/x)^R$. Since $(\Sigma\, (\mathfrak{l}/yx)^R y)\, x \subseteq \mathfrak{l}$, this implies that $\Delta_R^T \mathfrak{l} \ni x$. Therefore $\Delta_S^T (\Delta_R^S \mathfrak{l}) \subseteq \Delta_R^T \mathfrak{l}$ and the equality holds.

Let $R \leq R$. A left ideal $\mathfrak{l}$ is called a (*left*) *M-ideal* if $\Delta_R^R \mathfrak{l} = \mathfrak{l}$.

(3.7) *The intersection of any collection of M-ideals in a ring is also an M-ideal.*

Let $\mathfrak{l}_\alpha$ be M-ideals. By (3.2), $\Delta_R^R (\bigwedge \mathfrak{l}_\alpha) \subseteq \Delta_R^R \mathfrak{l}_\alpha = \mathfrak{l}_\alpha$. Hence $\bigwedge \mathfrak{l}_\alpha \subseteq \Delta_R^R (\bigwedge \mathfrak{l}_\alpha)$ by (3.1). Thus $\Delta_R^R (\bigwedge \mathfrak{l}_\alpha) = \bigwedge \mathfrak{l}_\alpha$.

(3.8) *Let $R \leq S$. Then $\Delta_R^S \mathfrak{l}$ is an M-ideal of S for every R-left submodule $\mathfrak{l}$ of S.*

In fact, $\Delta_S^S(\Delta_R^S \mathfrak{l}) = \Delta_R^S \mathfrak{l}$ by (3.6).

Theorem 2. *Let $R \leq S$. The Mappings $\mathfrak{l} \to \Delta_R^S \mathfrak{l}$ and $\mathfrak{L} \to \mathfrak{L} \cap R$ are mutually receiprocal and give a 1–1 correspondence between M–ideals $\mathfrak{l}$ of R and $\mathfrak{L}$ of S.*

Proof. If $\mathfrak{l}$ is an M–ideal of R then $\Delta_R^S \mathfrak{l}$ is an M–ideal of S by (3.8). Clearly $\Delta_R^S \mathfrak{l} \cap R = \Delta_R^R \mathfrak{l} = \mathfrak{l}$ by the definition. On the other hand, if $\mathfrak{L}$ is an M–ideal of S, then $\mathfrak{L} = \Delta_S^S \mathfrak{L} = \Delta_R^S \mathfrak{L} \supseteq \Delta_R^S(\mathfrak{L} \cap R) \supseteq \mathfrak{L}$ according to (3.5), (3.2) and (3.3). Hence $\mathfrak{L} = \Delta_R^S(\mathfrak{L} \cap R)$. Moreover, $\mathfrak{L} \cap R$ is an M–ideal of R since $\Delta_R^R(\mathfrak{L} \cap R) = \Delta_R^S(\mathfrak{L} \cap R) \cap R = \mathfrak{L} \cap R$.

(3.9) *Let $R \leq S$. If $\mathfrak{l}$ is an M–ideal of R, then $\Delta_R^S \mathfrak{l}$ is the maximal left ideal of S of which intersection with S is $\mathfrak{l}$.*

From (3.3) we see that $\Delta_R^S \mathfrak{l} = \Delta_R^S(\mathfrak{L} \cap R) \supseteq \mathfrak{L}$ if $\mathfrak{L} \cap R = \mathfrak{l}$.

In the following we make mention of two special types of M–ideals, i. e., the left annihilator ideals and the complemented left ideals.

By $l_R(A)$ $(r_R(A))$, we mean the left (right) annihilator ideal of A in R.

(3.10) *If $R \leq R$, then every left annihilator ideal in R is an M–ideal.*

By (3.4), $(\Delta_R^R\, l(x))\, x \subseteq \Delta_R^R\,(l(x)\, x) = 0$ for every $x \in R$. Since $\Delta_R^R\, l(x) \supseteq l(x)$, we have $\Delta_R^R\, l(x) = l(x)$. According to (3.7), every left annihilator ideal is an M–ideal.

(3.11) *Let $R \leq S$. If $\mathfrak{l}$ is a left annihilator ideal in R, then $\Delta_R^S \mathfrak{l}$ is also a left annihilator ideal in S.*

We assume $\mathfrak{l} = l_R(A)$. Then $l_S(A)$ is an M–ideal in S. Hence $l_S(A)$ $\Delta_R^S(l_S(A) \cap R) = \Delta_R^S\, l_R(A) = \Delta_R^S \mathfrak{l}$ by Theorem 2.

We may define a *right quotient ring* in an obvious way.

(3.12) *Let S be a left and right quotient ring of R. If $\mathfrak{L}$ is a left annihilator ideal in S, then $\mathfrak{L} \cap R$ is also a left annihilator ideal in R.*

Let $x \in r_R(\mathfrak{L} \cap R)$. Then $0 = \Delta_R^S((\mathfrak{L} \cap R)\, x) \supseteq (\Delta_R^S(\mathfrak{L} \cap R))\, x \supseteq \mathfrak{L} x$ by (3.4), (3.3). Hence $x \in r_R(\mathfrak{L})$. Therefore $r_R(\mathfrak{L} \cap R) = r_R(\mathfrak{L})$. Similarly we see that $l_R(r_S(\mathfrak{L}) \cap R) = l_R(r_S(\mathfrak{L}))$. Thus $l_R(r_R(\mathfrak{L} \cap R)) = l_R(r_R(\mathfrak{L})) = l_R(r_R(\mathfrak{L}) \cap R) = l_R(r_S(\mathfrak{L})) = l_S(r_S(\mathfrak{L})) \cap R = \mathfrak{L} \cap R$ and our assertion is proved.

