Tetsuya Asai
Nagoya Math. J.
Vol. 40 (1970), 193–211

ON A CERTAIN FUNCTION ANALOGOUS TO $\log|\eta(z)|$

TETSUYA ASAI

The purpose of this paper is to give the limit formula of the Kronecker's type for a non-holomorphic Eisenstein series with respect to a Hilbert modular group in the case of an arbitrary algebraic number field. Actually, we shall generalize the following result which is well-known as the first Kronecker's limit formula. From our view-point, this classical case is corresponding to the case of the rational number field $\boldsymbol{Q}$.

Let z be a point of the complex-upper-half-plane, and by $y(z)$ we denote the imaginary part of z, i.e., $y(z) = y > 0$ for $z = x + iy$. L denotes the group $SL(2, \boldsymbol{Z})$ ($\boldsymbol{Z}$: the ring of rational integers), and L_1 the subgroup consisting of all $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in L$ such that $\gamma = 0$. $\sigma\langle z\rangle$ stands for $(\alpha z + \beta)\cdot(\gamma z + \delta)^{-1}$ as usual. The non-holomorphic Eisenstein series with respect to L is defined by

$$E^*(z, s) = \sum_{\sigma \in L_1 \backslash L} y(\sigma\langle z\rangle)^s,$$

and this converges absolutely in the half plane $Re\ s > 1$. $E^*(z, s)$, as the function of s, is essentially an Epstein zeta function of the positive definite binary quadratic form. In fact, define another Dirichlet series by

$$E(z, s) = \frac{1}{2} \sum_{\substack{m, n = -\infty \\ (m, n) \neq (0, 0)}}^{\infty} \frac{y^s}{|mz + n|^{2s}},$$

then obviously, $E(z, s) = \zeta(2s) E^*(z, s)$, where $\zeta(s)$ is the Riemann zeta function. The series $E(z, s)$ can be holomorphically continued to the whole s-plane, and the continuation is regular except for one simple pole at $s = 1$ with the residue $\pi/2$. The Kronecker's limit formula gives the constant term in the Laurent expansion at $s = 1$ explicitly, i.e.,

$$\lim_{s \to 1}\left(E(z, s) - \frac{\pi/2}{s - 1}\right) = \frac{\pi}{2}(2C - \log 4 - \log y(z) + h(z)),$$

Received August 11, 1969.

where C is the Euler constant and $h(z)=-4\log|\eta(z)|$; $\eta(z)$ being the Dedekind η-function. It is well-known that the function $\eta(z)$ is a holomorphic cusp form of dimension $-\frac{1}{2}$ with respect to L, but we here notice only the following properties of the non-holomorphic function $h(z)$ itself.

I. $h(z)$ is a real-valued, real analytic function of two real variables x, y $(z = x + iy)$, and vanishes by the Laplace-Beltrami operator $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ of the complex-upper-half-plane.

II. $h(z)$ is a modular form with the automorphic factor $\log|\gamma z + \delta|^2$ with respect to L, i.e., $h(z) = \log|\gamma z + \delta|^2 + h(\sigma\langle z\rangle)$ for any $\sigma = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in L$.

Furthermore, as was recently remarked by Weil,

III. $h(z)$ is associated with the Dirichlet series $\zeta(s)\zeta(s+1)$ in the usual sense, i.e., essentially under Mellin transform.

Now, all the above result can be generalized to the case of any algebraic number field. Let F be an arbitrary algebraic number field and $\mathfrak{o}$ be the ring of integers of F, whose class number, we assume for simplicity, is equal to one. Let r_1, r_2 be the numbers of real and imaginary infinite places of F, respectively. For the upper-half-space corresponding to F, we need the product space $\mathscr{H} = H_c^{r_1} \times H_q^{r_2}$, where H_c and H_q are the complex-upper-half-plane and the quaternion-upper-half-space, respectively. H_q consists of all quaternion numbers $z = \begin{pmatrix} x & -y\\ y & \bar{x}\end{pmatrix}$ such that $y > 0$ while x is any complex number, and we also denote $y(z) = y$ for such z. Take the Hilbert modular group $\Gamma = SL(2, \mathfrak{o})$ which operates on $\mathscr{H}$ discontinuously, and let Γ_1 be the group of all $\sigma = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma$ such that $\gamma = 0$. For a point $\boldsymbol{z} = (z_j)$ of $\mathscr{H}$ and a complex number s with $Re\ s > 1$, the non-holomorphic Eisenstein series with respect to Γ is defined by

$$E^*(\boldsymbol{z}, s) = \sum_{\sigma \in \Gamma_1 \backslash \Gamma} N\boldsymbol{y}(\sigma\langle \boldsymbol{z}\rangle)^s,$$

here $\boldsymbol{y}(\boldsymbol{z}) = (y(z_j))$ and $N\boldsymbol{y}(\boldsymbol{z}) = \prod_{j=1}^{r_1+r_2} y(z_j)^{e_j}$; e_j being 1 or 2, according as the case of H_c or H_q. Similarly to the classical case, $E(\boldsymbol{z}, s) = \zeta_F(2s) E^*(\boldsymbol{z}, s)$ (where $\zeta_F(s)$ being the Dedekind zeta function of F) can be regarded as a generalized Epstein zeta function, and it can be holomorphically continued to the whole s-plane regularly except for a simple pole at $s = 1$. When we calculate the explicit limit formula of the Kronecker's type for $E(\boldsymbol{z}, s)$, we are naturally led to a certain new function $h(z)$ on $\mathscr{H}$. The function $h(z)$ is very analogous to $\log|\eta(z)|$, and really $h(z)$ satisfies all the conditions corresponding to the above I, II and III. (For this reason, we may call the function $h(z)$ "the harmonic modular form" on $\mathscr{H}$.)

The harmonic modular form $h(z)$ is in general expressed by the modified Bessel function in the Fourier expansion form, and so we may find the close relationship to the non-holomorphic automorphic functions defined by Maass. Actually, it seems possible to construct the theory of Maass' type in the case of Hilbert modular groups, but we shall not make further discussion on this subject in this paper.

For the purpose of emphasizing that the concept of the harmonic modular form is very naturally introduced, we would like to start our consideration by calculating the inverse Mellin transform of $\zeta_F(s)\zeta_F(s+1)$ in the case of F being the Gauss' number field (in § **1**). The Eisenstein series $E(z, s)$ in the general case will be defined in § **2**, and there we shall mention about the holomorphic continuation and the functional equation. In § **3** the main theorems about the Kronecker's limit formula will be proved, containing the discussion on the Dirichlet series associated with the harmonic modular form. Throughout this paper, we restrict our consideration only in the case of the class number one, but this does not essentially lose the generality. It, however, becomes some complicated in the general case; for instance, we must deal with many numbers of Eisenstein series and harmonic modular forms of a vector type.

In the case of totally real number fields our limit formula seems in substance the same one of Konno ([6]), or Katayama ([5]) in the real quadratic case. But it seems that they did not catch the harmonic modular form explicitly, and really, Hecke who originally studied on these problems did seek after "die zu $\log\eta(z)$ analogen Funktionen", though we choose the simpler way to seek after "die zu $\log|\eta(z)|$ analogen Funktionen" by contrast.

