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EXPONENTIAL ASYMPTOTICS IN THE SMALL PARAMETER EXIT PROBLEM

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1. Introduction

Let $\mathcal{M}$ be a d-dimensional Riemannian manifold of class C^∞ with Riemannian metric $g = (g_{ij})$ and let D be a connected domain in $\mathcal{M}$ having a non-empty smooth boundary ∂D and a compact closure $\bar{D}$. Suppose that $b^\varepsilon \in \mathfrak{X}(\mathcal{M}) = \{C^\infty\text{-vector fields on } \mathcal{M}\}$, $\varepsilon > 0$, are given and that $\{b^\varepsilon\}$ converges uniformly to $b \in \mathfrak{X}(\mathcal{M})$ on D' as $\varepsilon \downarrow 0$ for some neighborhood D' of D. Consider the diffusion process (x_t^ε, P_x) on D' with a small parameter $\varepsilon > 0$ generated by

$$\mathcal{L}^\varepsilon = \frac{\varepsilon^2}{2}\Delta + b^\varepsilon, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}$. Uniqueness of the process requires some boundary condition on $\partial D'$. However boundary conditions are not mentioned since the process is considered only before the time when it leaves a small neighborhood of $\bar{D}$. In this paper, we shall study the asymptotic behavior of the expectation of the first exit time τ^ε from the domain D; i.e.,

$$\tau^\varepsilon = \inf\{t > 0\,;\, x_t^\varepsilon \notin D\},$$

under the following assumptions:

(A_1) (gradient condition) there exists a potential function $U \in C^\infty(\bar{D})$ such that $b = -\dfrac{1}{2}\operatorname{grad} U$ on $\bar{D}$;

(A_2) the set of critical points $\mathcal{C} = \{x \in D\,;\, \operatorname{grad} U(x) = 0\}$ consists of finite number of connected components $K_1, \ldots, K_l$ (each of which is called compactum) such that, for arbitrary two points $x, y \in K_i$, there is an

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absolutely continuous function $\phi \in C_{01}^{x,y}(K_i)$ satisfying $\int_0^1 \| \dot{\phi}(t) \|^2 dt < \infty$;

(A_3) grad $U \neq 0$ on ∂D.

Here grad means the Riemannian gradient, $\| \cdot \| = \sqrt{g(\cdot,\cdot)}$ is the Riemannian norm and

$$C_{0T}^{x,y}(F) = \{\phi \in C([0, T], F)\,;\, \phi(0) = x,\ \phi(T) = y\}, \quad x, y \in F, \quad T > 0,$$

for an open or closed set F.

Introduce a quantity V_0 by

$$V_0 = \max_{x \in \mathscr{C}} \inf_{\phi \in C^{x,\partial D}} \max_{t \in [0,1]} \{U(\phi(t)) - U(x)\}, \tag{1.2}$$

where $C^{x,F} = \cup_{y \in F} C_{01}^{x,y}(\bar{D})$. By virtue of the theory of Freidlin and Wentzell [FW], one may expect

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log E_x[\tau^\varepsilon] = V_0, \quad x \in \Omega, \tag{1.3}$$

for a certain subdomain Ω of D. Indeed, one can see in [FW, Chapter 4] that, if the dynamical system determined by $-\dfrac{1}{2}$ grad U has a unique stable equilibrium position O and the domain D is attracted to O, (1.3) holds for $\Omega = D$. However, it is not clear whether (1.3) holds or not in case that D contains more than one compacta, although their theory [FW, Chapter 6] determines the exponential rates in terms of quasi-potentials and $\{\partial D\}$-graphs. In the present paper, by applying their results, we shall determine the subdomain Ω of D directly in terms of the potential U rather than the quasi-potentials in such a manner that (1.3) holds for all $x \in \Omega$ while the left hand side (LHS) of (1.3) is strictly less than V_0 for $x \in D \setminus \Omega$.

Let $\{\bar{x}_t(x)\,;\, t \geq 0,\ x \in \bar{D}\}$ be the flow determined by $-\dfrac{1}{2}$ grad U, i.e., $\bar{x}_t = \bar{x}_t(x)$ is a unique solution of the ordinary differential equation (ODE):

$$\frac{d\bar{x}_t}{dt} = -\frac{1}{2} \operatorname{grad} U(\bar{x}_t), \quad \bar{x}_0 = x. \tag{1.4}$$

We denote the ω-limit set of a point $x \in \bar{D}$ and the domain of the attraction of a connected open or closed set F in $\bar{D}$ with respect to this flow, respectively, by $\omega(x)$ and $\mathscr{D}(F)$: if $\bar{x}_t(x) \in \bar{D}$ for all $t > 0$,

$$\omega(x) = \{y \in \bar{D}\,;\, \bar{x}_{t_n}(x) \to y \text{ for some sequence } t_n \to \infty\},$$

otherwise $\omega(x) = \emptyset$, and $\mathscr{D}(F) = \{x \in \bar{D}\,; \omega(x) \subset F,\ \omega(x) \neq \emptyset\}$. Set $\mathbf{K} = \{K_1, \ldots, K_l\}$. $\mathbf{K}_{\mathrm{s}}$ and $\mathbf{K}_{\mathrm{u}}$ stand for the set of all stable compacta and that of all unstable ones, respectively, with respect to the flow mentioned above. Every non-empty ω-limit set is connected and consists of critical points of U. Namely, if $\omega(x) \neq \emptyset$, then we have $\omega(x) \subset K_i$ for some $K_i \in \mathbf{K}$. (See, e.g., Palis and de Melo [PD].)

