

## CROSSING CHANGE AND EXCEPTIONAL DEHN SURGERY

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### 1. Introduction

Thurston’s hyperbolic Dehn surgery theorem [11], [12] asserts that if a knot  $K$  in the 3-sphere  $S^3$  is hyperbolic (i.e.,  $S^3 - K$  admits a complete hyperbolic structure of finite volume), then all but finitely many Dehn surgeries on  $K$  yield hyperbolic 3-manifolds. By an *exceptional surgery* on a hyperbolic knot we mean a nontrivial Dehn surgery producing a non-hyperbolic manifold. Refer to [3], [6] for a survey on Dehn surgery on knots. We empirically know that ‘most’ knots are hyperbolic and ‘most’ hyperbolic knots have no exceptional surgeries. In this paper, we demonstrate the abundance of hyperbolic knots with no exceptional surgeries by showing that every knot is ‘close’ to infinitely many such hyperbolic knots in terms of crossing change.

We regard that two knots are the same if they are isotopic in  $S^3$ . For a knot  $K$  in  $S^3$ , let  $B_n(K)$  be the set of knots each of which is obtained by changing at most  $n$  crossings in a diagram of  $K$ .

**Theorem 1.1.** *For every knot  $K$  in  $S^3$ ,  $B_1(K)$  contains infinitely many hyperbolic knots each of which has no exceptional surgeries. In particular, an arbitrary knot can be deformed into a hyperbolic knot with no exceptional surgeries by a single crossing change.*

In Section 3, we raise some questions on the distribution of hyperbolic knots (with no exceptional surgeries).

### 2. Proofs

A (2-string) *tangle* is a pair  $(B, t)$  where  $B$  is a 3-ball and  $t$  is a pair of disjoint arcs properly embedded in  $B$ . We call  $(B, t)$  a *trivial tangle* if there is a homeomorphism from  $(B, t)$  to  $(D \times I, \{x, y\} \times I)$ , where  $D$  is a disk containing  $x$  and  $y$  in its interior. A tangle  $(B, t)$  is said to be *atoroidal* if  $B - \text{int } N(t)$  contains no incompressible tori. A tangle  $(B, t)$  is said to be  *$\partial$ -irreducible* if  $B - \text{int } N(t)$  is  $\partial$ -irreducible.

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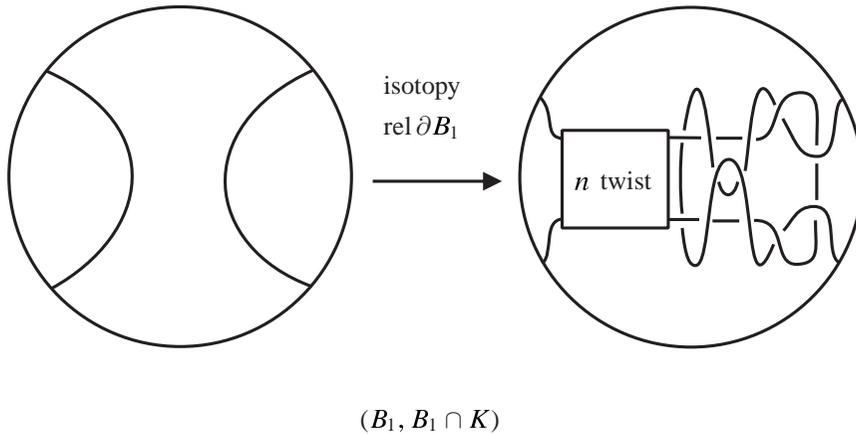


Fig. 1.

Let  $K$  be a knot in  $S^3$ . Suppose that  $S$  is a 2-sphere meeting  $K$  transversely in four points and separating  $S^3$  into two 3-balls  $B_1$  and  $B_2$ . Then  $(B_i, B_i \cap K)$  are tangles, and we say that  $K$  is decomposed into the union of two tangles  $(B_i, B_i \cap K)$ , where  $i = 1, 2$ .

