ON ANALYTIC FUNCTIONS ON SOME RIEMANN SURFACES

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Introduction

In the theory of functions meromorphic in $|z| < +\infty$, Iversen [4] proved the following: If $w = f(z)$ is meromorphic in $|z| < +\infty$ and has an essential singularity at $z = \infty$, then any inverse function-element of this function with the centre w_0 can be continued analytically to any point $w \neq w_0$, except possibly this point w, in any disc having the centre at the point w and containing the point w_0.

This fact plays important roles to study the properties of covering surfaces generated by the inverse functions of analytic functions. This property was discussed by many authors. Above all, Stoïlow [22] and Mori [10] contributed to extend the above Iversen theorem in more general cases.

In this article, we shall give an extension of the Iversen theorem in the case when the existence domain of a single-valued analytic function is a Riemann surface satisfying some condition. Such a Riemann surface belongs to O_{AB} but not to O_{HD}, where we use the following notations:

O_{HB} (or O_{AB}): the class of Riemann surfaces on which there exists no non-constant single-valued bounded harmonic (or analytic) function.

O_{HD} (or O_{AD}): the class of Riemann surfaces on which there exists no non-constant single-valued harmonic (or analytic) function whose Dirichlet integral taken over the Riemann surface is finite.

Our result contains Stoïlow's theorem and Mori's. Recently Heins [3] also dealt with such a problem.

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I. Subregions and the set of the class $N_{\mathfrak{B}}$

1. Let F be a Riemann surface and let G be a non-compact or compact domain on F whose relative boundary C with respect to F consists of at most an enumerable number of analytic curves being compact or non-compact and clustering nowhere in F. For simplicity, we shall call such a domain G a subregion on F.

If there exists no non-constant single-valued bounded analytic function $f(p)$ in a subregion G on F such that $f(p)$ is continuous on $G \cup C$ and that the real part of $f(p)$ vanishes at every point on C, we shall say that G belongs to the class SO_{AB}. And if there exists no non-constant single-valued bounded harmonic function in a subregion G which vanishes continuously at every point on C, then we may say that G belongs to the class SO_{HB}.

It is evident that, if G belongs to SO_{HB}, then G belongs to SO_{AB}. For, the real part of a single-valued bounded analytic function is a single-valued bounded harmonic function. In general the converse of the above statement does not hold. If the subregion G is simply connected, any analytic function in G is single-valued. Hence, in this case, G belongs to SO_{HB} if G belongs to SO_{AB}. Therefore, for a simply connected subregion G, $G \in SO_{HB}$ is equivalent to $G \in SO_{AB}$. It will be shown in Corollary of Theorem 6 that this fact does not hold for a subregion not being simply connected.

2. We consider a subregion G on F and construct a Riemann surface $\hat{G}$ by the process of symmetrization of G along C. There is given an indirectly conformal mapping of G on itself which leaves every point of C fixed. This surface $\hat{G}$ is called the double of G along C. If G is a compact subregion on F, then the double $\hat{G}$ is a compact Riemann surface.

We can prove the following

THEOREM 1. *The double $\hat{G}$ of a subregion G on F belongs to O_{AB} if and only if G belongs to SO_{AB}.*

Proof. First we suppose that G belongs to SO_{AB}. Denote by $\tilde{G}$ the image of G and by $\tilde{p}$ the image of a point p of $\hat{G}$ under the indirectly conformal mapping of $\hat{G}$ onto itself which leaves every point of C fixed. If $f(p)$ is a single-valued bounded analytic function on $\hat{G}$, the functions

$$F_1(p) = f(p) - \overline{f(\tilde{p})} \text{ and } F_2(p) = \frac{1}{i}(f(p) + \overline{f(\tilde{p})})$$

are also single-valued, bounded and analytic in G. And the real parts of $F_1(p)$ and $F_2(p)$ are both equal to zero at every point of C. Since G belongs to SO_{AB}, these functions $F_1(p)$ and $F_2(p)$ reduce to constants. Hence we can write

$$f(p) = \overline{f(\tilde{p})} + k_1 \text{ and } f(p) = -\overline{f(\tilde{p})} + k_2,$$

where k_1 and k_2 are the constants independent on the point p of G. From this fact, we see that $f(p)$ equals the constant $\frac{k_1 + k_2}{2}$. In other words, $\hat{G}$ belongs to O_{AB}.

Next we suppose that G does not belong to SO_{AB}. Then there exists a non-constant single-valued bounded analytic function $f_1(p) = u(p) + iv(p)$ in G which is continuous on $G \cup C$ and whose real part $u(p)$ vanishes at every point on C. It is easily seen that the function

$$f_2(p) = -u(\tilde{p}) + iv(\tilde{p}) \quad (p \in \tilde{G})$$

is non-constant, single-valued, bounded and analytic in $\tilde{G}$ and is continuous on $\tilde{G} \cup C$. Since the real part of $f_2(p)$ vanishes at every point on C, we can see by the well known reflection principle that the non-constant single-valued bounded function

$$f(p) = \begin{cases} f_1(p), & p \in G \cup C \\ f_2(p), & p \in \tilde{G} \cup C \end{cases}$$

is analytic on $\hat{G}$. Hence $\hat{G}$ does not belong to O_{AB}.

Thus our theorem is established.

As mentioned already, if G belongs to SO_{HB}, then G belongs to SO_{AB}. Hence we get the following which was proved in the previous paper [8].

COROLLARY. *If a subregion G on F belongs to SO_{HB}, the double $\hat{G}$ belongs to O_{AB}.*

3. Let us denote by NO_{HB} the class of subregions G with the relative boundary C on Riemann surfaces satisfying the following condition: There exists no non-constant single-valued bounded harmonic function in G which is continuous on $G \cup C$ and whose normal derivative vanishes at every point on C.

Then the following is obtained immediately.

THEOREM 2. *If a subregion G belongs to NO_{HB}, then G belongs to SO_{AB}, and hence, the double $\hat{G}$ belongs to O_{AB}.*

Proof. Let $f_1(p)=u(p)+iv(p)$ be a single-valued bounded analytic function in G which is continuous on $G \cup C$ and whose real part $u(p)$ equals zero at every point on C. By the same argument as in the proof of Theorem 1, we construct a single-valued bounded analytic function $f(p)$ on the double $\hat{G}$ of G along C. It is immediately seen that the normal derivative of the imaginary part of $f(p)$ must vanish at every point of C. Since $v(p)$ is a single-valued bounded harmonic function in G, it follows from the assumption that $v(p)$ reduces to a constant. Hence the function $f(p)$ must be a constant, which proves the first part of our theorem. The second part is evident from Theorem 1.

4. We shall state here some properties of sets of the class $N_{\mathfrak{B}}$ in the sense of Ahlfors-Beurling [1]. Following them, we denote by $N_{\mathfrak{B}}$ the class of closed sets in the complex plane, in whose complementary domains there exists no non-constant single-valued bounded analytic function. It is easily seen that the set of the class $N_{\mathfrak{B}}$ can contain no continuum. Hence, when we consider the set belonging to $N_{\mathfrak{B}}$, it is sufficient to consider the totally disconnected and bounded closed set. If two closed sets E_1 and E_2 satisfy the relation $E_1 \subset E_2$ and if E_2 belongs to $N_{\mathfrak{B}}$, then E_1 belongs also to $N_{\mathfrak{B}}$. From the definition, it is obvious that any set of the class $N_{\mathfrak{B}}$ is non-dense.