For given left ideal $\mathfrak{l}$ of R a left ideal of R is called a *complement* of $\mathfrak{l}$ if it is the maximal one among the left ideals having the zero intersections with $\mathfrak{l}$. We denote it by $\mathfrak{l}^c$. Of course, $\mathfrak{l}^c$ is not uniquely determined by $\mathfrak{l}$. A left ideal which is a complement of some left ideal is called a *complemented left ideal*. We use the notation $\mathfrak{l}^{cc}$ for $(\mathfrak{l}^c)^c$ containing $\mathfrak{l}$.

(3. 13) *Let* $R\leqq R$. *Any complemented left ideal of* R *is an* M*-ideal.*

In fact, $\Delta_R^R(\mathfrak{l}^c)=\mathfrak{l}^c$ since $\Delta_R^R(\mathfrak{l}^c)\cap\mathfrak{l}\subseteqq\Delta_R^R(\mathfrak{l}^c)\cap\Delta_R^R\mathfrak{l}=\Delta_R^R(\mathfrak{l}^c\cap\mathfrak{l})=0$ and $\mathfrak{l}^c\subseteqq\Delta_R^R(\mathfrak{l}^c)$.

(3. 14) *Let* $R\leqq S$. *If* $\mathfrak{l}$ *is a complemented left ideal in* R, *then* $\Delta_R^S\mathfrak{l}$ *is also a complemented left ideal in* S.

We may assume that $\mathfrak{l}=\mathfrak{l}^{cc}$. Clearly $\Delta_R^S\mathfrak{l}\cap\Delta_R^S(\mathfrak{l}^c)=\Delta_R^S(\mathfrak{l}\cap\mathfrak{l}^c)=0$. On the other hand, if $\mathfrak{L}'$ is a left ideal of S such that $\mathfrak{L}'\supset\Delta_R^S\mathfrak{l}$, then $\mathfrak{L}'\cap R\supset\Delta_R^S\mathfrak{l}\cap R=\mathfrak{l}$ by (3. 9) since $\mathfrak{l}$ is an M-ideal in R by (3. 13). Thus $\mathfrak{L}'\cap\Delta_R^S(\mathfrak{l}^c)\supseteqq(\mathfrak{L}'\cap R)\cap\mathfrak{l}^c\neq 0$ since $\mathfrak{l}^c$ is also an M-ideal in R. Therefore we have $\Delta_R^S\mathfrak{l}=(\Delta_R^S(\mathfrak{l}^c))^c$.

(3. 15) *Let* $R\leqq S$. *If* $\mathfrak{L}$ *is a complemented left ideal in* S, *then* $\mathfrak{L}\cap R$ *is also a complemented left ideal in* R.

We assume that $\mathfrak{L}=\mathfrak{L}^{cc}$. Let $\mathfrak{l}'$ be a left ideal of R such that $\mathfrak{L}\cap R\subseteqq\mathfrak{l}'$ and $\mathfrak{l}'\cap(\mathfrak{L}^c\cap R)=0$. Then $\Delta_R^S\mathfrak{l}'\cap\mathfrak{L}^c=\Delta_R^S\mathfrak{l}'\cap\Delta_R^S(\mathfrak{L}^c\cap R)=0$ and $\Delta_R^S\mathfrak{l}'\supseteqq\Delta_R^S(\mathfrak{L}\cap R)=\mathfrak{L}$. Hence $\Delta_R^S\mathfrak{l}'=\mathfrak{L}$. Thus $\mathfrak{L}\cap R\supseteqq\mathfrak{l}'$ by (3. 1). Therefore $\mathfrak{L}\cap R=\mathfrak{l}'$ and $\mathfrak{L}\cap R=(\mathfrak{L}^c\cap R)^c$.

4. In this section we discuss from our point of view the cose considered by R. E. Johnson [8].

A ring R is called a (*left*) *C-ring* if $R\leqq R$ and every M-ideal of R is a complemented left ideal.

From (3. 13), (3. 14), (3. 15) and Theorem 2 we obtain immediately the following proposition.

(4. 1) *Let* $R\leqq S$. R *is a C-ring if and only if* S *is a C-ring.*

We denote by $R^\triangle$ the set of all left ideals of R each of which has a nonzero intersection with every nonzero left ideal.

(4. 2) *Let* S *be an extension ring of* R. *If every nonzero* R*-left submodule has a nonzero intersection with* R, *then* $(R/x)^\triangle\in R^\triangle$ *for every* $x\in S$.

Let $\mathfrak{l}$ be a nonzero left ideal of R. If $l_R(x)\cap\mathfrak{l}\neq 0$, then evidently $(R/x)^R\cap\mathfrak{l}\neq 0$. And if $l_R(x)\cap\mathfrak{l}=0$ we see that $\mathfrak{l}x\neq 0$ and hence $\mathfrak{l}x\cap R\neq 0$. This implies $(R/x)^R\cap\mathfrak{l}\neq 0$ again. Therefore $(R/x)^R\in R^\triangle$.

(4. 3) *Let* $\mathfrak{l}$ *be a left ideal of a ring* R. *If* $x\in\mathfrak{l}^{cc}$, *then* $(\mathfrak{l}/x)^R\in R^\triangle$.