For every stable compactum K_i, we define a valley $\mathscr{V}(K_i)$ containing K_i in D. To do this, we set, for compact subsets F_1, F_2 of $\bar{D}$,

$$U(F_1) = \min_{x \in F_1} U(x), \tag{1.5a}$$

$$U_{F_1}(F_2) = \max_{x \in F_1} \inf_{\phi \in C^{x,F_2}} \max_{t \in [0,1]} \{U(\phi(t)) - U(x)\}. \tag{1.5b}$$

Then, $\mathscr{V}(K_i)$ is a connected component of $\{x \in D\,; U(x) < U(K_i) + U_{K_i}(\partial D)\}$ containing K_i. We denote the depth of valley $\mathscr{V}(K_i)$ by $\mathrm{Depth}\,\mathscr{V}(K_i)$: Depth $\mathscr{V}(K_i) = \sup_{x,y \in \mathscr{V}(K_i)} \{U(x) - U(y)\}$. Notice that Depth $\mathscr{V}(K_i) > 0$ for all $K_i \in \mathbf{K}_{\mathrm{s}}$ and that (1.2) is equivalent to $\max_{K_i \in \mathbf{K}_{\mathrm{s}}}$ Depth $\mathscr{V}(K_i) = V_0$.

Let us define the domain Ω mentioned above in (1.3). If there is no stable compactum in D, we put $\Omega = D$. In the case of $\#\mathbf{K}_{\mathrm{s}} \geq 1$, we define $\Omega = \bigcup_{k=0}^{\infty} \Omega_{k,0} \cap D$ by preparing subsets $\Omega_{k,j}$ and $\Omega_{k,j}^{(1)}$, $k, j = 0,1,\ldots,$ of D in the following manner. First, we write $\Omega_{0,0} = \emptyset$ and

$$\Omega_{0,0}^{(1)} = \bigcup_{K_i \in \mathbf{K}\mathrm{s}, \mathrm{Depth}\,\mathscr{V}(K_i) = V_0} \overline{\mathscr{V}(K_i)}.$$

Then, for each fixed $k = 0,1,\ldots,$ with noting that each $\Omega_{k,0}^{(1)}$, $k = 1,2,\ldots,$ is defined below from $\{\Omega_{k-1,j}\}_{j=0,1,\ldots}$, we construct $\Omega_{k,j}$ and $\Omega_{k,j}^{(1)}$, $j = 1,2,\ldots,$ by using induction on j as following:

$$\Omega_{k,j} = \overline{\Omega_{k,j-1}^{(1)}} \cup \bigcup_{K_i \in \mathbf{K}_{\mathrm{u}}, K_i \cap \overline{\Omega_{k,j-1}^{(1)}} \neq \emptyset} K_i, \quad j = 1,2,\ldots,$$

$$\Omega_{k,j}^{(1)} = D \cap \mathscr{D}(\Omega_{k,j}), \qquad j = 1,2,\ldots.$$

Finally, for $k = 1,2,\ldots$ and $j = 0$, $\Omega_{k,0}$ and $\Omega_{k,0}^{(1)}$ are defined by

$$\Omega_{k,0} = \bigcup_{j=0}^{\infty} \Omega_{k-1,j},$$
$$\Omega_{k,0}^{(1)} = \Omega_{k,0} \cup \bigcup_{K_i \in \mathbf{K}_{\mathrm{s}}: \overline{\mathscr{V}(K_i)} \cap \Omega_{k,0} \neq \emptyset} \overline{\mathscr{V}(K_i)}.$$

Here one notices that $\Omega_{k,j} \subset \Omega_{k,j+1}$ and $\Omega_{k,0} \subset \Omega_{k+1,0}$ for $k, j = 0,1,\ldots$ and that $\Omega_{k+1,0} = \Omega_{k,\#\mathbf{K}_{\mathrm{u}}}$ for $k = 0,1,\ldots$ and $\Omega = \Omega_{\#\mathbf{K}_{\mathrm{s}},0} \cap D$. We also note that Ω is closed in D since every $\Omega_{k,0}$, $k = 1,2,\ldots,$ is compact.

Now we formulate our main result.

THEOREM 1. *We have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log E_x[\tau^\varepsilon] = V_0 \tag{1.6}$$

for all $x \in \Omega$. When $x \notin \Omega$, the LHS of (1.6) *is strictly less than V_0.*

The proof essentially consists of two parts. In Section 2, Freidlin-Wentzell's quasi-potentials will be characterized by valleys of the potential U and the flow determined by $-\frac{1}{2}\operatorname{grad} U$. Then, the set Ω will be expressed in terms of valleys and quasi-potentials. We shall also show that the assumption (**A**) in [FW, p.169] is fulfilled, which guarantees the existence of the limit in (1.6). In Sections 3 and 4, we shall recall that the limit in (1.6) can be represented by using Freidlin-Wentzell's quasi-potentials and $\{\partial D\}$-graphs, and get the results by calculating the $\{\partial D\}$-graphs together with the estimates of quasi-potentials derived in Section 2. The main tool is the $\{\partial D\}$-graph with partially reversed arrows. Moreover two problems concerning the value of the LHS of (1.6) for $x \in D \setminus \Omega$ will be considered in Section 5. Namely, we shall show that, if the valley $\mathcal{V}$ is a bottom one in the sense that $\min_{x \in \mathcal{V}} U(x) = \min_{K_j \in \mathbf{K}_s} U(K_j)$, the LHS of (1.6) is equal to the depth of $\mathcal{V}$ for every $x \in \mathcal{V}$, and represent the values of the LHS of (1.6) for all $x \in D$ directly in terms of $U(x)$, $x \in D$, when $\mathcal{M}$ is one-dimensional Euclidean space. We notice that the technique in this paper is also applicable to getting the asymptotic behavior of the distribution, $P_x(x^\varepsilon_{\tau^\varepsilon_D} \in A)$, $A \subset \partial D$, of the exit position of x^ε_t from the boundary. (See [Su1] for details.)