It is well known that a non-constant single-valued analytic function $w=f(p)$ defined on a Riemann surface belonging to O_{AB} takes every value in the w-plane except possibly the values belonging to the set of $N_{\mathfrak{B}}$. Further, in the case of the totally disconnected bounded closed set E, Kametani [5] and Sario [19], [20] proved that E belongs to $N_{\mathfrak{B}}$ if and only if, for any domain D containing E, any single-valued bounded analytic function in a domain $D-E$ can be continued analytically throughout D and this function should be regular in D.

Let E be a bounded closed set in the complex w-plane. Denote by E^* the set of points $w \in E$ such that, for any neighbourhood U of the point $w \in E$, the closure of the intersection $E \cap U$ does not belong to $N_{\mathfrak{B}}$. We shall call the subset E^* of E the B-kernel of E.

Obviously the B-kernel E^* is closed. In fact, any limiting point of E^* belongs to E, since $E^* \subset E$ and E is closed. In any neighbourhood U of any limiting point of E^*, there exists at least a point w_0 of E^*. Since w_0 belongs to E^* and since U is a neighbourhood of w_0, the closure of the intersection $U \cap E$ does not belong to $N_{\mathfrak{B}}$. Thus, by the definition, any limiting point of E^* belongs to E^*.

If the closed set E belongs to $N_{\mathfrak{B}}$, then the B-kernel of E is empty.

5. For the later use, we shall prove the following

THEOREM 3. *If the closed set E does not belong to $N_{\mathfrak{B}}$, then the B-kernel E^* of E is not empty and, for any neighbourhood U of each point of E^*, the closure of the intersection $U \cap E^*$ does not belong to $N_{\mathfrak{B}}$.*

Proof. First we shall prove the first part of our theorem.

If E contains a continuum, the continuum is contained in E^* and so our assertion is evident. Hence we shall consider the case when E is a totally disconnected bounded closed set. Contrary to the assertion, suppose that the B-kernel E^* of E is empty. Then, for any point w of E, there exists a neighbourhood U of this point w such that the closure of $U \cap E$ belongs to $N_{\mathfrak{B}}$. By Kametani's lemma [5], we can find a neighbourhood U' of the point w in U such that the set $U' \cap E$ is closed. Since this set $U' \cap E$ is a closed subset of the closure of $U \cap E$ which belongs to $N_{\mathfrak{B}}$, the set $U' \cap E$ belongs to $N_{\mathfrak{B}}$.

Constructing a neighbourhood U' for each point w of E by the manner stated above, we get a system of neighbourhoods $\{U'\}$ covering the bounded closed set E. By Heine-Borel's theorem, we can find a finite number of neighbourhoods $\{U'_i\}$ among $\{U'\}$ such that the union of $\{U'_i\}$ covers the set E.

Let $f(w)$ be an arbitrary single-valued bounded analytic function in the complementary domain of E. Considering this function $f(w)$ in a domain $U'_i - (U'_i \cap E)$, we see by Kametani-Sario's theorem stated already that $f(w)$ can be continued analytically throughout U'_i and $f(w)$ should be bounded and analytic in U'_i. Repeating this for all U'_i, we obtain the fact that $f(w)$ can be continued analytically throughout the whole complex plane and should be regular in the whole plane. Hence $f(w)$ must be a constant. Thus the set E belongs to $N_{\mathfrak{B}}$. Therefore, the B-kernel E^* of the closed set E not belonging to $N_{\mathfrak{B}}$ is not empty.

Next we shall give a proof of the second part of our theorem. Suppose that there exist a point w^* of E^* and its neighbourhood U^* such that the closure of $U^* \cap E^*$ belongs to $N_{\mathfrak{B}}$. Then the closure of $U^* \cap E$ can contain no continuum and is totally disconnected. Hence we can find a neighbourhood U_0^* of w^* by Kametani's lemma such that U_0^* is contained in U^* and that the set $U_0^* \cap E$ containing the point w^* is a bounded closed set.

Let $f(w)$ be any non-constant single-valued bounded analytic function in a domain $U_0^* - (U_0^* \cap E)$. Then we can see that there exists a neighbourhood V of any point belonging to the set $U_0^* \cap (E - E^*)$ such that $V \subset U_0^*$ and the intersection $V \cap E$ is a closed set belonging to $N_{\mathfrak{B}}$ and that V contains no point of E^*. Considering $f(w)$ in V and mentioning Kametani-Sario's theorem, we can see that $f(w)$ can be continued analytically throughout V and we get the single-valued bounded analytic function $f(w)$ in V. Repeating this for each point of $U_0^* \cap (E - E^*)$, we see that $f(w)$ should be a single-valued bounded analytic function in a domain $U_0^* - (U_0^* \cap E^*)$. By our assumption, the set $E^* \cap U_0^*$ belongs to $N_{\mathfrak{B}}$ and, hence, $f(w)$ can be continued analytically throughout U_0^*. This shows that there exists a neighbourhood U_0^* of w^* such that the intersection $U_0^* \cap E$ belongs to $N_{\mathfrak{B}}$. Hence the point w^* must not belong to E^*, which is a contradiction. Thus our theorem is established.

Further, we can prove the following

THEOREM 4. *Let $\{E_k\}$ $(k = 1, 2, \ldots, n)$ be sets of $N_{\mathfrak{B}}$. Then the set $\bigcup_{k=1}^{n} E_k$ belongs also to $N_{\mathfrak{B}}$.*

Proof.[1] It is sufficient to prove the assertion for the case of $n = 2$. Since E_1 and E_2 belong to $N_{\mathfrak{B}}$, these two sets are totally disconnected and closed. Hence the set $E_1 \cup E_2$ is also totally disconnected and closed. We consider the domain D containing this set $E_1 \cup E_2$ entirely. Obviously the sets $D - (E_1 \cup E_2)$, $D - E_1$ and $D - E_2$ are domains. Let $f(w)$ be any single-valued bounded analytic function in the domain $D - (E_1 \cup E_2)$. For any point w of the set $E_1 - (E_1 \cap E_2)$, we can choose a neighbourhood U of the point w by Kametani's lemma such that U is contained in $D - E_2$ and the set $U \cap E_1$ is closed.

Since $U \cap E_1$ is the closed subset of E_1 belonging to $N_{\mathfrak{B}}$, $U \cap E_1$ is the set of $N_{\mathfrak{B}}$. Hence $f(w)$ can be continued analytically throughout U and should be regular and bounded in U. Repeating this, we see that $f(w)$ should be regular

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[1] The author's original proof using Theorem 3 was more complicated than this direct one which Professor Ohtsuka suggested to the author.

in $D-E_2$. Since E_2 belongs to $N_{\mathfrak{B}}$ from the assumption, $f(w)$ can be continued analytically throughout D and should be regular in D, which shows that the set $E_1 \cup E_2$ belongs also to $N_{\mathfrak{B}}$.

In the case when the sets E_k $(k=1, \ldots, n)$ are disjoint from each other, this theorem was proved by Kametani [5].