To see this let $\mathfrak{l}'$ be any nonzero left ideal of R. First we assume that $(\mathfrak{l}^c+\mathfrak{l}'x)\cap\mathfrak{l}=0$. Then $\mathfrak{l}'x\subseteqq\mathfrak{l}^c\cap\mathfrak{l}^{cc}=0$. Hence $(\mathfrak{l}/x)^R\cap\mathfrak{l}'\neq 0$. Next let $(\mathfrak{l}^c+\mathfrak{l}'x)\cap\mathfrak{l}\ni z\neq 0$ and $z=a+b$, $a\in\mathfrak{l}^c$, $b\in\mathfrak{l}'x$. Then $a=z-b\in\mathfrak{l}^c\cap(\mathfrak{l}+\mathfrak{l}'x)\subseteqq\mathfrak{l}^c\cap\mathfrak{l}^{cc}=0$. Thus $0\neq z=b\in\mathfrak{l}\cap\mathfrak{l}'x$ so that $(\mathfrak{l}/x)^R\cap\mathfrak{l}'\neq 0$. Therefore we see that $(\mathfrak{l}/x)^R\in R^\triangle$.

Theorem 3. *If* $R\leqq R$, *the following conditions are equivalent*:

(1) R *is a C-ring.*

(2) *If* $\mathfrak{l} \in R^{\triangle}$ *and* $\mathfrak{l}x=0$, *then* $x=0$.

(3) $R^{\triangle}=R^{\blacktriangle}$.

In this case, $\mathfrak{l}^{cc}$ *is uniquely determined for every left ideal* $\mathfrak{l}$, *and is in fact the smallest M-ideal* $\Delta_R^R \mathfrak{l}$ *containing* $\mathfrak{l}$.

Proof. (1)$\Rightarrow$(2) : If $x \neq 0$, then $l_R(x)$ is an M-ideal by (3.10), hence it is a complemented left ideal. Clearly $l_R(x) \neq R$. Hence $l_R(x) \notin R^{\triangle}$. (2)$\Rightarrow$(3) : It follows immediately from the definition that $R^{\blacktriangle} \subseteq R^{\triangle}$. Let $R^{\triangle} \ni \mathfrak{l}$ and let $\mathfrak{m}$ be a nonzero $\mathfrak{l}$-left submodule of R. Then $\mathfrak{l}\mathfrak{m}$ is a nonzero left ideal by the assumption. Hence $\mathfrak{l} \cap \mathfrak{m} \supseteq \mathfrak{l} \cap \mathfrak{l}\mathfrak{m} \neq 0$, which shows that the assumption of (4.2) is satisfied by R and $\mathfrak{l}$. Thus $(\mathfrak{l}/x)^{\mathfrak{l}} \in \mathfrak{l}^{\triangle}$ for every $x \in R$. It follows easily from this that $(\mathfrak{l}/x)^{\mathfrak{l}} \in R^{\triangle}$. If $0 \neq y \in R$, then $(\mathfrak{l}/x)^{\mathfrak{l}} y \neq 0$. This shows that there exists $a \in \mathfrak{l}$ such that $ay \neq 0$, $ax \in \mathfrak{l}$. Therefore $\mathfrak{l} \in R^{\blacktriangle}$ and hence $R^{\triangle} \subseteq R^{\blacktriangle}$. Thus $R^{\triangle}=R^{\blacktriangle}$. (3)$\Rightarrow$(1) : Let $\mathfrak{l}$ be a left ideal of R and let $x \in \mathfrak{l}^{cc}$. By (4.3) we see that $(\mathfrak{l}/x)^R \in R^{\triangle}=R^{\blacktriangle}$ or $x \in \Delta_R^R \mathfrak{l}$. This implies $\mathfrak{l}^{cc} \subseteq \Delta_R^R \mathfrak{l}$. Since $\mathfrak{l}^{cc}$ is an M-ideal by (3.13), $\Delta_R^R \mathfrak{l} \subseteq \Delta_R^R \mathfrak{l}^{cc}=\mathfrak{l}^{cc} \subseteq \Delta_R^R \mathfrak{l}$ and whence $\mathfrak{l}^{cc}=\Delta_R^R \mathfrak{l}$. In particular, if $\mathfrak{l}$ itself is an M-ideal, then $\mathfrak{l}^{cc}=\mathfrak{l}$ and $\mathfrak{l}$ is a complemented left ideal. Therefore R is a C-ring as required.

Here we note that (1) the assumption $R \leq R$ follows directly from the condition (2), and (2) means that R is a ring with zero singular ideal by the terminology of R. E. Johnson [8].

(4.4) *Let R be a C-ring. Then* $S(\geq)R$ *if and only if* $S \geq R$.

The "if" part is trivial. To see the "only if" part let $S \ni x(\neq 0), y$. Then $0 \neq ax \in R$ for some $a \in R$. By (4.2), $(R/ay)^R \in R^{\triangle}$. Since R is a C-ring, $(R/ay)^R ax \neq 0$ by (2) of Theorem 3. It follows from this that there is $c \in R$ such that ca, $cay \in R$ and $cax \neq 0$. Therefore $S \geq R$.

A unitary left module over a ring with a unit element is *injective* if it is a direct summand of every unitary extension module.[2] A necessary and sufficient condition for a unitary left R-module M to be injective is that any R-left homomorphism defined on a left ideal of R and having the values in M is obtained by the right multiplication of some element of M.[3] When a ring R is injective as an R-left module, we call it a (*left*) *injectiv ring.*

(4.5) (See R. E. Johnson [8]) *If* $\mathfrak{l}$ *is a left ideal of* R, *then* $\mathfrak{l}+\mathfrak{l}^c \in R^{\triangle}$.