The author got many hints especially from Kubota ([7]) and Siegel ([11]); from the former as to the quaternion-upper-half-space and the Eisenstein series corresponding to it, and from the latter as to the manner of dealing with the limit formula itself.

§ 1. Inverse Mellin transform of $\zeta_F(s)\zeta_F(s+1)$ in the Gauss' number field case

Recently, Weil ([13]) gave a new proof of the classical formula: $\log\eta(z)=$

$-\frac{1}{2}\log(-iz)+\log\eta\left(-\frac{1}{z}\right)$, by using the functional equation of $\zeta(s)\zeta(s+1)$, where $\zeta(s)$ is the Riemann zeta functoin. Really, he pointed out that the "modular form" $\log\eta(z)$ is associated with the Dirichlet series $\zeta(s)\zeta(s+1)$ in the usual sense, i.e., essentially under Mellin transform. We here consider a simple analogy of this fact, that is, we shall treat the problem what "modular form" is associated with the Dirichlet series $Z(s)Z(s+1)$, where $Z(s)$ is the Dedekind zeta function $\zeta_F(s)$ of the Gauss' number field $F=\boldsymbol{Q}(\sqrt{-1})$. This section will also play a role of introduction to the subsequent sections.

1-1. Let us consider the functions

$$\varphi(s)=Z(s)Z(s+1),\quad \Phi(s)=\pi^{-(2s+1)}\Gamma(s)\Gamma(s+1)\varphi(s).$$

From the functional equation of the zeta function $Z(s)$, we can derive

$$\Phi(s)=\Phi(-s).$$

Further, as is immediately observed from the properties of $Z(s)$, the function $\Phi(s)$ is holomorphic in the whole s-plane except one double pole at $s=0$ and two simple poles at $s=\pm 1$, and bounded in $\sigma\leq Re\ s\leq\sigma'$ $Im\ s\geq\varepsilon$ for any σ, σ', and $\varepsilon>0$. The residues of $\Phi(s)$ are $Z(2)/4\pi^2$, $-Z(2)/4\pi^2$ at $s=1$, -1 respectively, and $\Phi(s)+1/(16s^2)$ is holomorphic at $s=0$. Of course, for $Re\ s>1$, $\varphi(s)$ is expressed as the Dirichlet series:

$$\varphi(s)=\frac{1}{16}\sum_{\substack{\mu,\nu\in\boldsymbol{Z}[\sqrt{-1}]\\ \mu\nu\neq 0}}\left|\frac{\nu}{\mu}\right||\mu\nu|^{-(2s+1)}.$$

Here we should recall the Mellin transform formula of the modified Bessel function:

$$\int_0^\infty K_1(2ay)y^{2s+1}\frac{dy}{y}=\frac{1}{4}a^{-(2s+1)}\Gamma(s)\Gamma(s+1)$$

in $Re\ s>0$ and for any $a>0$ ([10], p. 91, for example). Hence we have the integral expression of $\Phi(s)$:

$$\Phi(s)=\int_0^\infty f(y)y^{2s}\frac{dy}{y},\quad Re\ s>1,$$

where the function $f(y)$ is defined by the absolutely convergent series:

$$f(y)=\frac{1}{4}\sum_{\substack{\mu,\nu\in\boldsymbol{Z}[\sqrt{-1}]\\ \mu\nu\neq 0}}\left|\frac{\nu}{\mu}\right|K_1(2\pi|\mu\nu|y)y,\quad y>0.$$

At the same time we also have the inversion formula:

$$f(y) = \frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s) y^{-2s} ds, \quad \sigma > 1.$$

In this expression, we can change the path of integration $Re\ s = \sigma$ to $Re\ s = -\sigma$. Namely, from the properties of the meromorphic function $\Phi(s)$ mentioned above, it follows that

$$\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s) y^{-2s} ds$$

$$= \frac{Z(2)}{2\pi^2}(y^{-2} - y^2) + \frac{1}{4}\log y + \frac{1}{\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \Phi(s) y^{-2s} ds.$$

Further, from the functional equation $\Phi(s) = \Phi(-s)$, we have

$$\frac{1}{\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \Phi(s) y^{-2s} ds = \frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s)\left(\frac{1}{y}\right)^{-2s} ds.$$

These relations shows that

$$g(y) = \frac{1}{4}\log y + g\left(\frac{1}{y}\right), \tag{1}$$

where we put

$$g(y) = \frac{Z(2)}{2\pi^2} y^2 + \frac{1}{4}\sum_{\substack{\mu,\nu\in \boldsymbol{Z}[\sqrt{-1}] \\ \mu\nu\neq 0}} \left|\frac{\nu}{\mu}\right| K_1(2\pi|\mu\nu|y)y, \quad y > 0.$$

Now in the classical case of Weil, the function corresponding to $g(y)$ can be holomorphically continued to be a modular form $-4\log\eta(z)$ on the complex-upper-half-plane. In our case, how can we make the function to be a modular form? This problem is not so obvious, for there are no holomorphic solutions on the complex-upper-half-plane. We can, however, give a natural solution of the above problem on the quaternion-upper-half-space.

Before presenting this modular form, we must recall something about the quaternion-upper-half-space. (It is described more precisely in Kubota [7].) The quaternion-upper-half-space H_q is a three dimensional hyperbolic space which is realized as the set of all quaternion numbers $z = \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix}$ such that x is any complex number and y is any positive real number. The group $SL(2, \boldsymbol{C})$ ($\boldsymbol{C}$: the complex number field) naturally operates on H_q as follows:

$$SL(2, \boldsymbol{C}) \ni \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : z \longrightarrow \sigma\langle z\rangle = (\alpha z + \beta)(\gamma z + \delta)^{-1},$$

where the complex numbers $\alpha, \cdots$ are identified with the quaternion numbers $\begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix}, \cdots$. The subgroup $SL(2, \boldsymbol{Z}[\sqrt{-1}])$ operates discontinuously. Further, we know that the Laplace-Beltrami operator

$$D_q = y^2 \left(4 \frac{\partial^2}{\partial x \partial \bar{x}} + \frac{\partial^2}{\partial y^2}\right) - y \frac{\partial}{\partial y}$$

is essentially a unique $SL(2, \boldsymbol{C})$-invariant differential operator on H_q.

We are now ready to answer the above problem:

THEOREM 1. *Define the function $h(z)$ on H_q by*

$$h(z) = \frac{8Z(2)}{\pi^2} y^2 + 4 \sum_{\substack{\mu, \nu \in \boldsymbol{Z}[\sqrt{-1}] \\ \mu\nu \neq 0}} \left| \frac{\nu}{\mu} \right| K_1(2\pi |\mu\nu| y) \cdot y \cdot e^{2\pi i Re(\mu\nu x)},$$

for $z = \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix} \in H_q$. Then the following properties hold:

I. *$h(z)$ is a real-valued, real analytic function on H_q of variables $x, \bar{x}, y$ and vanishes by the Laplace-Beltrami operator D_q of H_q.*

II. *$h(z)$ is a modular form with the automorphic factor $2\log(|\gamma x + \delta|^2 + |\gamma|^2 y^2)$ with respect to the group $SL(2, \boldsymbol{Z}[\sqrt{-1}])$, i.e., $h(z) = 2\log(|\gamma x + \delta|^2 + |\gamma|^2 y^2) + h(\sigma\langle z\rangle)$ for any $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \boldsymbol{Z}[\sqrt{-1}])$.*

We call the function $h(z)$ the harmonic modular form on H_q. Obviously, $h(z)|_{x=0} = 16\, g(y)$. Thus we can say as follows:

THEOREM 2. *The harmonic modular form $h(z)$ is associated with the Dirichlet series $Z(s)Z(s+1)$.*

1-2. *Proof of Theorem* 1. We begin with a lemma.