II. Riemann surfaces of the class O^0_{AB}

6. First we shall prove a theorem which plays an important role to study the behaviour of analytic functions defined on some Riemann surfaces.

Let $w=f(p)$ be a non-constant single-valued analytic function in a subregion G with the relative boundary C on a Riemann surface. We suppose that this function is continuous on $G \cup C$ and that, for a certain point $w=w^*$ in the w-plane and for a certain positive number ρ, the value of this function $f(p)$ at every point of G lies in an open disc (c_ρ) and further that the values of $f(p)$ on C fall on the circumference c_ρ of the disc (c_ρ), where (c_ρ) is the disc $|w-w^*|<\rho$ in the case of $w^* \neq \infty$ or the disc $|w| > \dfrac{1}{\rho}$ in the case of $w^*=\infty$.

THEOREM 5. *Let $f(p)$ be such a function as stated above and let E be the set of values in (c_ρ) which $f(p)$ does not take in G. If G belongs to SO_{AB}, then the intersection of E and any closed set in (c_ρ) belongs to $N_{\mathfrak{B}}$.*

Proof. Since E is closed with respect to (c_ρ), it is obvious that the intersection of E and any closed set in (c_ρ) is closed. Contrary to the assertion, suppose that there exists a closed set in (c_ρ) such that the intersection E' of this set and E does not belong to $N_{\mathfrak{B}}$. Denote by δ the domain being a connected component of the intersection of (c_ρ) and the complementary set of E' with respect to the whole w-plane and having the boundary c_ρ. Since the image of G on the w-plane by the function $w=f(p)$ is connected, it is contained in δ. By Sario's theorem [19], [20], there exists a non-constant single-valued bounded analytic function $\varphi(w)$ in δ which is continuous on $\delta \cup c_\rho$ and whose real part equals zero on c_ρ. The composed function $\varphi(f(p))$ is non-constant, single-valued, bounded and analytic in G and its real part vanishes continuously at every point on C. Hence G does not belong to SO_{AB}. Thus we have the theorem.

7. Let F be a Riemann surface and let $w=f(p)$ be a non-constant single-valued analytic function defined on F. The space formed by elements $q=[p, f(p)]$ defines a covering Riemann surface Φ spread over the w-plane and the point $q=[p, f(p)]$ of Φ has the projection $w=f(p)$ on the w-plane. The correspondence $p \leftrightarrow q$ gives a topological and conformal mapping between F and Φ. Two surfaces F and Φ are equivalent conformally to each other.

Denote by Φ_Δ any connected piece of Φ lying over the disc (c_ρ). Let Δ be the domain on F corresponding to Φ_Δ by the correspondence $p \leftrightarrow q$. If there exists at least one connected piece Φ_Δ above any disc (c_ρ) and if, for any Φ_Δ above any disc (c_ρ), there exists a path in Δ starting from any fixed point p_0 $(w^* \neq f(p_0))$ in Δ and tending to the inner point or to the ideal boundary of Δ such that $\lim f(p)=w^*$ along the path, where the point $w=w^*$ is the centre of the disc (c_ρ), then we shall say that Φ has the Iversen property.

8. Denote by O_{AB}^0 the class of Riemann surfaces whose all subregions belong to SO_{AB}. As will be stated later in No. 11, the class O_{AB}^0 is a subclass of O_{AB}.

First we shall prove the following which shows the existence of Riemann surfaces belonging to O_{AB}^0.

THEOREM 6. *If a Riemann surface F belongs to O_{HB}, then F belongs to O_{AB}^0.*

Proof. Suppose that there exists a subregion G on F such that G does not belong to SO_{AB}. There exists a non-constant single-valued bounded analytic function $w=f(p)$ in G which is continuous on $G \cup C$ and whose real part equals zero on C, where C is the relative boundary of G with respect to F. By elements $q=[p, f(p)]$ $(p \in G)$ the covering Riemann surface Φ_G is formed over the w-plane. The projection of Φ_G on the w-plane is a bounded domain and is contained in a sufficiently large finite disc $|w|<R$. For a certain positive number ε, we can describe two small discs (k_1) and (k_2) in the disc $|w|<R+\varepsilon$ as follows:

1) (k_1) and (k_2) are disjoint from each other,
2) the projection of Φ_G on the w-plane has common points with both discs (k_1) and (k_2),
3) both the closures of (k_1) and (k_2) have no point lying on the imaginary axis $\Re[w]=0$, and

4) (k_1) and (k_2) have points w_1 and w_2, respectively, which are exterior points of the projection of Φ_G.

In (k_i) $(i=1, 2)$ we describe a disc (k_i') whose centre is the point w_i and which has no common point with the projection of Φ_G. This is possible by virtue of 4). Denote by D_i the doubly connected domain obtained from (k_i) by deleting the closure of (k_i'). Let $\omega_i(w)$ be the harmonic function in D_i being equal to zero on the circumference of (k_i) and to 1 on the circumference of (k_i').

From the condition 2), it is seen that there exists at least one connected piece Φ_i $(i=1, 2)$ lying over (k_i). Denote by G_i the image of Φ_i under the mapping $p \leftrightarrow q$. As is easily seen from the above construction, G_i $(i=1, 2)$ is a subregion in G and the relative boundary C_i of G_i with respect to F is disjoint from C by virtue of 3) and C_i corresponds to the relative boundary of Φ_i with respect to Φ_G which lies over the circumference of (k_i). Further, G_1 and G_2 have no point in common by the condition 1).

The composed function $\omega_i(f(p))$ is a non-constant bounded harmonic function in G_i which is continuous on $G_i \cup C_i$ and vanishes at every point of C_i. Thus we see that subregions G_1 and G_2 do not belong to SO_{HB}. Hence, by the well known fact that, if there exist two subregions on a Riemann surface which are disjoint from each other and do not belong to SO_{HB}, the Riemann surface does not belong to O_{HB} (cf. Nevanlinna [15], Bader-Parreau [2], Mori [11]), our Riemann surface F can not belong to O_{HB}. Therefore, we get our theorem.

As a corollary of this theorem, we have

COROLLARY. *For subregions not being simply connected, the class SO_{HB} is a proper subclass of SO_{AB}.*

Proof. Already we stated in No. 1 that SO_{HB} is a subclass of SO_{AB}. Hence it is sufficient to prove that there exists a subregion not being simply connected and belonging to SO_{AB} and not to SO_{HB}. Let us consider a Riemann surface F which belongs to O_{HB} and has a positive boundary. The existence of such a Riemann surface was proved by Tôki [23]. Deleting from F a simply connected compact domain with an analytic boundary curve, we get a subregion belonging to SO_{AB} on account of Theorem 6. On the other hand, it is obvious that this subregion does not belong to SO_{HB}. Thus we get our assertion.

9. Using Theorem 5, we can prove the following interesting theorem.

THEOREM 7. *Let F be a Riemann surface belonging to O^0_{AB} and let $w=f(p)$ be a non-constant single-valued analytic function on F. If Φ is the covering Riemann surface formed by elements $q=[p, f(p)]$, then Φ has the Iversen property.*

Proof. We choose an arbitrary disc (c_ρ) with the centre $w=w^*$. There exists at least one connected piece Φ_Δ of Φ lying over the disc (c_ρ), since F belongs to O^0_{AB} and so to O_{AB} (see No. 11) and, hence, the projection of Φ on the w-plane is everywhere dense in the w-plane. Let Δ be the image of Φ_Δ on F by the mapping $p \leftrightarrow q$.