(4.6) *Let R be a C-ring with a unit element. If* $\mathfrak{l}$ *is an M-ideal of* $\overline{R}$, *then the R-left module* $\mathfrak{l}$ *is injective.*

2) See [2] Proposition 3.4.

3) See [2] Theorem 3.2.

Let $\mathfrak{l}'$ be a left ideal of R and θ an R-left homomorphism of $\mathfrak{l}'$ into $\mathfrak{l}$. We extend θ to an R-left homomorphism of $\mathfrak{l}'+\mathfrak{l}'^c$ into $\mathfrak{l}$ by making it vanish on $\mathfrak{l}'^c$. Then the extended θ belongs to $\mathfrak{A}_R$ since $\mathfrak{l}+\mathfrak{l}^c \in R^{\triangle}=R^{\blacktriangle}$. By (1.12) there is $a \in R$ such that $x\theta = xa$ for every $x \in \mathfrak{l}+\mathfrak{l}^c$. From $(\mathfrak{l}+\mathfrak{l}^c)a \subseteq \mathfrak{l}$ we see that $a \in \mathfrak{l}$ since $\mathfrak{l}$ is an M-ideal.

Theorem 4. *If R is a C-ring the following conditions are equivalent:*

(1) $\bar{R}=R$.

(2) *R is an injective ring.*

(3) *R is a regular ring*[4] *with unit element and has the property that if a family $\{x_\alpha + e_\alpha R\}$ of cosets of principal right ideals has the finite intersection property then the total intersection is non-void.*

Proof. (1)$\Rightarrow$(2) is a special case of (4.6). (2)$\Rightarrow$(3) The regularity of R is a result of R. E. Johnson.[5] This is easily shown by (4.5) and Theorem 3. Next, we assume that a family $\{x_\alpha + e_\alpha R\}$ has the finite intersection property. We set $\mathfrak{a} = \sum R(1-e_\alpha)$ and consider the correspondence $\theta: \sum u_{\alpha_i}(1-e_{\alpha_i})(\in \mathfrak{a}) \to \sum u_{\alpha_i}(1-e_{\alpha_i})x_{\alpha_i} = \sum u_{\alpha_i}(1-e_{\alpha_i})A_{\alpha_i}$. If $\sum u_{\alpha_i}(1-e_{\alpha_i})=0$, then $\sum u_{\alpha_i}(1-e_{\alpha_i})A_{\alpha_i} = \sum u_{\alpha_i}(1-e_{\alpha_i})x = 0$ where x is an element in $\bigcap A_{\alpha_i}$. It is easy to see that θ is an R-left homomorphism. By (2) there is an element u such that $z\theta = zu$ for every $z \in \mathfrak{a}$. Since $(1-e_\alpha)x_\alpha = (1-e_\alpha)u$ we know that $u \in x_\alpha + e_\alpha R$ or $u \in \bigcap A_\alpha$. (3)$\Rightarrow$(1) Let $\mathfrak{a}$ be a left ideal of R and θ an R-left homomorphism of $\mathfrak{a}$ into R. We set $A_\alpha = e_\alpha\theta + (1-e_\alpha)R$ for every idempotent $e_\alpha \in \mathfrak{a}$. For each finite subfamily $\{A_{\alpha_i}\}$ of the family $\{A_\alpha\}$ there exists an idempotent e_β such that $\sum R_{\alpha_i} = Re_\beta$. $e_\beta\theta - e_{\alpha_i}\theta = (1-e_{\alpha_i})e_\beta\theta \in (1-e_{\alpha_i})R$ and hence $e_\beta\theta \in A_{\alpha_i}$ for every $A_{\alpha_i} \in \{A_{\alpha_i}\}$. Thus $\{A_\alpha\}$ has the finite intersection property. Therefore there is $x \in \bigcap A_\alpha$ by our assumption. $e_\alpha\theta \in x + (1-e_\alpha)R$ and $e_\alpha\theta = e_\alpha x$. From $\mathfrak{a} = \sum Re_\alpha$ we see that $y\theta = yx$ for any $y \in \mathfrak{a}$. This implies $R = \bar{R}$ by Theorem 1.

The following (4.7)-(4.9) are corollaries of this Theorem.

(4.7) *Let R be a C-ring such that $R=\bar{R}$. Then a left ideal of R is a complemented left ideal if and only if it is a principal left ideal.*

The "only if" part is evident by (4.6) Since R is regular, every principal left ideal is a direct summand and hence it is a complemented left ideal.

(4.8) *If R is a C-ring, then the set of all complemented left ideals*

4) See [13].

5) See [8] Theorem 2.

of R forms a complete complemented modular lattice.[6]

In fact, by Theorem 2 and (4.7) the set of complemented left ideals of R forms a lattice isomorphic to that of principal left ideals of a regular ring with unit element. The completeness follows from (3.7).

In an obvious way, we may also define the notions of *right C-ring* and *right maximal quotient ring.*

(4.9) *Let R be a left and right C-ring and the left maximal quotient ring $\bar{R}$ be simultaneously the right maximal quotient ring.*[7] *Then a left ideal of R is a complemented left ideal if and only if it is a left annihilator ideal. The set of all left annihilator ideals and the set of all right annihilator ideals form the mutually dual isomorphic lattices.*

This follows easily from Theorem 2, (3.10)-(3.15) and (4.7).

An example of C-rings. Levitzki [10] called a ring to be a *semisimple I-ring* if every nonzero right ideal contains a nonzero idempotent. It is well known that this concept is right-left symmetric.