**Proposition 2.1.** *Every knot  $K$  in  $S^3$  is decomposed into the union of a trivial tangle and an atoroidal,  $\partial$ -irreducible tangle.*

*Proof.* This follows from Myers [8, Theorem 1.1]. Take an arc  $c$  in  $S^3$  such that  $c \cap K = \partial c$ , and  $c' = c \cap E(K)$  is a properly embedded arc in  $E(K) = S^3 - \text{int } N(K)$ ; then  $E(K) - \text{int } N(c') \cong S^3 - \text{int } N(K \cup c)$ . By [8, Theorem 1.1] we can choose  $c$  so that  $S^3 - \text{int } N(K \cup c)$  is  $\partial$ -irreducible and contains no incompressible tori. Let us choose a small regular neighborhood  $B_1$  of  $c$  in  $S^3$  so that  $(B_1, B_1 \cap K)$  is a trivial tangle. The 2-sphere  $\partial B_1$  decomposes  $K$  into the union of the trivial tangle  $(B_1, B_1 \cap K)$  and the tangle  $(B_2, B_2 \cap K)$  where  $B_2 = S^3 - \text{int } B_1$ . Since  $S^3 - \text{int } N(K \cup c) = B_2 - \text{int } N(B_2 \cap K)$ ,  $(B_2, B_2 \cap K)$  is an atoroidal,  $\partial$ -irreducible tangle.  $\square$

*Proof of Theorem 1.1.* By Proposition 2.1,  $K$  is decomposed into the union of a trivial tangle  $(B_1, B_1 \cap K)$  and an atoroidal,  $\partial$ -irreducible tangle  $(B_2, B_2 \cap K)$ . Isotope the trivial tangle  $(B_1, B_1 \cap K)$  fixing its boundary as in Fig. 1.

Let  $K_n$  be the knot obtained from  $K$  by changing a crossing of  $K$  as described in Fig. 2.

The tangle  $(B_1, B_1 \cap K)$  is changed to  $(B_1, B_1 \cap K_n)$ . Soma [10, Lemma 3] proved that  $(B_1, B_1 \cap K_n)$  is a nontrivial atoroidal tangle. It follows that  $K_n$  is decomposed into the union of the atoroidal,  $\partial$ -irreducible tangle  $(B_2, B_2 \cap K_n) = (B_2, B_2 \cap K)$  and a nontrivial atoroidal tangle  $(B_1, B_1 \cap K_n)$ . Applying [14, Theorem 3.3], we see that

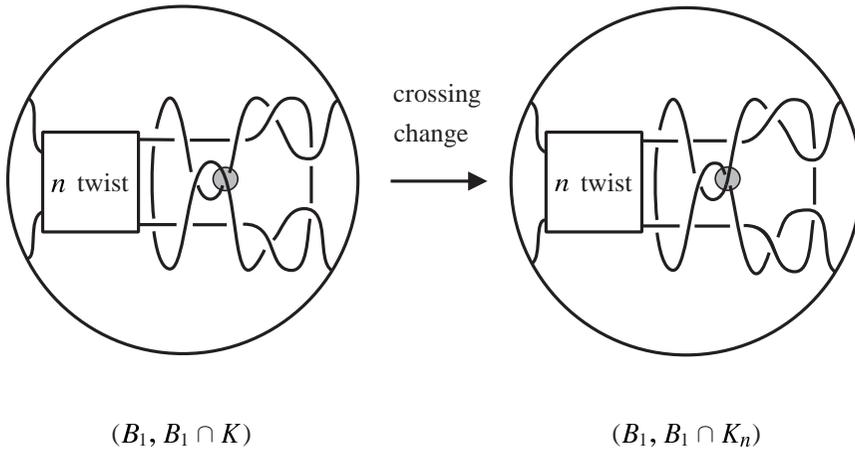


Fig. 2.

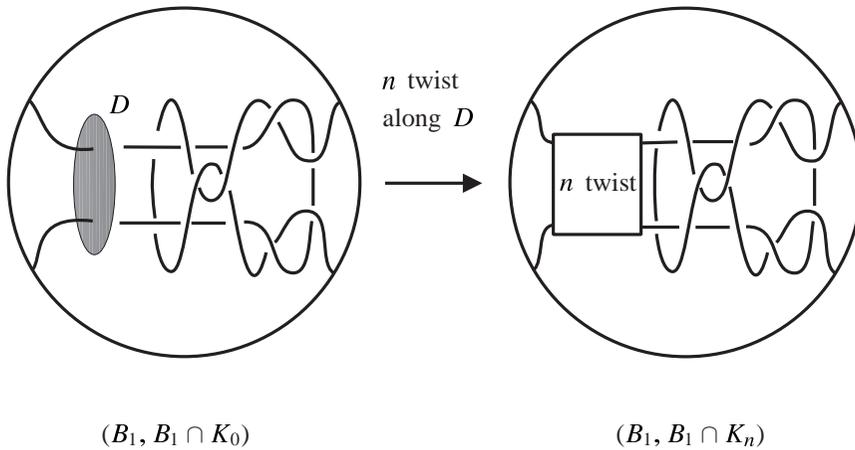


Fig. 3.

all nontrivial Dehn surgeries on  $K_n$  produce hyperbolic, Haken manifolds. Since every non-hyperbolic knot yields non-hyperbolic manifolds after infinitely many Dehn surgeries,  $K_n$  is a hyperbolic knot.