To establish the theorem, it is sufficient to prove that there exists a path in Δ starting from any fixed point p_0 $(w^* \neq f(p_0))$ in Δ and tending to a certain inner point of Δ or to the ideal boundary of Δ such that $\lim f(p) = w^*$ along the path. The relative boundary γ of Δ with respect to F consists of at most an enumerable number of analytic curves clustering nowhere in F. Hence Δ is a subregion on F and by our assumption $F \in O^0_{AB}$, Δ belongs to SO_{AB}.

At every point p of γ, the value $f(p)$ falls on the circumference of (c_ρ). Denoting by E the set of values in (c_ρ) which $f(p)$ does not take in Δ and so Φ_Δ does not cover, we see by Theorem 5 that the intersection of E and any closed set in (c_ρ) belongs to $N_{\mathfrak{B}}$. Therefore, we can choose and fix a point p_0 in Δ such that $w^* \neq f(p_0)$. Let q_0 be the image of p_0 on Φ_Δ by the mapping $p \leftrightarrow q$. Since the set belonging to $N_{\mathfrak{B}}$ can not contain a continuum and since Δ is the image of Φ_Δ under the mapping $p \leftrightarrow q$, there is a point q_1 $(\neq q_0)$ on Φ_Δ whose projection w_1 on the w-plane lies in the disc (c_{ρ_1}) with the centre w^*, where ρ_1 equals $\frac{\rho}{2}$. Let us denote by p_1 the image of q_1 by the mapping $p \leftrightarrow q$ and by l_0 a curve combining p_0 with p_1 in Δ.

Generally, we can take the connected piece Φ_{Δ_n} $(n \geqq 1)$ of $\Phi_{\Delta_{n-1}}$ $(\Delta_0 = \Delta)$ lying over the disc (c_{ρ_n}) $\left(\rho_n = \frac{\rho}{2^n}\right)$ and containing the point q_n. If Δ_n is the image on F of Φ_{Δ_n} by the mapping $p \leftrightarrow q$, Δ_n is a subregion belonging to SO_{AB} by our assumption and is contained in Δ_{n-1}. Let E_n be the set of values in (c_{ρ_n}) which $f(p)$ does not take in Δ_n. By Theorem 5, the intersection of E_n and any closed set in (c_{ρ_n}) belongs to $N_{\mathfrak{B}}$. Hence we can see that there exists a point q_{n+1} on Φ_{Δ_n} whose projection w_{n+1} lies in the disc $(c_{\rho_{n+1}})$ with the centre w^*. Denoting by p_{n+1} the image in Δ_n of q_{n+1} under the mapping $p \leftrightarrow q$, we get a curve l_n combining p_n with p_{n+1} in Δ_n.

The curve l in $\varDelta$ constructed as the union of curves $\{l_n\}$ $(n=0, 1, \ldots)$ tends to an inner point of $\varDelta$ or to the ideal boundary of $\varDelta$. It is easy to see that $\lim f(p) = w^*$ along l. Thus our proof is complete.

From Theorems 6 and 7, we can get the following which was proved by Mori [10].

THEOREM 8. *Suppose that a Riemann surface F belongs to O_{HB}. The covering Riemann surface Φ formed by elements $q=[p, f(p)]$ for a non-constant single-valued analytic function $w=f(p)$ on F has the Iversen property.*

10. From Theorem 5, we get the following fact.

Let F be a Riemann surface of the class O_{AB}^0 and let Φ be a covering surface conformally equivalent to F and spread over the w-plane. Then the intersection of any closed set in the disc (c_ρ) on the w-plane and the set of points, which any connected piece of Φ over (c_ρ) does not cover, is the set of $N_{\mathfrak{B}}$.

We can a little improve the above result. Under the same notations as above, let Φ_Δ be any connected piece of Φ lying over the arbitrary disc (c_ρ) on the w-plane. Denote by $\varDelta$ the image of Φ_Δ on F. We denote by $n(w)$ the number of points of Φ_Δ lying over the point $w(\in(c_\rho))$ and by E the set of points w in (c_ρ) such that

$$n(w) < Z_\rho = \sup_{w\in(c_\rho)} n(w) \quad (\leqq +\infty).$$

For any integer n, let E_n be the set of points w in (c_ρ) satisfying the inequality $n(w) \leqq n$. Then, this set E_n is closed with respect to (c_ρ) and $E_n \subset E_{n+1}$ $(n \geqq 1)$ and further

$$\bigcup_{n<Z_\rho} E_n = E.$$

Suppose that there exist an integer n $(<Z_\rho)$ and a closed set S in (c_ρ) such that $S\cap E_n$ does not belong to $N_{\mathfrak{B}}$. Let us denote by n_0 the smallest of such indices. Since $n_0 < Z_\rho$, the set of points w of (c_ρ) not belonging to E_{n_0} is a non-empty open set. Hence the boundary set B_{n_0} of $S\cap E_{n_0}$ with respect to (c_ρ) is not empty and does not belong to $N_{\mathfrak{B}}$. By Theorem 3, the B-kernel $B_{n_0}^*$ of B_{n_0} is not empty and does not belong to $N_{\mathfrak{B}}$. On the other hand, the closed set $B_{n_0}\cap E_{n_0-1}$ is contained in E_{n_0-1} and so belongs to $N_{\mathfrak{B}}$. Hence there exists at least one point w_0 of $B_{n_0}^*$ not belonging to $B_{n_0}\cap E_{n_0-1}$. Therefore, this point w_0 belongs to the set $B_{n_0}^*\cap(E_{n_0}-E_{n_0-1})$ and so $n(w_0)=n_0$.

Since w_0 belongs to $B^*_{n_0} \cap (E_{n_0} - E_{n_0-1})$, we can choose a sufficiently small disc (c) $(\subset (c_p))$ containing the point w_0 such that the closure e of the intersection of $B^*_{n_0}$ and any closed set in (c) does not belong to $N_{\mathfrak{B}}$ and Φ_Δ has exactly n_0 discs above (c), where ν-sheeted disc is counted as ν discs. On the other hand, w_0 belongs to B_{n_0} and so there exists a point w in (c) satisfying the inequality $n(w) > n_0$. Hence, besides these n_0 discs, Φ_Δ has at least one connected piece δ over (c). It is obvious from the above that δ does not cover the set e. In fact, any point of the set e is covered by Φ_Δ at most n_0 times and is already covered by the above mentioned n_0 discs.

Denoting by G' the image of δ on F by the mapping $p \leftrightarrow q$, we can see that G' is a subregion on F and the relative boundary of G' with respect to F corresponds to the boundary of δ lying over the circumference of (c). Since δ does not cover the set e, it is seen by Theorem 5 that G' does not belong to SO_{AB}.