This result will be applied in the collaborative papers [Su1], [Su2] to investigate metastable behaviors for a class of diffusion processes $\{x^\varepsilon_t\}$ of gradient type.

2. Properties of quasi-potentials

The action functional S_T is defined on $C([0, T], \mathcal{M})$, $T \geq 0$: $S_T(\phi) = \frac{1}{2}\int_0^T \| \dot{\phi}(t) - b(\phi(t)) \|^2 dt$ if $\phi \in C([0, T], \mathcal{M})$ is absolutely continuous, and $S_T(\phi) = +\infty$ otherwise. In particular, for an absolutely continuous $\phi \in C([0\ T], \bar{D})$,

$$S_T(\phi) = \frac{1}{2}\int_0^T \| \dot{\phi}(t) + \frac{1}{2}\operatorname{grad} U(\phi(t)) \|^2 dt. \tag{2.1}$$

Moreover we define

(2.2a) $$V_D(x, y) = \inf\{S_T(\phi) ; \phi \in C_{0T}^{x,y}(\bar{D}), T \geq 0\}, \quad x, y \in \bar{D},$$

which is called quasi-potential. We also denote, for compact subsets F_1, F_2 of $\bar{D}$,

(2.2b) $$V_D(x, F_2) = \inf_{y \in F_2} V_D(x, y),$$

(2.2c) $$V_D(F_1, y) = \inf_{y \in F_1} V_D(x, y),$$

(2.2d) $$V_D(F_1, F_2) = \inf_{x \in F_1, y \in F_2} V_D(x, y).$$

We state three lemmas without proofs: Lemmas 2.1 and 2.3 are written as a comment after [FW, Chapter 6, Lemma 1.1] and Lemma 5.2 in [FW, Chapter 6], respectively, and Lemma 2.2 can be shown by straightforward arguments.

LEMMA 2.1. *$V_D(x, y)$ is continuous for $x, y \in \bar{D}$. In particular, we have the following*:

(i) $V_D(x, y) < \infty$ *for all* $x, y \in \bar{D}$;

(ii) *the maps $x \mapsto V_D(x, F)$ and $y \mapsto V_D(F, y)$ are both continuous for every compact subset F of $\bar{D}$.*

LEMMA 2.2. *Let us suppose that compact subsets F_1, F_2 and $\mathscr{F}$ of $\bar{D}$ are mutually disjoint and have the property that every trajectory in $\bar{D}$ connecting F_1 and F_2 traverses $\mathscr{F}$; i.e., for every $\phi \in C([0,1], \bar{D})$ satisfying $\phi(0) \in F_1$ and $\phi(1) \in F_2$, there exists $t \in (0,1)$ so that $\phi(t) \in \mathscr{F}$. Then, we have $V_D(F_1, F_2) = \inf_{x \in \mathscr{F}}\{V_D(F_1, x) + V_D(x, F_2)\}$.*

LEMMA 2.3. *If α is an unstable compactum K_i or a regular point x of U, then either there exists a stable compactum K_j such that $V_D(\alpha, K_j) = 0$ or $V_D(\alpha, \partial D) = 0$.*

The next lemma is an easy consequence of the assumption (A_2).

LEMMA 2.4. *We have $V_D(x, y) = V_D(y, x) = 0$ for arbitrary two points x, y belonging to the same compactum K_i.*

Proof. From (A_2), there is an absolutely continuous $\phi \in C_{01}^{x,y}(K_i)$ such that $S_1(\phi) < +\infty$, where we recall $\mathbf{grad}\, U \equiv 0$ on K_i. If one sets $\psi(t) = \phi(t/T)$, $T > 0$, then $S_T(\psi) \leq S_1(\phi)/T$. This immediately verifies $V_D(x, y) = 0$ by letting

$T \to \infty$. □

The following two lemmas establish basic relations between the quasi-potential and the depth of the valley. Recall (1.5) for the notation $U_{F_1}(F_2)$.

LEMMA 2.5. *For all compact subsets F of $\bar{D}$, we have*

$$V_D(x, F) \geq U_{\{x\}}(F), \quad x \in D, \tag{2.3}$$

$$V_D(K_i, F) \geq U_{K_i}(F), \quad K_i \in \mathbf{K}. \tag{2.4}$$

In particular, if K_i is stable and satisfies $F \cap \mathcal{V}(K_i) = \emptyset$, then $V_D(K_i, F) \geq U_{K_i}(\partial D)$.