LEMMA. *For any $s \in \boldsymbol{C}$ and any non-zero $\alpha \in \boldsymbol{C}$ the function of $z = \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix} \in H_q$ defined by*

$$e_s(z, \alpha) = K_{2s-1}(|\alpha| y) y e^{iRe(\alpha x)}$$

is an eigenfunction of D_q, i.e.,

$$D_q e_s(z, \alpha) = 4s(s-1) e_s(z, \alpha).$$

Proof of Lemma. The function $w = K_u(v)$ satisfies the modified Bessel differential equation ([10], p. 66):

$$v^2 \frac{d^2w}{dv^2} + v\frac{dw}{dv} - (v^2 + u^2)w = 0.$$

This implies the lemma.

Further, as is immediately checked,

$$D_q y^{2s} = 4s(s-1)y^{2s}.$$

Combining this with the lemma for the case $s = 1$, we obtain

$$D_q h(z) = 0. \tag{2}$$

It is easy to see that $h(z)$ is real-valued. Thus the assertion I is proved.

To prove the assertion II, we have only to show the transformation formulas for the generators of $SL(2, \boldsymbol{Z}[\sqrt{-1}\,])$: $L = \begin{pmatrix} -i & \\ & i \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & i \\ & 1 \end{pmatrix}$ and $A = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. It is quite easy to see

$$h(z) = h(L\langle z\rangle) = h(T\langle z\rangle) = h(U\langle z\rangle).$$

For the transformation A, we must show $k(z) \equiv 0$, where

$$k(z) = h(z) - 2\log(|x|^2 + y^2) - h(A\langle z\rangle).$$

As is stated in the equation (1), we know

$$k(z)\,|_{x=0} = 0. \tag{3}$$

Since $A\langle z\rangle = (|x|^2 + y^2)^{-1}\begin{pmatrix} -\bar{x} & -y \\ y & -x \end{pmatrix}$ for $z = \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix}$, we can obtain by a simple calculation,

$$\begin{aligned} \frac{\partial}{\partial x} h(z)\Big|_{x=0} &= \frac{\partial}{\partial \bar{x}} h(z)\Big|_{x=0} = \frac{\partial}{\partial x} h(A\langle z\rangle)\Big|_{x=0} \\ &= \frac{\partial}{\partial \bar{x}} h(A\langle z\rangle)\Big|_{x=0} = 0. \end{aligned}$$

And so,

$$\frac{\partial}{\partial x} k(z)\Big|_{x=0} = \frac{\partial}{\partial \bar{x}} k(z)\Big|_{x=0} = 0. \tag{4}$$

Further, from (2) we get also

$$D_q k(z) = 0. \tag{5}$$

Since the function $k(z)$ is real analytic with respect to variables x, $\bar{x}$ and y, $k(z)$ has a power series expansion:

$$k(z) = \sum_{m,n=0}^{\infty} c_{m,n}(y) x^m \bar{x}^n, \quad z \in H_Q. \tag{6}$$

From (3), (4) and (6), we can first get

$$c_{0,0}(y) = c_{1,0}(y) = c_{0,1}(y) = 0. \tag{7}$$

From (5) and (6), we can derive

$$\left(y^2 \frac{\partial^2}{\partial y^2} - y \frac{\partial}{\partial y}\right) c_{m,n}(y) + 4y^2(m+1)(n+1) c_{m+1,n+1}(y) = 0, \tag{8}$$

for every $m, n \geq 0$. Combining (7) with (8), it follows inductively that

$$c_{m,n}(y) = 0 \text{ for every } m, n \geq 0. \tag{9}$$

This means that $k(z) \equiv 0$, hence we obtain the assertion II. Thus the proof of Theorem 1 is finished.

We shall later give another proof of Theorem 1 in the more general form as a result of the Kronecker's limit formula for the non-holomorphic Eisenstein series. But the direct proof given here may be interesting, because it is related to the method by Maass in the problem between non-holomorphic automorphic functions and determining the Dirichlet series by the functional equations ([9]).

§ 2. The non-holomorphic Eisenstein series of the Hilbert modular group

In this section we shall define a non-holomorphic Eisenstein series of a simple type with respect to a Hilbert modular group in the case of an arbitrary algebraic number field, and give the holomorphic continuation and the functional equation for the Eisenstein series. Furthermore, we shall get the explicit expression of the Eisenstein series, which can be essentially regarded as the Fourier expansion formula. For the sake of simplicity we really treat only the case that the class number of the field is one.

2-1. Let F be an arbitrary algebraic number field with the class number one, which has r_1 real conjugate fields and $2r_2$ imaginary conjugate fields, and so the degree n of the field F over the rational number field $\boldsymbol{Q}$ is equal to $r_1 + 2r_2$. We denote the conjugate maps $\alpha \to \alpha^{(j)}$; real ones for $1 \leq j \leq r_1$ and imaginary ones $\alpha^{(j)} = \bar{\alpha}^{(j+r_2)}$ for $r_1 + 1 \leq j \leq r_1 + r_2$. Let $\mathscr{H}$