Therefore, if F belongs to O^0_{AB}, then, for any integer $n < Z_p$ and for any closed set S in (c_p), the set $S \cap E_n$ belongs to $N_{\mathfrak{B}}$. Since the set of $N_{\mathfrak{B}}$ is non-dense as stated already, the set of all points in (c_p) satisfying $n(w) < Z_p$ is of the first category. In particular, if Z_p is finite, then, by Theorem 4, the intersection of this set and any closed set in (c_p) belongs to $N_{\mathfrak{B}}$. Thus we have

THEOREM 9. *Under the assumption of Theorem 7, it holds that, for any connected piece of Φ lying over an arbitrary disc (c_p) on the w-plane, the set of points w in (c_p) satisfying the inequality $n(w) < Z_p$ is of the first category. In particular, if $Z_p < +\infty$, the intersection of this set and any closed set in (c_p) belongs to $N_{\mathfrak{B}}$.*

Remark. In the case of $Z_p < +\infty$, this result coincides with a result of Kuramochi [6].

11. It is immediately seen that, if a Riemann surface F belongs to O^0_{AB}, then F belongs to O_{AB}. For, if F does not belong to O_{AB}, there exists a non-constant single-valued bounded analytic function $f(p)$ on F. Choosing a point p_0 on F arbitrarily, we consider the set of points p of F satisfying the inequality $\Re[f(p)] > \Re[f(p_0)]$. As is easily seen, this set is not empty and open. Let G be any connected component of this open set. On the relative boundary C of G with respect to F, $\Re[f(p_0)]$ equals $\Re[f(p_0)]$. It is easy to see that G is a subregion on F. The function $f(p) - \Re[f(p_0)]$ is non-constant, single-valued, bounded and analytic and is continuous on $G \cup C$ and the real part of this function vanishes at every point of C. Hence G does not belong to SO_{AB}.

On the other hand, Myrberg [14] gave a very important example of a covering Riemann surface Φ of infinite genus which belongs to O_{AB} and has not the Iversen property. Mentioning this and Theorem 7, we can see immediately that there exists a Riemann surface belonging to O_{AB} and not to O^0_{AB}. Therefore, the class O^0_{AB} is the proper subclass of O_{AB}.

Theorem 6 shows that the class O_{HB} of Riemann surfaces is contained in the class O^0_{AB}.

In the following, we shall give an example of Riemann surfaces belonging to O^0_{AB} and not to O_{HD} (see No. 18).

12. We suppose that F is an open Riemann surface. Let $\{F_n\}$ $(n = 0, 1, \ldots)$ be an exhaustion of F satisfying the following conditions:

1°) for each n, the domain F_n on F is compact with respect to F and the boundary Γ_n of F_n consists of a finite number of analytic closed curves,

2°) $\overline{F}_n = F_n \cup \Gamma_n \subset F_{n+1}$, $(n = 0, 1, \ldots)$,

3°) each connected component of $F - \overline{F}_n$ $(n = 0, 1, \ldots)$ is non-compact with respect to F and

4°) $\bigcup_{n=0}^{\infty} F_n = F$.

The open set $F_n - \overline{F}_{n-1}$ $(n \geqq 1)$ consists of a finite number of domains R^k_n $(k = 1, 2, \ldots, \nu = \nu(n))$. The boundary of R^k_n consists of analytic closed curves contained in $\Gamma_{n-1} \cup \Gamma_n$. We shall denote by α^k_{n-1} the part of the boundary of R^k_n on Γ_{n-1} and by β^k_n that on Γ_n. Let $u^k_n(p)$ be the harmonic function in R^k_n which vanishes at every point of α^k_{n-1} and equals $\log \mu^k_n$ on β^k_n and whose conjugate function $v^k_n(p)$ has the variation 2π on β^k_n, i.e.,

$$\int_{\beta^k_n} dv^k_n = 2\pi,$$

where the integral is taken in the positive sense with respect to R^k_n. The quantity $\log \mu^k_n$ is the so-called harmonic modulus of R^k_n. If we choose an additive constant of $v^k_n(p)$ suitably, the regular function $u^k_n(p) + iv^k_n(p)$ maps R^k_n with a finite number of suitable slits onto a slit-rectangle $0 < u^k_n < \log \mu^k_n$, $0 < v^k_n < 2\pi$ one-to-one conformally. Similarly we define the harmonic modulus

$\log \sigma_n$ of the open set $F_n - \bar{F}_{n-1}$ as follows. Let $u_n(p)$ be the harmonic function in $F_n - \bar{F}_{n-1}$ which is equal to zero on Γ_{n-1} and to $\log \sigma_n$ on Γ_n and whose conjugate function $v_n(p)$ has the variation 2π, that is,

$$\int_{\Gamma_n} dv_n = 2\pi.$$

The quantity $\log \sigma_n$ is the harmonic modulus of $F_n - \bar{F}_{n-1}$. If we choose an additive constant of $v_n(p)$ suitably, the regular function $u_n(p) + iv_n(p)$ maps R_n^k $(k = 1, 2, \ldots, \nu)$ with a finite number of suitable slits onto a slit-rectangle $0 < u_n < \log \sigma_n$, $b_k < v_n < a_k + b_k$ one to one conformally, where a_k $(k = 1, \ldots, \nu)$ and b_k $(k = 1, \ldots, \nu)$ are constants satisfying the following relations:

$$a_k = 2\pi \frac{\log \sigma_n}{\log \mu_n^k}, \qquad \sum_{k=1}^{\nu(n)} a_k = 2\pi$$

and

$$b_1 = 0, \qquad b_k = \sum_{i=1}^{k-1} a_i \quad (1 < k \leqq \nu).$$

Consequently, the function $u_n(p) + iv_n(p)$ maps $F_n - \bar{F}_{n-1}$ with a finite number of suitable slits onto a slit-rectangle $0 < u_n < \log \sigma_n$, $0 < v_n < 2\pi$ in a one to one conformal manner. From this, it follows that

$$\frac{1}{\log \sigma_n} = \sum_{k=1}^{\nu(n)} \frac{1}{\log \mu_n^k}. \tag{1}$$

Further, the function $u(p) + iv(p)$ defined by $u_n(p) + iv_n(p) + \sum_{j=1}^{n-1} \log \sigma_j$ for each $F_n - \bar{F}_{n-1}$ $(n \geqq 1)$ maps $F - \bar{F}_0$ with at most an enumerable number of suitable slits onto a strip domain $0 < u < R$, $0 < v < 2\pi$ with at most an enumerable number of slits one-to-one conformally, where

$$R = \sum_{j=1}^{\infty} \log \sigma_j.$$

This strip domain is the graph of F associated with the exhaustion $\{F_n\}$ in the sense of Noshiro [16]. We call R the length of this graph.

Sario [21] and Noshiro [16] proved that F has a null boundary if and only if there exists a graph of F whose length R is infinite.

13. Let γ_r be the niveau curve $u(p) = r$ $(0 < r < R)$ on F. The niveau curve γ_r consists of a finite number of closed analytic curves γ_r^i $(i = 1, \ldots, m = m(r))$. Putting

$$\Lambda(r) = \operatorname*{Max}_{1 \leqq i \leqq m} \int_{\gamma_r^i} dv,$$

we can formulate Pfluger's theorem [17] as follows.

THEOREM 10. *If there exists a graph of F such that the integral*

$$\int_0^R e^{4\pi \int_0^r \frac{dr}{\Lambda(r)}} dr \tag{2}$$

diverges, then F belongs to O_{AB}.