(4.10) *Every semisimple I-ring is a C-ring.*

Let $x \in R$ and $l_R(x) \in R^{\perp}$. If e is an idempotent in xR, then $0 = l_R(x)e \supseteq l_R(x) \cap Re$. Hence $Re = 0$ and $e = 0$. This shows $x = 0$.

5. The left maximal quotient ring $\bar{R}$ of a ring R is not always the right maximal quotient ring even if R is a both right and left C-ring. In the following we shall show this by treating a primitive with nonzero socle.

Let R be a primitive ring with nonzero socle and eR be its minimal right ideal. Then R may be regarded as a dence ring of linear transformations of the eRe-left module eR. We denote by L the ring of all linear transformations of eR.

(5.1) *L is the left maximal quotient ring of R.*

Indeed, since eR is a faithful R-right module, we see easily that $eR \leq R$. Hence $\overline{eR}$ is the (left) maximal quotient ring of R by (1.14). In eR every eRe-left submodule is a left ideal. Since eRe is a division ring we see that eR is completely reducible for left ideals. Hence $(eR)^{\triangle}$ consists of eR alone. Thus eR satisfies the condition (2) of Theorem 3 and this implies that eR is a C-ring. Therefore $(eR)^{\triangle} = (eR)^{\blacktriangle}$. It follows

6) This lattice is meet-homomorphic to that of all left ideals of R by (3.2) and Theorem 3. See [14].

7) On account of (1.5) and Theorem 1, this second assumption is, of course, equivalent to the condition that every left quotient ring of R is a right quotient ring of R and vice versa.

from this that $\overline{eR}$ is the ring of all endomorphisms of the eR- (or eRe-) left module eR and hence equal to L.

As an immediate corollary of (5.1) we obtain the following

(5.2) *Let R be a primitive ring with nonzero socle. Then R is also the right maximal quotient ring if and only if R is a simple ring with minimum condition.*

Next, we regard the minimal right ideal eR of R as a topological vecter space over eRe of which open base is the set of left annihilaters $l_{eR}(x)$ for all x in the socle $S(R)$ of R.[8] Then the right multiplication of any element in R is a continuous linear transformation of the space eR. We denote by $\tilde{R}$ the ring of all continuous transformations of eR. Then $\tilde{R}$ is also a primitive ring with nonzero socle and has the property that the socle of $\tilde{S}=S(R)\subseteq R\subseteq\tilde{R}\subseteq L$. This shows the part (3) of the following proposition.

(5.3) (1) *$\tilde{R}$ is the greatest one among the right quotient ring of R which is a subring of L.*

(2) *$(S(R)/S(\tilde{R}))^L=\tilde{R}$. In other words, $\tilde{R}$ is the left idealizer of $S(R)$ in L.*

(3) *$\tilde{R}$ is the greatest subring of L such that its intersection with the socle of L is $S(R)$.*

In fact, if $(S(R)/S(R))^L\ni x\neq 0$, then $0\neq xS(R)\subseteq S(R)$. Since $S(R)$ is a C-ring, it follows from this by (4.3) that $(S(R)/S(R))^L$ is a right quotient ring of $S(R)$. Clearly $R\subseteq(S(R)/S(R))^L$. Hence $(S(R)/S(R))^L$ is a right quotient ring of R. Now let A be any right quotient ring of R contained in L. Then A is, of course, that of $S(R)$. The right ideal of $S(R)$, which has $S(R)$ as its right quetient ring, is $S(R)$ itself alone since $S(R)$ is completely reducible for right ideals. Hence $A\subseteq(S(R)/S(R))^L$ by (1.10). Therefore $(S(R)/S(R))^L$ is the greatest right quotient ring of R contained in L. Next, let $x\in(S(R)/S(R))^L$ and $y\in S(R)$. Then $l_{eR}(xy)\,xy=0$; hence $l_{eR}(xy)\,x\in l_{eR}(y)$. This shows that $x\in\tilde{R}$. Thus $(S(R)/S(R))^L\subseteq\tilde{R}$. The converse inclusion is evident since $S(R)$ is the socle of $\tilde{R}$. This completes the proof.

6. First we prepare a certain number of terms we need. If the nilpotency indeces of nilpotent elements in a ring is bounded, the ring is called to be *of bounded index* and its least upper bound is called the *index* of the ring.[9] A (semisimple) I-ring is said to be *plain* if it is of

8) See [3], [7]. This topology is the *weak* topology.
9) See [6].

index 1.[10] It is well known that every idempotent in a ring of index 1 is central.[11] Thus,

(6.1) *A ring is plain if and only if every nonzero right ideal of R contains a nonzero central idempotent.*

The "only if" part follows immediately from the definition. If a ring R satisfies the condition, then R is evidently a semisimple I-ring. Let $0 \neq x \in R$. Then there is a nonzero central idempotent $e = xy$. Now $x^n y^n e = x^{n-1} e y^{n-1} e = x^{n-1} y^{n-1} e = \cdots = xye = e \neq 0$. This shows $x^n \neq 0$ and that R is plain.