denote the upper-half-space corresponding to the field F. Namely, $\mathscr{H}$ is the product space $\prod_{j=1}^{r_1+r_2} H_j$ of r_1 copies of the complex-upper-half-plane $H_c = H_j$ $(1 \leq j \leq r_1)$ and r_2 copies of the quaternion-upper-half-space $H_q = H_j$ $(r_1+1 \leq j \leq r_1+r_2)$. The product group $G = SL(2, \boldsymbol{R})^{r_1} \times SL(2, \boldsymbol{C})^{r_2}$ is naturally operating on the space $\mathscr{H}$ as follows: for any $\sigma = (\sigma_j) \in G$; $\sigma_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} \in SL(2, \boldsymbol{R})$ or $SL(2, \boldsymbol{C})$ according as $1 \leq j \leq r_1$ or $r_1+1 \leq j \leq r_1+r_2$, and for any $\boldsymbol{z} = (z_j) \in \mathscr{H}$; $z_j = x_j + iy_j$ or $z_j = \begin{pmatrix} x_j & -y_j \\ y_j & \bar{x}_j \end{pmatrix}$ according as $1 \leq j \leq r_1$ or $r_1+1 \leq j \leq r_1+r_2$, the operation is given by $\sigma\langle \boldsymbol{z} \rangle = (\sigma_j\langle z_j \rangle)$; $\sigma_j\langle z_j \rangle = (\alpha_j z_j + \beta_j)(\gamma_j z_j + \delta_j)^{-1}$, where $\alpha_j, \cdots$ being identified with $\begin{pmatrix} \alpha_j & \\ & \bar{\alpha}_j \end{pmatrix}, \cdots$ for $r_1+1 \leq j \leq r_1+r_2$. Let $\mathfrak{o}$ be the ring of integers in F. Then $\Gamma = SL(2, \mathfrak{o})$ is a discontinuous subgroup of G under the identification $\Gamma \ni \sigma = (\sigma^{(j)})$, here $\sigma^{(j)} = \begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix}$ for $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We shall also use the following notations: for any $z_j = x_j + iy_j$ or $\begin{pmatrix} x_j & -y_j \\ y_j & \bar{x}_j \end{pmatrix} \in H_j$, we denote $y(z_j) = y_j$, and for $\boldsymbol{z} = (z_j) \in \mathscr{H}$, the vector $\boldsymbol{y}(\boldsymbol{z}) = (y(z_j))$. And then, $N\boldsymbol{y}(\boldsymbol{z})$ denotes the "norm" $\prod_{j=1}^{r_1+r_2} y(z_j)^{e_j}$. Here the symbol e_j means equal to 1 or 2, according as the case of $1 \leq j \leq r_1$ or $r_1+1 \leq j \leq r_1+r_2$. This symbol e_j will be frequently used later for abbreviation. Further, for any $\mu, \nu \in F$ with $(\mu, \nu) \neq (0, 0)$ and any $\boldsymbol{z} = (z_j) \in \mathscr{H}$, the vector $\boldsymbol{y}(\mu, \nu; \boldsymbol{z})$ is defined by $\boldsymbol{y}(\mu, \nu; \boldsymbol{z}) = (y(\mu^{(j)}, \nu^{(j)}; z_j))$ and $N\boldsymbol{y}(\mu, \nu; \boldsymbol{z}) = \prod_{j=1}^{r_1+r_2} y(\mu^{(j)}, \nu^{(j)}; z_j)^{e_j}$, where $y(\mu^{(j)}, \nu^{(j)}; z_j) = \dfrac{y_j}{|\mu^{(j)} x_j + \nu^{(j)}|^2 + |\mu^{(j)}|^2 y_j^2}$ for $z_j = x_j + iy_j$ or $z_j = \begin{pmatrix} x_j & -y_j \\ y_j & \bar{x}_j \end{pmatrix}$. Then, $\boldsymbol{y}(\sigma\langle \boldsymbol{z} \rangle) = \boldsymbol{y}(\gamma, \delta; \boldsymbol{z})$ for any $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, in fact, we can check $y(\sigma^{(j)}\langle z_j \rangle) = y(\gamma^{(j)}, \delta^{(j)}; z_j)$ by a simple calculation. Through this paper, all the above notations are fixed once and for all.

Now we are going to define a non-holomorphic Eisenstein series with respect to $\Gamma = SL(2, \mathfrak{o})$. Let Γ_1 be the subgroup of Γ consisting of all $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\gamma = 0$. The Eisenstein series, converging absolutely in the half plane $Re\ s > 1$, is defined by

$$E^*(\boldsymbol{z}, s) = \sum_{\sigma \in \Gamma_1 \backslash \Gamma} N\boldsymbol{y}(\sigma\langle \boldsymbol{z} \rangle)^s. \tag{10}$$

On the other hand, it is convenient for the later use to define another series by

$$E(\boldsymbol{z}, s) = \sum_{\{\mu, \nu\} \neq \{0, 0\}} N\boldsymbol{y}(\mu, \nu; \boldsymbol{z})^s, \tag{11}$$

where $\{\mu, \nu\}$ in the summation runs over all non-associated pairs $(\mu, \nu) \in \mathfrak{o} \times \mathfrak{o}$ except $\{0, 0\}$, and here two pairs (μ, ν) and (μ_1, ν_1) are called associated if both relations $\mu_1 = \varepsilon\mu$ and $\nu_1 = \varepsilon\nu$ hold for a same unit ε of $\mathfrak{o}$. Between these two series, the relation

$$E(\boldsymbol{z}, s) = \zeta_F(2s) E^*(\boldsymbol{z}, s) \tag{12}$$

holds, in fact, this follows easily from the expression of the Dedekind zeta function of F:

$$\zeta_F(s) = \sum_{\mathfrak{o} | (\mu) \neq 0} |N\mu|^{-s}, \quad Re\ s > 1.$$

Thus the series (10) is almost the same as the series (11), and so we shall call the series (11) the non-holomorphic Eisenstein series, too. Actually, the series (11) can be regarded as a type of the Epstein zeta function, and for this advantage we hereafter deal with the Eisenstein series (11) mainly.

The Eisenstein series $E(\boldsymbol{z}, s)$ converges absolutely also in the half plane $Re\ s > 1$, and it can be holomorphically continued to the whole s-plane; and it becomes regular except only a simple pole at $s = 1$. This can be shown directly by using the binary Hecke's theta formula in the similar manner of Tamagawa ([12]). It, however, is convenient for our purpose to show it by the some different method, and this method is like one to calculate the Fourier expansion formula of the Eisenstein series in the $\boldsymbol{Q}$-case by Maass.

2-2. Before the calculation, it should be recalled the assumption that the class number of F is one. In particular, we can justly denote by $\mathfrak{o}^* = (\omega)$ the inverse different, and then $N\mathfrak{o}^{*-1} = |N\omega|^{-1} = \varDelta$ is nothing but the absolute value of the discriminant of F.

We first decompose the summation in (11) as follows:

$$\sum_{\{\mu, \nu\} \neq \{0, 0\}} = \sum_{\substack{\mathfrak{o} | (\nu) \neq 0 \\ \mu = 0}} + \sum_{\mathfrak{o} | (\mu) \neq 0} \sum_{\mathfrak{o} \ni \nu} .$$

Then, for $Re\ s > 1$, we have

$$\begin{aligned} E(\boldsymbol{z}, s) &= \sum_{\mathfrak{o} | (\nu) \neq 0} N\boldsymbol{y}(0, \nu; \boldsymbol{z})^s + \sum_{\mathfrak{o} | (\mu) \neq 0} \sum_{\mathfrak{o} \ni \nu} N\boldsymbol{y}(\mu, \nu; \boldsymbol{z})^s \\ &= N\boldsymbol{y}(\boldsymbol{z})^s \zeta_F(2s) \\ &\quad + \sum_{\mathfrak{o} | (\mu) \neq 0} \sum_{\mathfrak{o} \in \nu} \prod_{j=1}^{r_1 + r_2} \frac{(e_j \pi)^{e_j s}}{\Gamma(e_j s)} \int_0^\infty e^{-e_j \pi t_j\, y(\mu^{(j)}, \nu^{(j)};\, z_j)^{-1}} t_j^{e_j s} \frac{dt_j}{t_j} \end{aligned}$$

$$= N\boldsymbol{y}(\boldsymbol{z})^s\zeta_F(2s)$$

$$+\left(\frac{\pi^s}{\Gamma(s)}\right)^{r_1}\left(\frac{(2\pi)^{2s}}{\Gamma(2s)}\right)^{r_2}\sum_{\mathfrak{D}|(\mu)\neq 0}\int_0^\infty\cdots\int_0^\infty e^{-\pi\sum_j e_j t_j y_j|\mu^{(j)}|^2}\theta(\boldsymbol{t})\prod_{j=1}^{r_1+r_2} t_j^{e_j s}\frac{dt_j}{t_j}\,.$$