Proof. We shall prove only (2.4) since (2.3) is obtained in a quite parallel manner. Let $x_0 \in K_i$ be fixed arbitrarily. For $\delta > 0$, (2.2) and Lemma 2.4 verify the existence of an absolutely continuous $\phi \in C([0, T], \bar{D})$, $T \geq 0$, so that $\phi(0) = x_0$, $\phi(T) \in F$ and

$$V_D(K_i, F) \geq S_T(\phi) - \delta.$$

From (1.5), one can find $0 \leq T_0 \leq T$ satisfying

$$U(\phi(T_0)) - U(\phi(0)) \geq U_{\{x_0\}}(F).$$

On the other hand, with the help of the definition (2.1) of the action functional $S_T(\phi)$ and the gradient condition (A_1), we have

$$\begin{aligned} S_T(\phi) &\geq \int_0^{T_0} g(\dot{\phi}(t), \operatorname{grad} U(\phi(t)))dt \\ &= U(\phi(T_0)) - U(\phi(0)). \end{aligned}$$

From these estimates, we obtain

$$V_D(K_i, F) \geq U_{\{x_0\}}(F)$$

by letting $\delta \downarrow 0$. Since it holds for every $x_0 \in K_i$, (2.4) is now derived. □

COROLLARY 2.6. *We have $U(x) = U(y)$ for arbitrary two points x, y belonging to the same compactum.*

Proof. If x, y are belonging to the same compactum, one has $V_D(x, y) = V_D(y, x) = 0$ from Lemma 2.4. By applying Lemma 2.5, this implies $U_{\{x\}}(\{y\}) =$

$U_{\{y\}}(\{x\}) = 0$, which is equivalent to $U(x) = U(y)$. □

LEMMA 2.7. *Let each of α and β be a point of $\bar{D}$ or a compactum in $\mathbf{K}$. Then we have*

$$V_D(\alpha, \beta) - V_D(\beta, \alpha) = U(\beta) - U(\alpha). \tag{2.5}$$

In particular, if $V_D(\alpha, \beta) = 0$, then $V_D(\beta, \alpha) = U(\alpha) - U(\beta)$.

Proof. We shall treat only the case where both of α and β are compacta, because the other cases are shown similarly. Write $\alpha = K_i$ and $\beta = K_j$. For an arbitrary $\delta > 0$, there exists an absolutely continuous $\phi \in C([0, T], \bar{D})$, $T \geq 0$, such that $\phi(0) \in K_i$, $\phi(T) \in K_j$ and

$$V_D(K_i, K_j) \geq S_T(\phi) - \delta.$$

Put $\psi(t) = \phi(T - t)$, $0 \leq t \leq T$. Then, we have

$$\begin{aligned} S_T(\phi) - S_T(\psi) &= \int_0^T g(\dot{\phi}(t), \operatorname{grad} U(\phi(t)))dt \\ &= U(\phi(T)) - U(\phi(0)). \end{aligned}$$

On the other hand, since $\psi(0) \in K_j$ and $\psi(T) \in K_i$, $V_D(K_j, K_i) \leq S_T(\psi)$. Hence, by letting $\delta \downarrow 0$, we get

$$V_D(K_i, K_j) - V_D(K_j, K_i) \geq U(K_j) - U(K_i).$$

By reversing the symbols K_i and K_j, it holds that

$$V_D(K_i, K_j) - V_D(K_j, K_i) \leq U(K_j) - U(K_i),$$

and now (2.5) is obtained. □

The next lemma gives an important property of regular points.

LEMMA 2.8. *Let $x \in D$ be a regular point of U,* namely, $\operatorname{grad} U(x) \neq 0$ and *suppose $\bar{x}_t(x) \in \bar{D}$ for $0 \leq t \leq T$. Then, we have $V_D(x, y) > 0$ for every point $y \in \bar{D} \setminus \{\bar{x}_t(x)\,;\, 0 \leq t \leq T\}$ such that $U(y) > U(\bar{x}_T(x))$. Recall that $\bar{x}_t(x)$ is the solution of the ODE* (1.4).

Proof. Set $\rho_0 = \inf_{0 \leq t \leq T} \rho(\bar{x}_t(x), y) > 0$, where $\rho(\cdot, \cdot)$ denotes the Riemannian distance on $\mathcal{M}$. From Lemma 2.1 of [FW, Chapter 4], we know that

$$I_{T'} = \inf\{S_{T'}(\phi)\,;\phi \in C([0, T'], \mathcal{M}),\ \phi(0) = x,\ \max_{0\le t\le T'} \rho(\bar{x}_t(x), \phi(t)) > \rho_0/2\} > 0$$

for every $0 < T' \le T$. Since $I_{T'}$ is a non-increasing function of T',

$$\inf\{S_{T'}(\phi)\,;\ \phi \in C_{0T'}^{x,y}(\bar{D}),\ 0 \le T' \le T\} \ge I_T > 0.$$

Let $0 < T_1 < T_2 < T$ satisfy $U(y) > U(\bar{x}_{T_1}(x))$. Then, by using the same argument of Lemma 2.2 in [FW, Chapter 4], one can find $a > 0$ such that $S_{T'}(\phi) \ge a(T' - T_2)$ for every $T' > T_2$ and $\phi \in C([0, T'], \bar{D})$ with $\phi(0) = x$ and $U(\phi(t)) \ge U(\bar{x}_{T_1}(x))$ during $0 \le t \le T'$. Hence, combining this with Lemma 2.5, we obtain

$$\inf\{S_{T'}(\phi)\,;\ \phi \in C_{0T'}^{x,y}(\bar{D}),\ T' \ge T\} \ge \min\{a(T - T_2),\ U(y) - U(\bar{x}_{T_1}(x))\} > 0$$

and the proof is completed. □

COROLLARY 2.9. *If x satisfies $\omega(x) = \emptyset$ and F is a compact subset of $\bar{D}$ satisfying $\omega(y) \ne \emptyset$ for all $y \in F$, then $V_D(x, F) > 0$.*