It remains to show that  $\{K_n\}_{n \in \mathbb{Z}}$  contains infinitely many distinct knots. Note that  $K_n$  is obtained from  $K_0$  by twisting  $n$  times along the disk  $D$  in Fig. 3.

CLAIM 2.2. The circle  $\partial D$  does not bound a disk  $D'$  in  $S^3$  which intersects  $K_0$  in at most one point.

Proof. Since the algebraic intersection number of  $K_0$  and  $D$  is zero, it is sufficient to show that there is no disk  $D'$  satisfying  $\partial D' = \partial D$  and  $D' \cap K_0 = \emptyset$ . Suppose for a contradiction that we had such a disk  $D'$ . Let  $A$  be an obvious annulus in  $B_1 - B_1 \cap K_0$  connecting  $\partial D$  and an essential simple loop in  $\partial B_1 - \text{int } N(B_1 \cap K_0)$ . Then the existence of a (possibly singular) disk  $A \cup D'$  would imply the compressibility of  $F = \partial B_1 - \text{int } N(B_1 \cap K_0)$  in  $E(K_0)$ . This is a contradiction.  $\square$

By applying [5, Theorem 3.2] we see that  $\{K_n\}_{n \in \mathbb{Z}}$  consists of infinitely many distinct knots. This completes the proof of Theorem 1.1.  $\square$

REMARKS. (1) In Theorem 1.1 we can choose a crossing change of  $K$  so that the resulting hyperbolic knot is concordant to  $K$  and has the same Alexander invariant as  $K$ . In fact, by [9, Lemma 3.3] the crossing change given in the proof of Theorem 1.1 does not change knot concordance class and Alexander invariant.

(2) Kawauchi [4] proved that  $B_1(K)$  contains infinitely many hyperbolic knots as an application of the imitation theory. In [4, Theorem 3.1], let  $L''$  be the knot obtained from  $L$  by nugatory crossing change (so that  $L \cong L''$ ), then Theorem 3.1(1), (2) implies that an imitation  $L^*$  of  $L$  is a hyperbolic knot and obtained from  $L$  by a single crossing change.

### 3. Questions

On the distribution of hyperbolic knots in the knot table, Adams [1] conjectures that the proportion of hyperbolic knots among all prime knots with minimal crossing number less than  $n$  approaches 1 as  $n \rightarrow \infty$ . For any knot projection  $G$  of  $n$  crossings, we can obtain  $2^n$  knot diagrams by indicating over-under relation at each crossing point in  $G$ . Weeks [13] asks if “under reasonable conditions” most knot diagrams obtained from a given knot projection represent hyperbolic knots; we paraphrase this question as follows.

**Question 1.** Let  $f(G)$  be the proportion of diagrams representing hyperbolic knots among all the knot diagrams obtained from a knot projection  $G$ . Let  $\mathcal{D}_n$  be the set of knot projections  $G$  in  $S^2$  of  $n$  crossings such that if  $C$  is a simple closed curve in  $S^2$  meeting  $G$  transversely in two non-crossing points, then a component of  $G - C$  is an embedded arc in  $S^2$ . We denote by  $g(r, n)$  the proportion of knot projections  $G \in \mathcal{D}_n$  satisfying  $f(G) > r$ . Then for each  $r$  with  $0 < r < 1$ , does  $g(r, n)$  approach 1 as  $n \rightarrow \infty$ ?

On the distribution of hyperbolic knots with no exceptional surgeries we raise the following questions.

- Question 2.** (1) Is there a hyperbolic knot  $K$  such that each hyperbolic knot in  $B_1(K)$  has no exceptional surgeries?
- (2) If  $K$  is a trivial knot, then  $B_1(K)$  contains hyperbolic, twist knots, each admitting exceptional surgeries [2]. Is there a nontrivial knot  $K$  such that  $B_1(K)$  contains infinitely many hyperbolic knots with exceptional surgeries?

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