Here $\theta(\boldsymbol{t})$ stands for the Hecke's theta function:

$$\theta(\boldsymbol{t}) = \sum_{\mathfrak{D}\ni\nu} e^{-\pi\left(\sum_j e_j\frac{t_j}{y_j}|\mu^{(j)}x_j+\nu^{(j)}|^2\right)},$$

hence by the transformation formula it becomes

$$\theta(\boldsymbol{t}) = \varDelta^{-\frac{1}{2}}(N\boldsymbol{y}(\boldsymbol{z}))^{\frac{1}{2}}\prod_j t_j^{-\frac{e_j}{2}}\sum_{\mathfrak{D}^*\ni\nu_1} e^{-\pi\sum_j e_j\frac{y_j}{t_j}|\nu_1^{(j)}|^2+2\pi i\sum_j e_j Re(\mu^{(j)}\nu_1^{(j)}x_j)}$$

Since the summation can be changed over again as follows:

$$\sum_{\mathfrak{D}|(\mu)\neq 0}\ \sum_{\mathfrak{D}^*\ni\nu_1} = \sum_{\substack{\mathfrak{D}|(\mu)\neq 0\\ \nu_1=0}} + \sum_{\substack{\{\mu,\nu\}',\mu\nu\neq 0\\ \nu_1=\nu\omega}}, \quad {}^{*)}$$

it follows that

$$E(\boldsymbol{z},s) = N\boldsymbol{y}(\boldsymbol{z})^s\zeta_F(2s) + N\boldsymbol{y}(\boldsymbol{z})^{1-s}\varDelta^{-\frac{1}{2}}\left(\frac{\pi^{\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^{r_1}\left(\frac{2\pi\Gamma(2s-1)}{\Gamma(2s)}\right)^{r_2}\zeta_F(2s-1)$$

$$+N\boldsymbol{y}(\boldsymbol{z})^{\frac{1}{2}}\varDelta^{-\frac{1}{2}}\left(\frac{\pi^s}{\Gamma(s)}\right)^{r_1}\left(\frac{(2\pi)^{2s}}{\Gamma(2s)}\right)^{r_2}\sum_{\substack{\{\mu,\nu\}'\\ \mu\nu\neq 0}} e^{2\pi i S(\mu\nu\omega\boldsymbol{x})}\prod_j\int_0^\infty e^{-e_j\pi y_j\left(t_j|\mu^{(j)}|^2+\frac{|\nu^{(j)}\omega^{(j)}|^2}{t_j}\right)}\times$$

$$\times\, t_j^{e_j\left(s-\frac{1}{2}\right)}\frac{dt_j}{t_j},$$

where we put $S(\mu\nu\omega\boldsymbol{x}) = \sum_{j=1}^{r_1+r_2} e_j Re((\mu\nu\omega)^{(j)}x_j)$ for abbreviation. Further, one should recall the integral expression of the modified Bessel function (see [10], p. 85, for example):

$$2\left|\frac{b}{a}\right|^u K_u(2|ab|) = \int_0^\infty e^{-\left(a^2t+\frac{b^2}{t}\right)}t^u\frac{dt}{t},$$

for any non-zero real numbers a, b. Consequently, we obtain the explicit expression of the Eisenstein series:

$$(13)\qquad E(\boldsymbol{z},s)=N\boldsymbol{y}(\boldsymbol{z})^s\zeta_F(2s)+N\boldsymbol{y}(\boldsymbol{z})^{1-s}\varDelta^{-\frac{1}{2}}\left(\frac{\pi^{\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(2s)}\right)^{r_1}\left(\frac{2\pi\Gamma(2s-1)}{\Gamma(2s)}\right)^{r_2}\zeta_F(2s-1)$$

---

*) We should remark that the symbol $\{\ ,\ \}'$ has a slightly different meaning from $\{\ ,\ \}$, i.e. $\{\mu,\nu\}'$ is a class of pairs $(\mu,\nu)\in\mathfrak{D}\times\mathfrak{D}$ under the equivalence relation $(\mu,\nu)\sim(\mu\varepsilon,\nu\varepsilon^{-1})$ for a unit ε.

$$+2^{r_1+r_2}\Delta^{-s}\left(\frac{\pi^s}{\Gamma(s)}\right)^{r_1}\left(\frac{(2\pi)^{2s}}{\Gamma(2s)}\right)^{r_2}\sum_{\substack{\{\mu,\nu\}'\\ \mu\nu\neq 0}}\left|\frac{N\nu}{N\mu}\right|^{s-\frac{1}{2}}e^{2\pi i S(\mu\nu\omega x)}$$

$$\times\prod_{j=1}^{r_1+r_2}K_{e_j\left(s-\frac{1}{2}\right)}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j)y_j^{\frac{e_j}{2}}.$$

Now it is not difficult to see that the right-hand-side of the expression (13) defines a holomorphic function on the whole s-plane. This shows that the holomorphic continuation of $E(z,s)$ is accomplished. Furthermore, from two well-known functional equations:

$$G(s)\zeta_F(s)=G(1-s)\zeta_F(1-s),$$

where $G(s)$ denotes the gamma-factor of the Dedekind zeta function, i.e.,

$$G(s)=\Delta^{\frac{s}{2}}\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2},$$

and

$$K_u(v)=K_{-u}(v),$$

we can easily derive the functional equation for the Eisenstein series $E(z,s)$:

$$G(2s)E(z,s)=G(2(1-s))E(z,1-s),$$

where the gamma-factor $G(s)$ is the same one defined above.

§ 3. The Kronecker's limit formula for the non-holomorphic Eisenstein series

We are now ready to mention about the Kronecker's limit formula. As explained in § 2, the Eisenstein series $E(z,s)$ has a simple pole at $s=1$. We here want to give the explicit form of the residue and the constant term in the Laurent expansion of the Eisenstein series at $s=1$. The residue can be immediately obtained from the residue of the Dedekind zeta function, and there appear the function which is quite analogous to $\log|\eta(z)|$ in the constant term.

3-1. As we have already had the expression (13), there remains no difficulty in the calculation in the sequel.

The first and the third terms of the right-hand-side of the formula (13) are both regular at $s=1$. Namely, in the neighbourhood of $s=1$, we have

$$(14)\qquad N\boldsymbol{y}(\boldsymbol{z})^s\zeta_F(2s)=N\boldsymbol{y}(\boldsymbol{z})\zeta_F(2)+O(|s-1|),$$

(15)

$$2^{r_1+r_2}\Delta^{-s}\Big(\frac{\pi^s}{\Gamma(s)}\Big)^{r_1}\Big(\frac{(2\pi)^{2s}}{\Gamma(2s)}\Big)^{r_2}\sum_{\substack{\{\mu,\nu\}'\\ \mu\nu\neq 0}}\left|\frac{N\nu}{N\mu}\right|^{s-\frac{1}{2}}e^{2\pi i S(\mu\nu\omega\boldsymbol{x})}\prod_{j=1}^{r_1+r_2}K_{e_j\left(s-\frac{1}{2}\right)}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j)y_j^{\frac{e_j}{2}}$$

$$=2^{r_1+3r_2}\Delta^{-1}\pi^n\sum_{\substack{\{\mu,\nu\}'\\ \mu\nu\neq 0}}\left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}}e^{2\pi i S(\mu\nu\omega\boldsymbol{x})}\prod_{j=1}^{r_1+r_2}K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j)y_j^{\frac{e_j}{2}}+O(|s-1|).$$