Proof. Let $T = \inf\{t > 0\,;\bar{x}_t(x) \notin \bar{D}\}$. If one denotes

$$\mathcal{F} = \{z \in \bar{D}\,;\delta/2 \le \inf_{0\le t\le T} \rho(\bar{x}_t(x), z) \le \delta\} \tag{2.6}$$

for sufficiently small $\delta > 0$, three compact subsets $\{x\}$, F and $\mathcal{F}$ of $\bar{D}$ are mutually disjoint. From Lemma 2.8, we can obtain $\inf_{z\in\mathcal{F}} V_D(x, z) > 0$, where we use a sufficiently smooth function $\tilde{U}$ on a neighborhood of $\bar{D}$ satisfying $\tilde{U} = U$ on $\bar{D}$. Hence, since every trajectory in $\bar{D}$ connecting x and F traverses $\mathcal{F}$, by applying Lemma 2.2 we get

$$\begin{aligned} V_D(x, F) &= \inf_{z\in\mathcal{F}} \{V_D(x, z) + V_D(z, F)\} \\ &\ge \inf_{z\in\mathcal{F}} V_D(x, z) \\ &> 0. \end{aligned}$$ □

COROLLARY 2.10. *If $x \in K_i$ and $y \notin K_i$, then either $V_D(x, y) > 0$ or $V_D(y, x) > 0$.*

Proof. From Lemma 2.5, it suffices to show the case where $U(x) = U(y)$. Let $T > 0$ satisfy $\bar{x}_t(y) \in \bar{D}$ for $0 \le t \le T$. By choosing a sufficiently small $\delta > 0$, we can suppose that $\{y\}$, $\{x\}$ and $\mathcal{F}$ are mutually disjoint, where we define $\mathcal{F}$ by (2.6) in which x should be replaced by y. For $0 < T' < T$ Lemmas 2.5 and 2.8

imply, respectively,

$$\inf_{z\in\mathscr{F}:U(z)\leq U(\bar{x}_{T'}(y))} V_D(z, x) \geq U(x) - U(\bar{x}_{T'}(y)) > 0,$$
$$\inf_{z\in\mathscr{F}:U(z)>U(\bar{x}_{T'}(y))} V_D(y, z) > 0.$$

Hence, combining these estimates with Lemma 2.2, we obtain

$$V_D(y, x) = \inf_{z\in\mathscr{F}} \{V_D(y, z) + V_D(z, x)\} > 0. \qquad \square$$

We define a subdomain $\tilde{\Omega}$ of D in terms of quasi-potentials. Set $\tilde{\Omega} = D$ if there is no stable compactum. In the case of $\# \mathbf{K}_s \geq 1$, determine $\tilde{\Omega}_k^{(1)}$, $k = 0,1,\ldots$, and $\tilde{\Omega}_k$, $k = 1,2,\ldots$, inductively, by

$$\tilde{\Omega}_0^{(1)} = \bigcup_{K_i\in \mathbf{Ks}\cdot\mathrm{Depth}\,\mathscr{V}(K_i)=V_0} \overline{\mathscr{V}(K_i)},$$
$$\tilde{\Omega}_k = \{x \in \bar{D}\,;\, V_D(x, \tilde{\Omega}_{k-1}^{(1)}) = 0\}, \quad k = 1,2,\ldots,$$
$$\tilde{\Omega}_k^{(1)} = \tilde{\Omega}_k \cup \bigcup_{K_i\in \mathbf{Ks}\cdot\overline{\mathscr{V}(K_i)}\cap\tilde{\Omega}_k\neq\emptyset} \overline{\mathscr{V}(K_i)}, \quad k = 1,2,\ldots.$$

We remark that Lemma 2.1 (ii) implies the compactness of the sets $\tilde{\Omega}_k$ and $\tilde{\Omega}_k^{(1)}$. Noting that the sequence $\{\tilde{\Omega}_k\}_{k=1,2,\ldots}$ is not decreasing and that $\tilde{\Omega}_{k_0} = \tilde{\Omega}_{k_0+1} = \cdots$ for $k_0 \geq \# \mathbf{K}_s$, we define $\tilde{\Omega} = \tilde{\Omega}_{k_0} \cap D$.

PROPOSITION 2.11. *We have* $\Omega = \tilde{\Omega}$.