Although the proof is obtained by Pfluger's argument, we shall state the outline of the proof.

Let $f(p) = U(p) + iV(p)$ be a single-valued bounded analytic function on F. If we denote by $D(r)$ the Dirichlet integral of $f(p)$ taken over the region bounded by γ_r and containing F_0, we get

$$D(r) = \int_{\gamma_r} U dV = \sum_{i=1}^{m} \int_{\gamma_r^i} U \frac{\partial U}{\partial u} dv.$$

Since every γ_r^i is closed, it follows by Wirtinger's inequality that

$$\int_{\gamma_r^i} U^2 dv \leqq \frac{(\Lambda(r))^2}{4\pi^2} \int_{\gamma_r^i} \left(\frac{\partial U}{\partial v}\right)^2 dv.$$

Hence, using the Schwarz inequality, we have

$$\begin{aligned}
\sum_{i=1}^{m} \int_{\gamma_r^i} U \frac{\partial U}{\partial u} dv &\leqq \sum_{i=1}^{m} \sqrt{\int_{\gamma_r^i} U^2 dv \int_{\gamma_r^i} \left(\frac{\partial U}{\partial u}\right)^2 dv} \\
&\leqq \frac{\Lambda(r)}{2\pi} \sum_{i=1}^{m} \sqrt{\int_{\gamma_r^i} \left(\frac{\partial U}{\partial v}\right)^2 dv \int_{\gamma_r^i} \left(\frac{\partial U}{\partial u}\right)^2 dv} \\
&\leqq \frac{\Lambda(r)}{4\pi} \int_{\gamma_r} \left[\left(\frac{\partial U}{\partial u}\right)^2 + \left(\frac{\partial U}{\partial v}\right)^2\right] dv.
\end{aligned}$$

Therefore, it holds that

$$4\pi \frac{dr}{\Lambda(r)} \leqq \frac{dD(r)}{D(r)},$$

whence, by the integration, we obtain

$$D(0) e^{4\pi \int_0^r \frac{dr}{\Lambda(r)}} \leqq D(r),$$

where $D(0)$ is the Dirichlet integral of $f(p)$ taken over F_0. On the other hand, we get

$$\int_0^r D(r)dr = \frac{1}{2}\left(\int_{\gamma_r} U^2 dv - \int_{\Gamma_0} U^2 dv\right) \leqq \frac{1}{2}\int_{\gamma_r} U^2 dv,$$

for it holds that

$$\frac{d}{dr}\left(\int_{\gamma_r} U^2 dv\right) = 2\int_{\gamma_r} U\frac{\partial U}{\partial u}dv = 2D(r).$$

Thus, putting $M = \sup_{p\in F}|f(p)|$, we have

$$D(0)\int_0^R e^{4\pi\int_0^r \frac{dr}{\Lambda(r)}}dr \leqq \pi M^2.$$

If the integral (2) diverges, we get that $D(0)=0$. Thus the function $f(p)$ must reduce to a constant.

14. Here we shall show that the above theorem implies Mori's result [12] which is the modification of Pfluger's theorem [18].

We consider an exhaustion $\{F_n\}$ $(n=0, 1, \ldots)$ of F and use the same notations as in No. 12. Putting $\log \mu_n = \underset{1\leqq k\leqq \nu(n)}{\mathrm{Min}} \log \mu_n^k$ and $N(n) = \underset{1\leqq j\leqq n}{\mathrm{Max}}\, \nu(j)$, we have

THEOREM 11. (Pfluger-Mori). *If*

$$\limsup_{n\to\infty}\left\{\sum_{j=1}^n \log \mu_j - \frac{1}{2}\log N(n)\right\} = +\infty,$$

then F belongs to O_{AB}.

Proof. First we construct the graph of F associated with the exhaustion $\{F_n\}$ $(n=0, 1, \ldots)$ and calculate the integral

$$\int_0^R e^{\kappa\int_0^r \frac{dr}{\Lambda(r)}}dr,$$

where κ is a positive constant. It is evident that

$$\int_0^R e^{\kappa\int_0^r \frac{dr}{\Lambda(r)}}dr = \lim_{n\to\infty}\sum_{j=1}^n \int_{\tau_{j-1}}^{\tau_j} e^{\kappa\int_0^r \frac{dr}{\Lambda(r)}}dr,$$

where

$$\tau_j = \sum_{i=0}^j \log \sigma_i \quad \text{and} \quad \tau_0 = \log\sigma_0 = 0.$$

If we put

$$I_j = \int_{\tau_{j-1}}^{\tau_j} e^{\kappa g(r)}dr, \quad (j=1, 2, \ldots),$$

where

$$g(r) = \int_0^r \frac{dr}{\Lambda(r)}$$

for $0 < r \leqq \tau_1$ and

$$g(r) = \sum_{s=1}^{j-1} \int_{\tau_{s-1}}^{\tau_s} \frac{dr}{\Lambda(r)} + \int_{\tau_{j-1}}^{r} \frac{dr}{\Lambda(r)}$$

for $\tau_{j-1} < r \leqq \tau_j$ $(j > 1)$, then it holds that

$$\int_0^R e^{\kappa \int_0^r \frac{dr}{\Lambda(r)}} dr = \lim_{n \to \infty} \sum_{j=1}^{n} I_j.$$

Since it is easily seen by the construction of the graph of F that $\Lambda(r)$ is not greater than $2\pi \dfrac{\log \sigma_s}{\log \mu_s}$ in the interval (τ_{s-1}, τ_s) of r. Hence

$$g(r) \geqq \frac{1}{2\pi} \frac{\log \mu_1}{\log \sigma_1} r$$

for $0 < r \leqq \tau_1$ and

$$\begin{aligned} g(r) &\geqq \sum_{s=1}^{j-1} \int_{\tau_{s-1}}^{\tau_s} \frac{\log \mu_s}{2\pi \log \sigma_s} dr + \int_{\tau_{j-1}}^{r} \frac{\log \mu_j}{2\pi \log \sigma_j} dr \\ &= \frac{1}{2\pi} \sum_{s=1}^{j-1} \frac{\log \mu_s}{\log \sigma_s} (\tau_s - \tau_{s-1}) + \frac{1}{2\pi} \frac{\log \mu_j}{\log \sigma_j} (r - \tau_{j-1}) \\ &= \frac{1}{2\pi} \sum_{s=1}^{j-1} \log \mu_s - \frac{1}{2\pi} \frac{\log \mu_j}{\log \sigma_j} \tau_{j-1} + \frac{1}{2\pi} \frac{\log \mu_j}{\log \sigma_j} r \end{aligned}$$

for $\tau_{j-1} < r \leqq \tau_j$ $(j > 1)$. Therefore, we obtain the following estimation of I_j:

$$I_1 \geqq \frac{\log \sigma_1}{K \log \mu_1} (e^{K \log \mu_1} - 1)$$

and

$$\begin{aligned} I_j &\geqq e^{K \sum_{s=1}^{j-1} \log \mu_s - K \frac{\log \mu_j}{\log \sigma_j} \tau_{j-1}} \frac{\log \sigma_j}{K \log \mu_j} \left(e^{K \frac{\log \mu_j}{\log \sigma_j} \tau_j} - e^{K \frac{\log \mu_j}{\log \sigma_j} \tau_{j-1}}\right) \\ &= \frac{\log \sigma_j}{K \log \mu_j} e^{K \sum_{s=1}^{j-1} \log \mu_s} (e^{K \log \mu_j} - 1) \end{aligned}$$