The second term in (13) has a simple pole at $s=1$, and the following formulas are well-known or easily to be checked:

$$N\boldsymbol{y}(\boldsymbol{z})^{1-s}=1-\log N\boldsymbol{y}(\boldsymbol{z})\cdot(s-1)+O(|s-1|^2),$$

$$\left(\frac{\pi^{\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^{r_1}=\pi^{r_1}(1-2r_1\log 2\cdot(s-1))+O(|s-1|^2),$$

$$\Big(\frac{2\pi\Gamma(2s-1)}{\Gamma(2s)}\Big)^{r_2}=\Big(\frac{2\pi}{2s-1}\Big)^{r_2}=(2\pi)^{r_2}(1-2r_2(s-1))+O(|s-1|^2),$$

$$\zeta_F(2s-1)=\frac{2^{r_1+r_2}\pi^{r_2}R}{2w\sqrt{\Delta}}\frac{1}{s-1}+A_0+O(|s-1|),$$

where w is the number of roots of the unity in F and R denotes the regulator of F. $A_0=\lim\limits_{s\to 1}\Big(\zeta_F(s)-\dfrac{2^{r_1+r_2}\pi^{r_2}R}{w\sqrt{\Delta}}\dfrac{1}{s-1}\Big)$ is the constant which is not completely clarified yet in the general case. From these formulas the second term becomes

$$(16)\qquad N\boldsymbol{y}(\boldsymbol{z})^{1-s}\Delta^{-\frac{1}{2}}\left(\frac{\pi^{\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^{r_1}\Big(\frac{2\pi\Gamma(2s-1)}{\Gamma(2s)}\Big)^{r_2}\zeta_F(2s-1)$$

$$=\frac{2^{n-1}\pi^n R}{w\Delta}\frac{1}{s-1}+\frac{2^{r_2}\pi^{r_1+r_2}A_0}{\sqrt{\Delta}}-\frac{2^{n-1}\pi^n R}{w\Delta}(\log N\boldsymbol{y}(\boldsymbol{z})+2r_1\log 2-2r_2)$$

$$+O(|s-1|).$$

Combining these expansion formulas (14), (15) and (16) we obtain

$$(17)\qquad E(\boldsymbol{z},s)=\frac{2^{n-1}\pi^n R}{w\Delta}\Big\{\frac{1}{s-1}+\frac{w\Delta}{2^{n-1}\pi^n R}\zeta_F(2)N\boldsymbol{y}(\boldsymbol{z})$$

$$-\log N\boldsymbol{y}(\boldsymbol{z})-2r_1\log 2-2r_2+\frac{w\sqrt{\Delta}A_0}{2^{r_1+r_2-1}\pi^{r_2}R}$$

$$+\frac{2^{r_2+1}w}{R}\sum_{\substack{\{\mu,\nu\}'\\ \mu\nu\neq 0}}\left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}} e^{2\pi i S(\mu\nu\omega x)}\prod_{j=1}^{r_1+r_2} K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j)y_j^{\frac{e_j}{2}}\Bigg\}$$

$$+O(|s-1|).$$

For convenience' sake, we formulate the result as follows:

THEOREM 3. *For the non-holomorphic Eisenstein series of the Hilbert modular group defined by* (11), *the following limit formula is valid:*

$$\lim_{s\to 1}\left\{E(z,s)-\frac{2^{n-1}\pi^n R}{w\Delta}\frac{1}{s-1}\right\}$$

$$=\frac{2^{n-1}\pi^n R}{w\Delta}(\alpha_0-2r_1\log 2-2r_2-\log Ny(z)+h(z)).$$

Here the constant α_0 *is given by*

$$\alpha_0=2\cdot\lim_{s\to 1}\left\{\frac{w\sqrt{\Delta}}{2^{r_1+r_2}\pi^{r_2}R}\zeta_F(s)-\frac{1}{s-1}\right\},$$

and the function $h(z)$ *on* $\mathscr{H}=H_c^{r_1}\times H_q^{r_2}$ *is defined by*

$$(18)\qquad h(z)=\frac{w\Delta}{2^{n-1}\pi^n R}\zeta_F(2)Ny(z)$$

$$+\frac{2^{r_2+1}w}{R}\sum_{\substack{\{\mu,\nu\}'\\ \mu\nu\neq 0}}\left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}} e^{2\pi i S(\mu\nu\omega x)}\prod_{j=1}^{r_1+r_2} K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j)y_j^{\frac{e_j}{2}},$$

where $z=(z_j)\in\mathscr{H}$; $z_j=x_j+iy_j$ *for* $1\leq j\leq r_1$ *and* $z_j=\begin{pmatrix}x_j & -y_j\\ y_j & x_j\end{pmatrix}$ *for* $r_1+1\leq j\leq r_1+r_2$, *and* $S(\mu\nu\omega x)$ *stands for* $\sum_{j=1}^{r_1+r_2}e_jRe((\mu\nu\omega)^{(j)}x_j)$.

Now let us show that the function $h(z)$ defined by (18) can be regarded the generalization of $\log|\eta(z)|$ of the classical case. For this purpose, we first return back to the properties of the Eisenstein series itself. The Eisenstein series $E(z,s)$ has the following properties:

$$(19)\qquad E(\sigma\langle z\rangle,s)=E(z,s)\ \text{for any}\ \sigma\in\Gamma=SL(2,\mathfrak{o}),$$

$$(20)\qquad D_jE(z,s)=e_j^2s(s-1)E(z,s)\ \text{for}\ 1\leq j\leq r_1+r_2,$$

where D_j is the Laplace-Beltrami operator of the j-th component space H_j of $\mathscr{H}$, i.e.,

$$D_j = y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right), \quad e_j = 1 \text{ for } 1 \leq j \leq r_1, \text{ and}$$

$$D_j = y_j^2\left(4\frac{\partial^2}{\partial x_j \partial \bar{x}_j} + \frac{\partial^2}{\partial y_j^2}\right) - y_j\frac{\partial}{\partial y_j}, \quad e_j = 2 \text{ for } r_1 + 1 \leq j \leq r_1 + r_2.$$

In fact, (19) follows immediately from the expressions (10) and (12), and the property (20) follows from (10), (12) and the relations:

$$D_j y_j^s = e_j^2 s(s-1) y_j^s \quad \text{for} \quad 1 \leq j \leq r_1 + r_2.$$

Combining these properties with the Laurent expansion of the Eisenstein series:

$$E(\boldsymbol{z}, s) = \frac{a}{s-1} + b(\boldsymbol{z}) + O(|s-1|),$$

we can derive

$$b(\sigma\langle \boldsymbol{z}\rangle) = b(\boldsymbol{z}) \text{ for any } \sigma \in \Gamma, \tag{21}$$

and

$$D_j b(\boldsymbol{z}) = e_j^2 a \quad \text{for} \quad 1 \leq j \leq r_1 + r_2. \tag{22}$$

If we use the relations

$$D_j \log N\boldsymbol{y}(\boldsymbol{z}) = -e_j^2 \quad \text{for} \quad 1 \leq j \leq r_1 + r_2,$$

the formula (22) can be stated as

$$D_j(b(\boldsymbol{z}) + a\log N\boldsymbol{y}(\boldsymbol{z})) = 0 \quad \text{for} \quad 1 \leq j \leq r_1 + r_2. \tag{23}$$

Since Theorem 3 gives the explicit expression of the constant a and the function $b(\boldsymbol{z})$, we can find the properties of the function $h(\boldsymbol{z})$.