If a two-sided ideal of a ring R is the total matrix ring, of finite degree, over a plain ring with unit element, then it is called a *matrix ideal* of R.[12] Of course, the unit element of any matrix ideal is central in R and hence every matrix ideal is a direct summand of R. A ring is called *semisimple weakly reducible* if every nonzero two-sided ideal contains a nonzero matrix ideal.[13] Levitzki [12] has proved the following facts:

(1) Every semisimple weakly reducible ring is a semisimpl I-ring [12, Theorem 3.1];

(2) Every semisimple I-ring of bounded index is semisimple weakly reducible [12, Theorem 3.3];

(3) Every semisimple I-ring, of which each primitive image is a simple ring with minimum condition, is semisimple weakly reducible [12, Theorem 3.4]. We note teat this assumption is satisfied by every semisimple I-ring with a polynomial identity.[14]

To investigate the maximal quotient ring of a semisimple weakly reducible ring it seems pertinent to re-construct it by a special manner.

A family B of central idempotents in a ring R is called a *B-family* if the following conditions are satisfied:

(B1) Let f be a central idempotent in R. If $ef = f$ for some $e \in B$, then $f \in B$.

(B2) For every nonzero central idempotent f in R there exists a nonzero idempotent e in B such that $ef = f$.

We say that a mapping θ of a B-family B into the ring R is an *H-mapping* if θ satisfies the condition (H) that if $e, f \in B$ and $ef = f$ then $(e\theta)f = f\theta$.

The totality of H-mappings defined on a B-family B forms a ring H_B by the operations $e(\theta + \varphi) = e\theta + e\varphi$ and $e(\theta\varphi) = (e\theta)(e\varphi)$. It is easy to

10) See [12].
11) See [4], Lemma 1.
12), 13) See [12].
14) See [10] and [11].

see that the intersection of any pair of B-families is also a B-family. Now we say that two H-mapping are equivalent if their restrictions to some B-family coincide. Then this relation is reflexive, symmetric and transitive, and the set of equivalence classes forms evidently a ring R°. We note that for every $x \in R$ and every B-family B the mapping x_B: $e \to ex$ $(e \in B)$ belongs to H_B.

(6.2) *Let R be a semisimple weakly reducible ring. Identifying each $x \in R$ with the class $\bar{x}_B \in R^\circ$ containing x_B we can regard R° as an extension ring of R. Then $R^\circ \simeq R$ over R.*

(1) Let B be a B-family. If $x \in R$ is nonzero, then $Bx \neq 0$. In fact, we assume $\bigwedge_{e \in B}(1-e)R \neq 0$. Then $\bigwedge(1-e)R$ would contain a nonzero matrix ideal and hence a nonzero central idempotent. By (B2) some nonzero $g \in B$ would be contained in $\bigwedge(1-e)R$. Then $gR \subseteq \bigwedge(1-e)R \subseteq (1-g)R$ and $g=0$, which is a contradiction. This shows that $\bigwedge(1-e)R=0$. If $x \neq 0$, then $x \notin (1-e)R$ or $ex \neq 0$ for some $e \in B$.

From (1) it is easy to see that the identification in (6.2) is allowable.

(2) Let $\mathfrak{m} \in R^\blacktriangle$. Then the set $B_\mathfrak{m}$ of central idempotents contained in $\mathfrak{m}$ forms a B-family. In fact, $B_\mathfrak{m}$ satisfies evidently (B1). Let e be a nonzero central idempotent. The Re contains a nonzero matrix ideal T_n over a plain ring T. Since T_n is a direct summand of R it follows from (1) of the proof of (2.1) that $T_n \cap \mathfrak{m} \in T_n^\blacktriangle$. By (2) of the proof of (2.3) there is $\mathfrak{m}' \in T^\blacktriangle$ such that $\mathfrak{m}_n' \subseteq T_n \cap \mathfrak{m}$. By (6.1) $\mathfrak{m}'$ contains a noozero central idempotent f. By (1.3) f is central in T and hence in R. Now $f \in \mathfrak{m}_n' \subseteq T_n \cap \mathfrak{m} \subseteq Re \cap \mathfrak{m}$. This implies $fe=f$ and $f \in \mathfrak{m}$. Therefore $B_\mathfrak{m}$ satisfies ($B2$) and it is a B-family.

(3) Let $e \in B$ and $\theta \in H_B$. Then $e\theta = e\bar{\theta}$ where $\bar{\theta}$ is the class $\in R^\circ$ containing θ. In fact, if $e, f \in B$, then $e(e\theta)=e\theta$ and $(fe)(e\theta)=(fe)\theta$ by (H). Hence $f(e\theta)=(fe)(e\theta)=(fe)\theta=(ef)(f\theta)=(fe_B)(f\theta)$.

(4) The extension R° of R satisfies the condition (1) of Theorem 1. In fact, we let $\mathfrak{m} \in R^\blacktriangle$ and let φ be an R-left homomorphism of $\mathfrak{m}$ into R. Then the restriction θ of φ to $B_\mathfrak{m}$ is clearly an H-mapping. On the other hand, R is a C-ring since it is a semisimple I-ring. Hence $R^\blacktriangle = R^\vartriangle$ by Theorem 3. From (1), (2) it is easy to see that $\Sigma_{B_\mathfrak{m} \ni e} Re \in R^\vartriangle = R^\blacktriangle$. For any element $\Sigma x_i e_i$ in ΣRe, $(\Sigma x_i e_i)\varphi = \Sigma x_i(e_i\varphi) = \Sigma x_i(e_i\theta) = \Sigma x_i(e_i\bar{\theta}) = (\Sigma x_i e_i)\bar{\theta}$.

(5) Let $0 \neq \bar{\theta} \in R^\circ$ and let $\theta \in H_B$ be a representative of $\bar{\theta}$. Since $B\theta \neq 0$, we see that $e\bar{\theta} = e\theta \neq 0$ for some $e \in B$. This shows $R \leq R^\circ$ by (4.4). Therefore $R^\circ \sim \bar{R}$ over R by (4) and Theorem 1.