for $j > 1$, where K equals $\dfrac{\kappa}{2\pi}$, whence, putting $\log \mu_0 = 0$, we have

$$I_j \geqq \frac{\log \sigma_j}{K \log \mu_j} \left(e^{K \sum_{s=0}^{j} \log \mu_s} - e^{K \sum_{s=0}^{j-1} \log \mu_s}\right)$$

for each j. By the formula (1), we get

$$\log \sigma_j \geqq \frac{\log \mu_j}{\nu(j)},$$

and, hence, by $N(n) \geqq 1$,

$$(3) \qquad \sum_{j=1}^{n} I_j \geqq \sum_{j=1}^{n} \frac{1}{K\nu(j)} \left(e^{K\sum_{s=0}^{j} \log \mu_s} - e^{K\sum_{s=0}^{j-1} \log \mu_s}\right)$$

$$\geqq \frac{1}{KN(n)} \left(e^{K\sum_{s=1}^{n} \log \mu_s} - 1\right)$$

$$= \frac{1}{K}\left(e^{K\sum_{s=1}^{n} \log \mu_s - \log N(n)} - \frac{1}{N(n)}\right)$$

$$\geqq \frac{1}{K}\left(e^{K\sum_{s=1}^{n} \log \mu_s - \log N(n)} - 1\right).$$

In our case when $\kappa = 4\pi$, K is equal to 2. By the assumption of the theorem, the integral (2) diverges and so, by Theorem 10, F belongs to O_{AB}.

15. We proved in the previous paper [9] the following theorem of Phragmén-Lindelöf type.

Suppose that $f(p)$ is a single-valued regular function in a non-compact region G on an open Riemann surface and that the real part of $f(p)$ is equal to zero on the relative boundary of G. Construct a graph of F with the length R and denote by $M(r)$ the maximum of the absolute values of the real part of $f(p)$ on the common part θ_r of G and the niveau curve γ_r. If

$$\liminf_{r \to R} \frac{(M(r))^2}{\int_{r_0}^{r} e^{2\pi \int_{r_0}^{r} \frac{dr}{\Theta(r)}} dr} = 0,$$

where $\Theta(r) = \operatorname{Max}_i \int_{\theta_r^i} dv$ and θ_r^i is a component of θ_r and r_0 is a suitable number such that θ_{r_0} is not empty, then $f(p)$ reduces to a constant.

This and Theorem 7 imply the following which is slightly different from Kuramochi's result [7].

THEOREM 12. *Let F be an open Riemann surface satisfying the following condition: there exists a graph with the length R of F such that the integral*

$$(4) \qquad \int_0^R e^{2\pi \int_0^r \frac{dr}{\Lambda(r)}} dr$$

diverges. Then F belongs to O^0_{AB} and, hence, the covering Riemann surface Φ formed by elements $q=[p, f(p)]$ for a non-constant single-valued analytic function $w=f(p)$ on F has the Iversen property.

16. Let F^* be a compact Riemann surface of positive genus g. We cut F^* along simple closed analytic loop-cuts L_i $(i=1, \ldots, h\ ;\ 1 \leqq h \leqq g)$ disjoint from each other and not dividing F^* into two or more parts and denote by F_0 the resulting surface. We take infinitely many same samples as F_0 and construct a Schottky covering surface of F^* by connecting these samples along opposite shores of L_i $(i=1, \ldots, h)$ in the well known way.

In this construction of F, first we fix a sample F_0 and denote by R_1^k $(k=1, \ldots, 2h)$ the samples which we connect to F_0. Denoting by F_1 the resulting surface, we connect $2h(2h-1)$ samples to F_1 and we denote by R_2^k $(k=1, \ldots, 2h(2h-1))$ these samples. Thus we get the surface F_2. In general we connect $2h(2h-1)^{n-1}$ samples to F_{n-1}. We denote by R_n^k $(k=1, \ldots, 2h(2h-1)^{n-1})$ these samples and get the surface F_n.

The sequence $\{F_n\}$ $(n=0, 1, \ldots)$ gives an exhaustion of F.

17. Now we consider such an exhaustion $\{F_n\}$ $(n=0, 1, \ldots)$ of F and construct the graph of F associated with this exhaustion. The harmonic modulus $\log \sigma_{n+1}$ of $F_{n+1}-\overline{F}_n$ $(n \geqq 0)$ is evaluated as follows. Since $F_1-\overline{F}_0$ is the union of all R_1^k $(k=1, \ldots, 2h)$, we can get from (1)

$$\frac{1}{\log \sigma_1}=\sum_{k=1}^{2h} \frac{1}{\log \mu_1^k},$$

where $\log \mu_1^k$ is the harmonic modulus of R_1^k. By the construction of F, it is easy to see that $F_{n+1}-\overline{F}_n$ $(n \geqq 0)$ is the union of all R_{n+1}^k $(k=1, \ldots, 2h(2h-1)^n)$. It is immediate from the construction of F that among R_{n+1}^k $(k=1, \ldots, 2h(2h-1)^n)$, there exist $(2h-1)^n$ samples which are connected to F_n along shores corresponding to the same shore of the same L_i. And so we have

$$\log \mu_{n+1}=\min_{1 \leqq k \leqq 2h(2h-1)^n} \log \mu_{n+1}^k=\min_{1 \leqq k \leqq 2h} \log \mu_1^k=\log \mu_1$$

for each n and

$$\frac{1}{\log \sigma_{n+1}}=(2h-1)^n \sum_{k=1}^{2h} \frac{1}{\log \mu_1^k}=(2h-1)^n \frac{1}{\log \sigma_1}.$$

Hence the length $R=\sum_{n=1}^{\infty}\log\sigma_n$ of this graph is equal to $\log\sigma_1\sum_{n=0}^{\infty}\frac{1}{(2h-1)^n}$.

By Sario-Noshiro's result [21], [16], the Schottky covering surface F in question has a null boundary if $h=1$. And, by Myrberg-Tsuji's theorem [13], [25], F has a positive boundary and does not belong to O_{HD} if $h \geqq 2$.

On the other hand, Sario [19] and Tsuji [25] have shown that F belongs always to the class O_{AD}.

Since this does not always imply that the Schottky covering surface F in question belongs to O_{AB}^0, we can not immediately see that the covering Riemann surface Φ formed by elements $q=[p, f(p)]$ for any non-constant single-valued analytic function $w=f(p)$ on F has the Iversen property. But we can give a sufficient condition for it.

THEOREM 13. *If h equals 1 or if the minimum* $\log\mu_1$ *of the moduli* $\log\mu_1^k$ *of* R_1^k $(i=1,\dots,2h)$ *is not smaller than* $\log(2h-1)$, *then F belongs to* O_{AB}^0 *and so Φ has the Iversen property.*

Proof. In the case when $h=1$, F has a null boundary. Hence, as is well known, F belongs to O_{HB}. Thus, in this case, Theorem 8 implies our assertion.