Thus we can conclude that

THEOREM 4. *The function $h(\boldsymbol{z})$ defined by (18) in Theorem 3 satisfies the following properties:*

I. *$h(\boldsymbol{z})$ is a real-valued, real analytic function on $\mathscr{H}$ of $2r_1 + 3r_2$ variables; x_j, y_j $(1 \leq j \leq r_1)$ and $x_j, \bar{x}_j, y_j$ $(r_1 + 1 \leq j \leq r_1 + r_2)$. And $h(\boldsymbol{z})$ vanishes by any invariant differential operator on $\mathscr{H}$ which is represented as a polynomial without constant terms of the Laplace-Beltrami operators D_j's of the component spaces of $\mathscr{H}$.*

II. *$h(\boldsymbol{z})$ is a modular form with the automorphic factor $J(\sigma, \boldsymbol{z}) = \log \prod_{j=1}^{r_1+r_2} (|\gamma^{(j)} x_j + \delta^{(j)}|^2 + |\gamma^{(j)}|^2 y_j^2)^{e_j}$ with respect to the discontinuous group $\Gamma = SL(2, \mathfrak{o})$, i.e.,*

$$h(z) = J(\sigma, z) + h(\sigma\langle z\rangle) \quad \textit{for any} \quad \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma.$$

We call the function $h(z)$ the harmonic modular form on $\mathscr{H}$ for the reason of these two properties. It should be remarked that the harmonic modular form $h(z)$ can be expressed without the aid of the modified Bessel function if the number field F is totally real. Really, we know that $K_{\frac{1}{2}}(v) = \sqrt{\frac{\pi}{2v}}\, e^{-v}$, but $K_1(v)$ is not an elementary function. Also it should be noted that the defining expression (18) of $h(z)$ can be essentially regarded as the Fourier expansion of one. In fact, because of the propety II, $h(z)$ is invariant by translations $x_j \to x_j + \mu^{(j)}$ for all $\mu \in \mathfrak{o}$, hence $h(z)$ has the Fourier expansion of the series $e^{2\pi i S(\mu\omega x)}$, as is readily derived from the very expression (18).

3-2. We lastly consider the Dirichlet series associated with the harmonic modular form $h(z)$. In order to clarify the meaning of "associated" we must start by recalling the Mellin transform in our case.

Let $\boldsymbol{R}_+$ be the multiplicative group of all positive real numbers, and Y be the product group of $r_1 + r_2$ copies of $\boldsymbol{R}_+$, i.e.,

$$Y = \{\boldsymbol{y} = (y_j);\ y_j \in \boldsymbol{R}_+,\ 1 \leq j \leq r_1 + r_2\}.$$

Let U denote the group generated only by the fundamental units $\varepsilon_1, \cdots, \varepsilon_r$ $(r = r_1 + r_2 - 1)$ of the number field F. The groups $\boldsymbol{R}_+$ and U operate on Y in natural way, i.e.,

$$\boldsymbol{R}_+ \ni a : \boldsymbol{y} = (y_j) \longrightarrow a\boldsymbol{y} = (\sqrt[n]{a}\, y_j),$$
$$U \ni \varepsilon : \boldsymbol{y} = (y_j) \longrightarrow \varepsilon\boldsymbol{y} = (|\varepsilon^{(j)}|\, y_j).$$

Then we can define the group Λ consisting of all $\boldsymbol{R}_+$- and U-invariant characters of Y. Namely, $\Lambda \ni \lambda$ if and only if λ is a continuous homomorphism of Y into $\{u \in \boldsymbol{C};\ |u| = 1\}$ with the property:

$$\lambda(a\boldsymbol{y}) = \lambda(\varepsilon\boldsymbol{y}) = \lambda(\boldsymbol{y}) \quad \text{for any} \quad a \in \boldsymbol{R}_+ \quad \text{and} \quad \varepsilon \in U. \tag{24}$$

As is well-known (Hecke [2]), the group Λ is isomorphic to $\boldsymbol{Z}^r$; the product group of r copies of the additive group $\boldsymbol{Z}$ of all rational integers. More precisely, any character λ in Λ is uniquely parametrized by an integral vector $\mathfrak{m} = (m_1, \cdots, m_r) \in \boldsymbol{Z}^r$ as follows:

$$(25) \qquad \lambda(\boldsymbol{y}) = \lambda_{\mathrm{m}}(\boldsymbol{y}) = \prod_{j=1}^{r_1+r_2} y_j^{2\pi i \sum_{k=1}^{r} m_k e_j^{<k>}},$$

where $e_j^{<k>}$ is the element of the matrix

$$\begin{pmatrix} \frac{e_1}{n} & \cdots\cdots & \frac{e_{r_1+r_2}}{n} \\ e_1^{<1>} & \cdots\cdots & e_{r_1+r_2}^{<1>} \\ \vdots & & \vdots \\ e_1^{<r>} & \cdots\cdots & e_{r_1+r_2}^{<r>} \end{pmatrix} = \begin{pmatrix} 1 & \log|\varepsilon_1^{(1)}| & \cdots\cdots & \log|\varepsilon_r^{(1)}| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\varepsilon_1^{(r_1+r_2)}| & \cdots & \log|\varepsilon_r^{(r_1+r_2)}| \end{pmatrix}^{-1}$$

Furthermore, for any $\boldsymbol{y} = (y_j) \in Y$, we denote by $N\boldsymbol{y}$ the norm $\prod_{j=1}^{r_1+r_2} y_j^{e_j}$. Of course, the symbol e_j is 1 or 2, according as $1 \leq j \leq r_1$ or $r_1 + 1 \leq j \leq r_1 + r_2$, as before. In particular, for the property (24) of $\lambda \in \Lambda$ the ideal character $\lambda(\mu)$ is well-defined by $\lambda(\mu) = \lambda((|\mu^{(j)}|))$ for any ideal (μ) of F, and this is nothing but the Grössen-character defined by Hecke ([2]). We should also note that $N((|\mu^{(j)}|)) = |N\mu|$.

Now we can say about the Mellin transform of the harmonic modular form $h(\boldsymbol{z})$ of (18). Let $h_0(\boldsymbol{z})$ be the function excluding the constant term of $h(\boldsymbol{z})$ in the Fourier expansion, i.e.,

$$h_0(\boldsymbol{z}) = h(\boldsymbol{z}) - \frac{w\Delta}{2^{n-1}\pi^n R} \zeta_F(2) N\boldsymbol{y}(\boldsymbol{z}).$$

It should be remarked that we can let the point $\tilde{\boldsymbol{y}} = (\tilde{y}_j)$ of $\mathscr{H}$ correspond to any element $\boldsymbol{y} = (y_j)$ of Y, by $\tilde{y}_j = iy_j$ or $\begin{pmatrix} & -y_j \\ y_j & \end{pmatrix}$ according as $1 \leq j \leq r_1$ or $r_1 + 1 \leq j \leq r_1 + r_2$. Hence we can regard the function

$$h_0(\tilde{\boldsymbol{y}}) = \frac{2^{r_2+1}w}{R} \sum_{\substack{\{\mu,\nu\}' \\ \mu\nu \neq 0}} \left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}} \prod_{j=1}^{r_1+r_2} K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j) y_j^{\frac{e_j}{2}}$$

as the function on Y. By a simple calculation we can derive the expression:

$$(26) \qquad h_0(\tilde{\boldsymbol{y}}) = \frac{2^{r_2+1}w^2}{R} \sum_{\substack{\mathfrak{D}|(\mu)\neq 0 \\ \mathfrak{D}|(\nu)\neq 0}} \sum_{U \ni \varepsilon} \left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}} \prod_{j=1}^{r_1+r_2} K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\omega\varepsilon)^{(j)}|y_j) y_j^{\frac{e_j}{2}}.$$