Theorem 5. *Let R be a semisimple weakly reducible ring.*

(1) *The left maximal quotient ring $\bar{R}$ of R is also the right maximal quotient ring of R;*

(2) *If R is of index n, then so is $\bar{R}$;*

(3) *If R satisfies a polynomial identity, then $\bar{R}$ satisfies the same polynomial identity;*

(4) *$\bar{R}$ is also semisimple weakly reducible.*

Proof. By the left-right symmetry of our method in (5.2) we see that R° is also the right maximal quotient ring. (2) Let $\bar{\theta}\in R^\circ$ be nilpotent and $\theta\in H_B$ be its representative. Then θ is nilpotent and hence so is $e\theta$ for every $e\in B$. $e\theta^n=(e\theta)^n=0$. Thus $\bar{\theta}^n=0$ and $\bar{\theta}^n=0$. This shows that the index of R° (or $\bar{R}$) is at most n and hence is equal to n. (3) H_B may be regarded as a subdirect sum of Re for all $e\in B$. The identity holds in each Re. Hence it holds in H_B and in its limit R°. (4) Let $\mathfrak{a}$ be a nonzero two-sided ideal of $\bar{R}$. Then $\mathfrak{a}\frown R$ is nonzero and contains a nonzero matrix ideal $Re=T_n$ over a plain ring T. Since e is central in R it follows by (1.3) that e is central also in $\bar{R}$. $\bar{R}=e\bar{R}\oplus(1-e)\bar{R}$. By (2.2), $e\bar{R}\simeq\overline{e\bar{R}\frown R}=\overline{eR}=\overline{(T_n)}$. Hence $e\bar{R}\simeq(\bar{T})_n$ by (2.3). Now T is regular (Theorem 4) and of index 1 ((2) of this Theorem), and hence plain. Thus $e\bar{R}$ is a nonzero matrix ideal of $\bar{R}$ and is contained in $\mathfrak{a}$. This shows that $\bar{R}$ is semisimple weakly reducible.

7. In this section we consider some matrix rings as an application of Theorem 5.

A ring is called *strongly regular* if for any element x there is an element y such that $x^2y=x$. A necessary and sufficient condition for a ring to be strongly regular is that it is regular ring of index 1.[15]

(7.1) *Every plain ring R is embedded isomorphically into a strongly regular ring.*

In fact, the regular ring $\bar{R}$ is of index 1 by Theorem 5.

(7.2) *If R is a nonzero plain ring, then R_n is of index n.*

Every strongly regular ring is a subdirect sum of division rings.[16] Thus $R\subseteq\Sigma_{\oplus}^{c}P^{(\alpha)}$, $P^{(\alpha)}$ divisiom rings. Then $R_n\subseteq(\Sigma_{\oplus}^{c}P^{(\alpha)})_n\sim\Sigma_{\oplus}^{c}P_n^{(\alpha)}$. Since $P_n^{(\alpha)}$ is of index n the index of R_n is at most n. On the other hand, for any nonzero idempotent $e\in R$, $\Sigma_{i=1}^{n-1}ee_{ii+1}$ is of index n. Therefore R_n is of index n.

(7.3) *Let R be a semisimple I-ring. Then R is of index n if and*

14) See [10]. and [11].
15) See [4], Lemma 3.
16) See [4], Theorem 3.

only if R is a subdirect sum of its matrix ideals $T_{n_\alpha}^{(\alpha)}$ over plain rings $T^{(\alpha)}$ and Max $n_\alpha = n$.

"If" part: The index of R is evidently at most n. And some $T_{n_\alpha}^{(\alpha)}$ contains a nilpotent element of index n by (7.2). "Only if" part: From the assumtion we see that R is a semisimple weakly reducible ring. Hence it follows from a result of Levitski [12, Theorem 3.1] that R is a subdirect sum of its matrix ideals $T_{n_\alpha}^{(\alpha)}$. By (7.2), Max $n_\alpha = n$.

Theorem 6. *A ring R is semisimple I-ring if and only if so is the total matrix ring R_n. In this case, R is of index m if and only if R_n is of index mn.*

Proof. (1) Let R be a semisimple I-ring. We assume that AR_n contains no nonzero idempotent where $A = \sum a_{ij}e_{ij} \in R_n$. Then $(xa_{ij}ye_{11}) R_n = (xe_{1i}) A(ye_{j1}) R_n$ contains no nonzero idempotent for any $x, y \in R$. Let $e = xa_{ij}yz$ be an idempotent. Then $ee_{11} = (xa_{ij}ye_{11})(ze_{11})$ is also an idempotent. Hence $ee_{11} = 0$ and $e = 0$. This implies $xa_{ij}y = 0$. Therefore $a_{ij} = 0$ and $A = 0$. It follows from this that S_n is a semisimple I-ring.

(2) Let R_n be a semisimple I-ring and $\mathfrak{l}$ a nonzero left ideal of R. The $\sum \mathfrak{l} e_{i1}$ is a nonzero left ideal of R_n. Hence it contains a nonzero idempotent $\sum x_{i1}e_{i1}$. $\sum x_{i1}e_{i1} = (\sum x_{i1}e_{i1})^2 = \sum x_{i1}x_{11}e_{i1}$. Therefore x_{11} is a nonzero idempotent in $\mathfrak{l}$ which shows that R is a semisimple I-ring.