Proof. If there is no stable compactum in D, the statement is obvious. So we assume $\# \mathbf{K}_s \geq 1$. Claim that $\Omega_{1,0} = \tilde{\Omega}_1$. It is obvious that $\Omega_{1,0} \subset \tilde{\Omega}_1$. In order to prove $\Omega_{1,0} \supset \tilde{\Omega}_1$, it is sufficient to show $V_D(K_i, \tilde{\Omega}_0^{(1)}) > 0$ for every compactum K_i in $D \setminus \Omega_{1,0}$. Indeed, let $x \in D \setminus \Omega_{1,0}$ be a regular point. Then, if $\omega(x) = \emptyset$, we know $V_D(x, \tilde{\Omega}_0^{(1)}) > 0$ from Corollary 2.9, and, if $\omega(x) \subset K_i$ and $V_D(x, \tilde{\Omega}_0^{(1)}) = 0$, by using a similar argument to Corollary 2.9 or 2.10 we get $V_D(\bar{x}_T(x), \tilde{\Omega}_0^{(1)}) = 0$ for all $T > 0$ from Lemmas 2.1, 2.2 and 2.8 and, consequently, $V_D(K_i, \tilde{\Omega}_0^{(1)}) = 0$ from Lemma 2.1. First, suppose that $K_i \in \mathbf{K}$ satisfies $U(K_i) = \min\{U(K_j)\,; K_j \subset D \setminus \Omega_{1,0}\}$. For a stable compactum $K_i \subset D \setminus \Omega_{1,0}$, one has $V_D(K_i, \tilde{\Omega}_0^{(1)}) \geq$ Depth $\mathscr{V}(K_i) > 0$ from $\Omega_0^{(1)} = \tilde{\Omega}_0^{(1)}$ and $K_i \cap \tilde{\Omega}_0^{(1)} = \emptyset$. If K_i is unstable, there is an open neighborhood G of K_i such that $\omega(y) = \emptyset$, which implies $V_D(y, \tilde{\Omega}_0^{(1)}) > 0$ from Corollary 2.9, for all $y \in G \setminus K_i : U(y) \leq U(K_i)$. Hence, by a parallel method to Corollary 2.10 with using Lemmas 2.1, 2.2 and 2.5, $V_D(K_i, \tilde{\Omega}_0^{(1)}) > 0$ is obtained. Next, take $K_i \in \mathbf{K}$ such that $V_D(K_j, \tilde{\Omega}_0^{(1)}) > 0$ for all $K_j \subset D \setminus \Omega_{1,0}$:

$U(K_j) < U(K_i)$. Then, one can find an open neighborhood G of K_i such that every $y \in G \setminus K_i : U(y) \leq U(K_i)$ satisfies either $\omega(y) = \emptyset$ or $\omega(y) \subset K_j$ for some $K_j \subset D \setminus \Omega_{1,0} : U(K_j) < U(K_i)$, namely, $V_D(y, \tilde{\Omega}_0^{(1)}) > 0$ from Lemma 2.8. Lemmas 2.1, 2.2 and 2.5 also verify $V_D(K_i, \tilde{\Omega}_0^{(1)}) > 0$. Hence, we obtain $\Omega_{1,0} = \tilde{\Omega}_1$ by induction.

Since one can show that $\Omega_{k,0} = \tilde{\Omega}_k$ implies $\Omega_{k+1,0} = \tilde{\Omega}_{k+1}$ for $k = 1,2,\ldots$, by using the methods explained above, the proof is immediately concluded by induction. □

3. Summaries of Freidlin and Wentzell's results

We recall Freidlin-Wentzell's $\{\partial D\}$-graph. (See also [FW, Chapter 6].) Let L be a finite set and let W be a subset of L. A graph consisting of arrows $\alpha \rightarrow \beta$ ($\alpha \in L \setminus W$, $\beta \in L$, $\alpha \neq \beta$) is called a W-graph on L if it satisfies the following conditions:

(1) every $\alpha \in L \setminus W$ is the initial point of exactly one arrow;

(2) there are no closed cycles in the graph.

We note that condition (2) can be replaced by the next one:

(2′) for every $\alpha \in L \setminus W$ there exists a sequence of arrows leading from it to some $\beta \in W$.

We denote by $G^L(W)$ the set of W-graphs on L and, for $\alpha \in L \setminus W$ and $\beta \in W$, $G^L_{\alpha\beta}(W)$ stands for the set of W-graphs on L each of which contains the sequence of arrows leading from α to β. For $\alpha \in L \setminus W$, we set, if $\# [L \setminus W] \geq 2$,

$$G^L(\alpha \not\rightarrow W) = G^L(W \cup \{\alpha\}) \cup \bigcup_{\beta \in L \setminus W, \beta \neq \alpha} G^L_{\alpha\beta}(W \cup \{\beta\})$$

and, if $\# [L \setminus K] = 1$, $G^L(\alpha \not\rightarrow W) = \emptyset$.

Let us define

$$W_D = \min_{g \in G^{\mathbf{K}^*}(\partial D)} \sigma(g), \tag{3.1}$$

$$M_D(x) = \min_{g \in G^{\mathbf{K}^* \cup \{x\}}(x \not\rightarrow \{\partial D\})} \sigma(g), \quad x \in D, \tag{3.2}$$

$$M_D(K_i) = \min_{g \in G^{\mathbf{K}^*}(K_i \not\rightarrow \{\partial D\})} \sigma(g), \quad K_i \in \mathbf{K}, \tag{3.3}$$

where $\mathbf{K}^* = \{K_1, \ldots, K_l, \partial D\}$ and

$$\sigma(g) = \sum_{(\alpha \rightarrow \beta) \in g} V_D(\alpha, \beta)$$

for a graph g. From Lemma 2.4 and Corollary 2.10, our system satisfies the assumption **(A)** in [FW, p.169]. Hence, under the assumptions (A_1)-(A_3), we have the next theorems stated in [FW, Chapter 6, §5].

THEOREM 3.1. *Let us assume* $\# \mathbf{K}_s \geq 1$. *We have*

$$W_D = \min_{g \in G^{\mathbf{K}_s^*}\{\partial D\}} \sigma(g), \tag{3.4}$$

$$W_D = \min_{g \in G^{\mathbf{K}_s^* \cup \{x\}}\{\partial D\}} \sigma(g), \qquad \text{for } x \in D, \tag{3.5}$$

$$M_D(x) = \min_{g \in G^{\mathbf{K}_s^* \cup \{x\}}(x \nrightarrow \{\partial D\})} \sigma(g), \quad \text{for } x \in D, \tag{3.6}$$

$$M_D(K_i) = \min_{g \in G^{\mathbf{K}_s^*}(K_i \nrightarrow \{\partial D\})} \sigma(g), \quad \text{for } K_i \in \mathbf{K}_s, \tag{3.7}$$

where $\mathbf{K}_s^* = \mathbf{K}_s \cup \{\partial D\}$.