Obviously, the function $h_0(\tilde{\boldsymbol{y}})\bar{\lambda}(\boldsymbol{y})N\boldsymbol{y}^s$ is invariant under the operations of $U(s \in \boldsymbol{C})$, and so the following integral will be well-defined:

$$(27) \qquad I(s, \lambda) = \int_{Y/U} h_0(\tilde{\boldsymbol{y}})\bar{\lambda}(\boldsymbol{y}) N\boldsymbol{y}^s \, d^\times \boldsymbol{y},$$

where the measure $\prod_{j=1}^{r_1+r_2} \frac{dy_j}{y_j}$ is denoted by $d^\times \boldsymbol{y}$ in short. Really, the integral (27) is absolutely converging in $Re\ s>1$. Thus the family of integrals $I(s, \lambda)$; $\lambda \in \Lambda$ and the harmonic modular form $h(z)$ are reciprocally associated with each other, essentially under Mellin transform and the inverse transform. (See also Herrmann ([4]).)

In our case, there appears the Dirichlet series of the classical type, in fact, by using the expression (26) it follows that for $Re\ s>1$,

$$I(s, \lambda) = \frac{2^{r_2+1}w^2}{R} \sum_{\substack{\mathfrak{D}|(\mu)\neq 0 \\ \mathfrak{D}|(\nu)\neq 0}} \left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}} \prod_{j=1}^{r_1+r_2} \int_0^\infty K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\omega)^{(j)}|y_j) y_j^{e_j\left(s+\frac{1}{2}\right)-2\pi i \sum_{k=1}^{r} m_k e_j^{<k>}} \frac{dy_j}{y_j}.$$

Owing to the formula of the modified Bessel function ([10], p. 91, for example):

$$\int_0^\infty K_u(2at)t^s \frac{dt}{t} = \frac{1}{4} a^{-s} \Gamma\left(\frac{s-u}{2}\right) \Gamma\left(\frac{s+u}{2}\right),$$

for any $a>0$ and $Re\ s>|Re\ u|$, we can derive

$$I(s, \lambda) = \frac{2^{r_2+1}w^2}{R} \sum_{\substack{\mathfrak{D}|(\mu)\neq 0 \\ \mathfrak{D}|(\nu)\neq 0}} \left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}} \prod_{j=1}^{r_1+r_2} \frac{1}{4}(e_j\pi|(\mu\nu\omega)^{(j)}|)^{-e_j\left(s+\frac{1}{2}\right)+i\alpha_j} \Gamma\left(\frac{e_j s - i\alpha_j}{2}\right) \Gamma\left(\frac{e_j(s+1)-i\alpha_j}{2}\right),$$

here α_j stands for $2\pi \sum_{k=1}^{r} m_k e_j^{<k>}$. Namely,

$$I(s, \lambda) = \frac{w^2}{2^{2r_1+r_2-1}R} \lambda(\omega) G(s, \lambda) G(s+1, \lambda) \zeta_F(s, \lambda) \zeta_F(s+1, \lambda), \tag{28}$$

where $\zeta_F(s, \lambda)$ is the zeta function with the Grössen-character $\lambda = \lambda_{\mathrm{m}}$:

$$\zeta_F(s, \lambda) = \sum_{\mathfrak{D}|(\mu)\neq 0} \lambda(\mu)|N\mu|^{-s}, \quad Re\ s>1,$$

and $G(s, \lambda)$ is its gamma-factor, i.e.,

$$G(s, \lambda) = (\Delta \cdot \pi^{-2n} \cdot 2^{-2r_2})^{\frac{s}{2}} \prod_{j=1}^{r_1} \Gamma\left(\frac{s-i\alpha_j}{2}\right) \prod_{j=r_1+1}^{r_1+r_2} \Gamma\left(s - \frac{i\alpha_j}{2}\right).$$

From the relations (27) and (28), we may say that

THEOREM 5. *The harmonic modular form $h(z)$ is associated with the family of Dirichlet series $\zeta_F(s, \lambda)\zeta_F(s+1, \lambda)$; $\lambda \in \Lambda$.*

If we use the the transformation formula (the property II of Theorem

4) for $h(z)$ in the case of $\sigma = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, then we can deduce the holomorphic continuation of $I(s, \lambda)$ or $\zeta_F(s, \lambda)\zeta_F(s+1, \lambda)$, and the functional equation

$$I(s, \lambda) = I(-s, \bar{\lambda}). \tag{29}$$

On the other hand, the functional equation (29) follows also from one for $\zeta_F(s, \lambda)$:

$$G(s, \lambda)\zeta_F(s, \lambda) = \bar{\lambda}(\omega)G(1-s, \bar{\lambda})\zeta_F(1-s, \bar{\lambda}). \tag{30}$$

Actually, if we start from the equations (30), we may obtain the property II of Theorem 4, in principle, without any help of the Eisenstein series. Moreover, the property I can be also proved directly by the method as in **§1**, and consequently all the assertions of Theorem 4 can be directly verified. This method is closely connected with Maass' theory of non-holomorphic automorphic functions.

References

[1] E. Hecke, Über die Konstruktion der Klassenkörper reeller quadratischer Körper mit Hilfe von automorphen Funktionen, Math. Werke, 64–68.

[2] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen, Math. Werke, 215–234.

[3] E. Hecke, Analytische Funktionen und algebraische Zahlen, I, II, Math. Werke, 336–360, 381–404.

[4] O. Herrmann, Über Hilbertsche Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, Math. Ann. **127,** 357–400, (1954).

[5] K. Katayama, Kronecker's limit formulas and their applications, J. Fac. Sci., Univ. of Tokyo, Sect. I, **13,** 1–44, (1966).

[6] S. Konno, On Kronecker's limit formula in a totally imaginary quadratic field over a totally real algebraic number field, J. Math. Soc. Japan, **17,** 411–424, (1965).

[7] T. Kubota, Über diskontinuierlicher Gruppen Picardschen Typus und zugehörige Eisensteinsche Reihen, Nagoya Math. J. **32,** 259–271, (1968).

[8] T. Kubota, On automorphic functions and the reciprocity law in a number field, Lectures in Math. **2,** Kyoto Univ. (1969).

[9] H. Maass, Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. **121,** 141–183, (1949).

[10] Magnus, Oberhettinger and Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed. Springer-Verlag, 1966.

[11] C.L. Siegel, Analytische Zahlentheorie II, Vorlesung an der Univ. Göttingen, 1963/64.

[12] T. Tamagawa, On some extensions of Epstein's Z-series, Proc. International Symposium on algebraic number theory, Tokyo-Nikko, 1955, 259–261.

[13] A. Weil, Sur une formule classique, J. Math. Soc. Japan, **20,** 400–402, (1968).

Mathematical Institute,
Nagoya University