Next we consider the case of $h \geqq 2$. If we consider the graph of F associated with the exhaustion $\{F_n\}$ $(n=0, 1, \dots)$ which was constructed in No. 16, then it is easy to see that $\nu(j)$ equals $2h(2h-1)^{j-1}$ and $\log\mu_j$ equals $\log\mu_1$. Therefore, we get, using the inequality (3) for the case of $\kappa=2\pi$,

$$\int_0^R e^{2\pi\int_0^r\frac{dr}{\Lambda(r)}}dr \geqq \sum_{j=1}^{\infty}\frac{1}{\nu(j)}e^{\sum_{s=0}^{j-1}\log\mu_s}(e^{\log\mu_j}-1)$$
$$=\sum_{j=1}^{\infty}\frac{1}{2h(2h-1)^{j-1}}e^{(j-1)\log\mu_1}(e^{\log\mu_1}-1)$$
$$=\frac{1}{2h}(\mu_1-1)\sum_{j=1}^{\infty}\left(\frac{\mu_1}{2h-1}\right)^{j-1},$$

where $\mu_0=1$. Since $\mu_1 \geqq 2h-1$ by the assumption, the integral (4) must be divergent. From Theorem 12, we have our assertion.

As is easily seen from the argument of the above proof, we can get a criterion for F to belong to O_{AB}^0 which is similar to Theorem 11. We shall state it without proof.

THEOREM 14. *Under the same notations as in Theorem* 11, *if*

$$\limsup_{n\to\infty}\{\sum_{j=1}^{n}\log\mu_j - \log N(n)\} = +\infty,$$

then F belongs to O^0_{AB}.

This is closely related with Kuramochi's theorem [6].

18. Here we shall give an example of a Schottky covering surface satisfying the condition of Theorem 13. We consider the ring domain $R: 1<|z|<12$ on the complex z-plane. Identifying its boundary point $e^{i\theta}$ with the boundary point $12e^{i\theta}$ and introducing local parameters in the usual manner, we get a closed Riemann surface T of genus 1. T is nothing but a torus. By this identification, the boundary curves $|z|=1$ and $|z|=12$ of R correspond to two shores of a loop-cut L_1 of T which is an analytic closed curve in T and does not devide T. If we cut T along L_1 and denote by T' the resulting surface, there exists a one-to-one conformal mapping $z=z(p)$ $(p\in T')$ between T' and R and, by this mapping, two shores of L_1 correspond to the circles $|z|=1$ and $|z|=12$.

Consider two small circular closed discs d_1 and d_2 in the ring domain $3<|z|<4$ such that d_1 and d_2 are disjoint from each other. Denoting by D_1 and D_2 the images of d_1 and d_2 on T' under the mapping $z=z(p)$ and deleting D_1 and D_2 from the torus T, we get an open Riemann surface T_0 of genus 1 whose boundary consists of two analytic closed curves γ_1 and γ_2, where γ_i $(i=1,2)$ is the boundary of D_i $(i=1,2)$. Cutting T_0 along L_1, we denote by T_0' the resulting surface which is the domain obtained by deleting D_1 and D_2 from T'. This domain T_0' is mapped one-to-one conformally on the domain obtained by deleting d_1 and d_2 from R under the mapping $z=z(p)$.

We construct a double of T_0 along $\gamma_1\cup\gamma_2$ and denote it by F^*. It is easy to see that F^* is a closed Riemann surface of genus 3. We consider the above loop-cut L_1 on F^* and denote by L_2 the image of L_1 under the indirectly conformal mapping of F^* on itself which leaves every point of $\gamma_1\cup\gamma_2$ fixed. Then L_2 is also a loop-cut of F^* and is disjoint from L_1 and does not devide F^*. Cutting F^* along L_1 and L_2, we have a Riemann surface F_0 which is of genus 1 and has a boundary consisting of four analytic closed curves. These curves are shores of loop-cuts L_1 and L_2. It is obvious that F_0 is a double of T_0' along $\gamma_1\cup\gamma_2$. By the manner stated in No. 16, we construct a Schottky covering surface F of F^* from F_0. This surface F is of infinite genus.

Now we shall evaluate the quantity $\log\mu_1$ corresponding to the above F_0.

Let $u'(p)$ be the harmonic function in T_0' which equals zero on one shore L_1' of L_1 and to $\log \mu'$ on the other shore L_1'' of L_1 and on $\gamma_1 \cup \gamma_2$ and whose conjugate function $v'(p)$ satisfies the condition $\int_{L_1'} dv = 2\pi$. If we construct the harmonic function $u''(p)$ in T_0' which equals to zero on L_1'' and to $\log \mu''$ on $L_1' \cup \gamma_1 \cup \gamma_2$ and whose conjugate function $v''(p)$ satisfies $\int_{L_1''} dv = 2\pi$, then it is evident that

$$\log \mu_1 \geqq \operatorname{Min}(\log \mu', \log \mu'').$$

Since T_0' is mapped conformally on the domain $R-(d_1 \cup d_2)$ by $z=z(p)$ and since L_1' corresponds to a circle $|z|=1$ or $|z|=12$, it is immediately seen that $\operatorname{Min}(\log \mu', \log \mu'') \geqq \log 3$. Hence we have

$$\log \mu_1 \geqq \log 3.$$

In our example F of a Schottky covering surface, the genus of F^* is 3 and the number h in Theorem 13 is 2 and, further, $\log \mu_1 \geqq \log 3$. Therefore, the surface F of infinite genus belongs to O_{AB}^0 and not to O_{HD}.

By the similar consideration, we can obtain the existence of a Schottky covering surface F of planar character belonging to O_{AB}^0 and having a positive boundary. Mapping such a surface F on the planar domain, we get the domain whose boundary is of positive logarithmic capacity and belongs to $N_{\mathfrak{B}}$.

19. Summarizing the statements in Nos. 11 and 18, we have the following

THEOREM 15. *For Riemann surfaces of infinite genus, it holds that*

$$O_{HB} \subsetneqq O_{AB}^0 \subsetneqq O_{AB},\quad O_{HD} \not\supset O_{AB}^0 \quad \textit{and} \quad O_{HD} \not\subset O_{AB}^0.$$

For Riemann surfaces of finite genus, the following holds:

$$O_G \subsetneqq O_{AB}^0 \subset O_{AB}.$$

Proof. For Riemann surfaces of infinite genus, it is seen from the above that

$$O_{HB} \subset O_{AB}^0 \subsetneqq O_{AB}, \quad O_{HD} \not\supset O_{AB}^0.$$

Suppose that $O_{HB} = O_{AB}^0$. Since O_{HB} is a subclass of O_{HD} by Virtanen's result [26], O_{AB}^0 must be a subclass of O_{HD}, which is a contradiction. Hence O_{HB} is a proper subclass of O_{AB}^0.

Further, O_{HD} is not a subclass of O_{AB} by Tôki's example [24]. Hence O_{HD}

is not a subclass of O^0_{AB}.

For Riemann surfaces of finite genus, our inclusion-relations are evident.

Remark 1. In the case of Riemann surfaces of finite genus, it is still open whether O^0_{AB} is a proper subclass of O_{AB} or not.

Remark 2. From the above theorem, we see that, without restriction for genus of Riemann surfaces, Mori's result, Theorem 8, can not imply Theorem 7.

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