THEOREM 3.2. *We have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log E_x[\tau^\varepsilon] = W_D - M_D(x)$$

uniformly in x *belonging to every compact subset of* D.

Remark 3.3. Theorem 3.2 guarantees the existence of the limit in the LHS of (1.6).

4. Proof of Theorem 1

In this section we shall show Theorem 1. By combining Theorem 3.2 with Proposition 2.11, the next theorem immediately verifies Theorem 1.

THEOREM 4.1. *We have*

$$W_D - M_D(x) = V_0, \quad x \in \tilde{\Omega}, \tag{4.1}$$

$$W_D - M_D(x) < V_0, \quad x \notin \tilde{\Omega}. \tag{4.2}$$

Let us suppose that there is no stable compactum in D. Fix an arbitrary $x \in D$. For $g \in G^{\mathbf{K}^* \cup \{x\}}(x \nrightarrow \{\partial D\})$ attaining the minimum in the right hand side (RHS) of (3.2), we consider a $\{\partial D\}$-graph on $\mathbf{K}^* \cup \{x\}$ derived from g by exchanging one arrow starting from x with an arrow $(x \rightarrow \partial D)$. Since $V_D(x, \partial D) =$

0, from Lemma 5.3 in [FW, Chapter 6] we obtain $W_D - M_D(x) = 0$ and this completes the proof of Theorem 4.1 when $\# \mathbf{K}_s = 0$. Therefore we assume $\# \mathbf{K}_s \geq 1$ throughout the rest of this section.

For a graph g in $G^{\mathbf{K}_s^*}\{\partial D\}$, $G^{\mathbf{K}_0^*}\{\partial D\}$, $G^{\mathbf{K}_s^*}(K_i \nrightarrow \{\partial D\})$ or $G^{\mathbf{K}_0^*}(K_i \nrightarrow \{\partial D\})$, we introduce a notation $K_i \xrightarrow{g} K_j$ for $K_i, K_j \in \mathbf{K}_0^*$ if g contains a sequence of arrows leading from K_i to K_j; we also use the notation $K_i \overset{g}{\nrightarrow} K_j$ if g does not contain such a sequence of arrows. Here, taking the formulae (3.5) and (3.6) into account, we set $K_0 = \{x_0\}$ $(x_0 \in \bar{D})$, $K_{l+1} = \partial D$, $\mathbf{K}_0^* = \mathbf{K}_s \cup \{K_0\} \cup \{K_{l+1}\}$.

Let g be a $\{\partial D\}$-graph in $G^{\mathbf{K}_s^*}\{\partial D\}$ on $\mathbf{K}_s^*$ and $K_i \in \mathbf{K}_s$. For a sequence of arrows $(K_i \to K_{i_1})$, $(K_{i_1} \to K_{i_2}), \ldots, (K_{i_n} \to \partial D) \in g$, we set

$$\mathbf{n} = \min\{p \geq 0 \,;\, K_{i_{p+1}} \not\subset \mathcal{V}(K_i)\},$$

where we write $K_{i_0} = K_i$ and $K_{i_{n+1}} = \partial D$ simply. Then, we call $K_{i_\mathbf{n}}$ the last compactum of g in a valley $\mathcal{V}(K_i)$ from K_i. For a graph g in $G^{\mathbf{K}_s^*}(K_i \nrightarrow \{\partial D\})$, there is a unique compactum (except ∂D) which does not become the initial point of any arrows. We call it the end compactum of g.

LEMMA 4.2. *Let a $\{\partial D\}$-graph $g \in G^{\mathbf{K}_s^*}\{\partial D\}$ attain the minimum in the RHS of* (3.4). *Then, for each valley $\mathcal{V}$, the last compactum of g in $\mathcal{V}$ does not depend on any particular choice of stable compacta in $\mathcal{V}$.*

Proof. Suppose that there exist more than one last compacta of g in $\mathcal{V} = \mathcal{V}(K_i)$, $K_i \in \mathbf{K}_s$. Let K_i be a last compactum. We consider a connected compact subdomain $\mathcal{V}_\gamma = \{x \in \mathcal{V} \,;\, U(x) \leq \gamma\}$ of $\mathcal{V}$ for $\max_{x \in \mathcal{V} \cap \mathscr{C}} U(x) < \gamma < U(K_i) + U_{K_i}(\partial D)$, and set

$$\mathcal{V}_\gamma^{(1)} = \{x \in \mathcal{V}_\gamma \,;\, \text{there exists } K_j \in \mathbf{K}_s \text{ in } \mathcal{V} \text{ such that } K_j \xrightarrow{g} K_\mathrm{i} \text{ and } V_D(x, K_j) = 0\},$$
$$\mathcal{V}_\gamma^{(2)} = \{x \in \mathcal{V}_\gamma \,;\, \text{there exists } K_j \in \mathbf{K}_s \text{ in } \mathcal{V} \text{ such that } K_j \overset{g}{\nrightarrow} K_\mathrm{i} \text{ and } V_D(x, K_j) = 0\}.$$