(3) Let R be a semisimple I-ring of index m. Then by (7.3) R is a subdirect sum of its matrix ideals $T_{n_\alpha}^{(\alpha)}$ and Max $n_\alpha = m$. Hence R_n is a subdirect sum of its matrix ideal $T_{n_\alpha n}^{(\alpha)}$ and Max $n_\alpha n = mn$. By (7.3) this means that R_m is a semisimple I-ring of index mn.

(4) Let R_n be a semisimple I-ring of index mn. Then R is also a semisimple I-ring by (2). Since R_n contains a subring isomorphic to R, we see that R is of bounded index. Hence the index of R is m by (3).

As a corollary of this Theorem, we have

(7.4) *Let R be a ring with a unit element and assume that some homomorphic image of some two-sided ideal of R is a nonzero semisimple I-ring of bounded index. Then $R_n \simeq R_m$ implies $n = m$.*

The minimum of the indeces of those rings, each of which is a nonzero semisimple I-ring of bounded index and is a homomorphic image of some two-sided ideal of R, is denoted by $\rho(R)$, $\rho(R_n)$ and $\rho(R_m)$ are similarly defined. Then $\rho(R_n) = n\rho(R)$ and $\rho(R_m) = m\rho(R)$. Therefore $n = m$ if $R_n = R_m$.

(7.5) *Assume that a ring R satisfies the condition of* (7.4). *Let M be a unitary R-module with a basis consisting of k elements. Then any other basis is also finite and consists of k elements.*

This is evident from (7.4) since the R-endomorphism ring of M is R_k.

We note that every ring with a unit element, which is semisimple weakly reducible modulo its radical, satisfies the assumption in (7.4) and (7.5).

8. In this supplementary section we take a glance at the extended centralizer defined in [8]. We denote the extended centralizer over a module M as $E(M)$ and the family of submodules of M each of which has a nonzero intersection with every nonzero submodule of M as $M^{\triangle}$.

Theorem 7. *$E(N)\subseteq E(M)$ for every submodule N of M.*

We omit the detailed proof. It is easy to see that (1) $E(N)=E(M)$ if $N\in M^{\triangle}$ and (2) $E(N)\subseteq E(M)$ if N is a direct summand of M. Now let N be an arbitrary submodule of M. Then $N+N^c\in M^{\triangle}$, where N^c is a maximal one among submodules having zero intersections with N. Hence $E(M)=E(N\oplus N^c)\supseteq E(N)$.

(8.1) *Let K be a module and M the direct sum of n isomorphic copies $\{K_i\}$ of K. Then $E(M)\simeq(E(K))_n$.*

Let θ_i be an isomorphism of K onto K_i. For any submodule H of K we denote the sum $\Sigma H\theta_i$ as H^*. Then we know that $H^*\in M^{\triangle}$ and that for any $N\in M^{\triangle}$ there is a submodule H of K such that $H^*\subseteq N$. From these facts we can prove the Theorem by the usual method.

Finally we add a simple application:

(8.2) *Let R be a semisimple I-ring of bounded index and have a unit element. We assume that a unitary R-module M has a basis consisting of n elements. Then any basis of any free submedule N of M consists of at most n elements.*

Owing to (8.1) we have $E(M)\simeq(E(R))_n$. Moreover, $E(R)=\bar{R}$ since R is a C-ring by (4.10). Let r be the index of R. Then the index of $E(R)$ is also r by Theorem 5. Hence that of $E(M)$ is rn by Theorem 6. Let t be a natural number which is not greater than the cardinal number of the given basis elements of N. Then N contains a submodule L which has a basis consisting of t elements. Now Theorem 7 assures that $E(L)\subseteq E(M)$. Since $E(L)\simeq(E(R))_t=(\bar{R})_t$, we know that the index of $E(L)$ is rt. Therefore $rt\leq rn$ whence $t\leq n$. This proves the proposition.

(Received March 19, 1956)

References

[1] K. Asano: Ueber die Quotientenbildung von Schiefringen, J. Math. Soc. Japan **1**, 73–78 (1949).

[2] H. Cartan and S. Eilenberg: Homological algebra (1955), Princeton.

[3] J. Dieudonné: Sur le socle d'un anneau et les anneux simples infinis, Bull. Soc. Math. France **71**, 46–75 (1943).

[4] A. Forsythe and N. H. McCoy: The commutativity of certain rings, Bull. Amer. Math. Soc. **52**, 523–526 (1946).

[5] N. Jacobson: The radical and semisimplicity for arbitrary rings, Amer. J. Math. **67**, 300–320 (1946).

[6] ———: Structure theory for algebras of bounded degree, Ann. of Math. **46**, 695–707 (1945).

[7] ———: On the theory of primitive rings, Ann. of Math. **48**, 8–21 (1947).

[8] R. E. Johnson: The extended centralizer of a ring over a module, Proc. Amer. Math. Soc. **2**, 891–895 (1951).

[9] ———: Prime rings, Duke Math. J. **18**, 799–809 (1951).

[10] I. Kaplansky: Rings with a polynomial identity, Bull. Amer. Math. Soc. **54**, 575–580 (1948).

[11] J. Levitzki: A theorem on polynomial identities, Proc. Amer. Math. Soc. **1**, 334–341 (1950).

[12] ———: On the structure of algebras and related rings, Trans. Amer. Math. Soc. **74**, 384–409 (1953).

[13] J. von Neumann: Continuous geometry, II, Princeton (1937).

[14] Y. Utumi: On complemented modular lattices meet-homomorphic to a modular lattice, Kodai Math. Sem. Rep. No. 4, 99–100 (1952).