Then, since both $\mathcal{V}_\gamma^{(1)}$ and $\mathcal{V}_\gamma^{(2)}$ are non-empty closed subsets of $\mathcal{V}_\gamma$ and Lemma 2.3 varifies $\mathcal{V}_\gamma^{(1)} \cup \mathcal{V}_\gamma^{(2)} = \mathcal{V}_\gamma$, we have $\mathcal{V}_\gamma^{(1)} \cap \mathcal{V}_\gamma^{(2)} \neq \emptyset$; i.e., there exist $x_1 \in \mathcal{V}_\gamma$ and $K_{j_0}, K_{j_1} \in \mathbf{K}_s$ in $\mathcal{V}_\gamma$ so that $K_{j_0} \xrightarrow{g} K_\mathbf{i}$, $K_{j_1} \overset{g}{\nrightarrow} K_\mathbf{i}$ and that $V_D(x_1, K_{j_0}) = V_D(x_1, K_{j_1}) = 0$. From Lemmas 2.2 and 2.7, one knows

$$\begin{aligned} V_D(K_{j_1}, K_{j_0}) &\leq V_D(K_{j_1}, x_1) + V_D(x_1, K_{j_0}) \\ &= U(x_1) - U(K_{j_1}) \\ &\leq \gamma - U(K_{j_1}). \end{aligned} \tag{4.3}$$

Let K_j be the last compactum of g in $\mathcal{V}$ from K_{j_1}. For a sequence of arrows $(K_{j_1} \to$

K_{j_2}), $(K_{j_2} \to K_{j_3})$, . . . , $(K_{j_{n-1}} \to K_{\mathbf{j}})$, $(K_{\mathbf{j}} \to K_{j_{n+1}}) \in g$, $\tilde{g}$ denotes a $\{\partial D\}$-graph obtained from g by replacing these n arrows with n arrows $(K_{\mathbf{j}} \to K_{j_{n-1}}), \ldots, (K_{j_2} \to K_{j_1})$, $(K_{j_1} \to K_{j_0})$. Since Lemma 2.5 verifies $V_D(K_{\mathbf{j}}, K_{j_{n+1}}) \geq U_{K_{\mathbf{j}}}(\partial D)$, by using Lemma 2.7 and (4.3) we have

$$\begin{aligned}
&\sigma(\tilde{g}) - \sigma(g) \\
&= \sum_{k=1}^{n-1} \{V_D(K_{j_{k+1}}, K_{j_k}) - V_D(K_{j_k}, K_{j_{k+1}})\} + V_D(K_{j_1}, K_{j_0}) - V_D(K_{j_n}, K_{j_{n+1}}) \\
&\leq U(K_{j_1}) - U(K_{\mathbf{j}}) + \gamma - U(K_{j_1}) - U_{K_{\mathbf{j}}}(\partial D) \\
&= \gamma - \{U(K_{\mathbf{j}}) + U_{K_{\mathbf{j}}}(\partial D)\} \\
&< 0,
\end{aligned}$$

where $K_{j_n} = K_{\mathbf{j}}$. But this contradicts the assumption that g attains the minimum in the RHS of (3.4). □

PROPOSITION 4.3. *We have*

$$W_D - M_D(x_0) \geq V_0 \tag{4.4}$$

for all $x_0 \in \tilde{\Omega}$.

Proof. Let a $\{\partial D\}$-graph $g \in G^{\mathbf{K}_s^*}\{\partial D\}$ attain the minimum in the RHS of (3.4) and be fixed throughout the proof. Consider $K_* \in \mathbf{K}_s$ satisfying $U_{K_*}(\partial D) = V_0$ and the last compactum $K_{\mathbf{i}}$ of g in the valley $\mathcal{V}(K_*)$. For a sequence of arrows $(K_* \to K_{i_1})$, $(K_{i_1} \to K_{i_2})$, . . . , $(K_{i_{n-1}} \to K_{\mathbf{i}})$, $(K_{\mathbf{i}} \to K_{i_{n+1}}) \in g$, we define $g_0 \in G^{\mathbf{K}_s^*}(K_* \nrightarrow \{\partial D\})$ from g by deleting these $n+1$ arrows and adding n arrows $(K_{\mathbf{i}} \to K_{i_{n-1}}), \ldots, (K_{i_2} \to K_{i_1})$, $(K_{i_1} \to K_*)$. Then, Lemmas 2.5 and 2.7 imply

$$\begin{aligned}
\sigma(g) - \sigma(g_0) &= \sum_{k=1}^{n} \{V_D(K_{i_{k-1}}, K_{i_k}) - V_D(K_{i_k}, K_{i_{k-1}})\} + V_D(K_{\mathbf{i}}, K_{i_{n+1}}) \\
&\geq U(K_{\mathbf{i}}) - U(K_*) + U_{K_{\mathbf{i}}}(\partial D) \\
&= V_0,
\end{aligned}$$

where $K_{i_0} = K_*$ and $K_{i_n} = K_{\mathbf{i}}$. Since Lemma 4.2 proves $g_0 \in G^{\mathbf{K}_s^*}(K_i \nrightarrow \{\partial D\})$ for all stable compacta $K_i \subset \mathcal{V}(K_*)$, one obtains the estimate

$$W_D - M_D(K_i) \geq V_0 \tag{4.5}$$

for all stable compacta K_i satisfying Depth $\mathcal{V}(K_i) = V_0$. On the other hand, for every $x_0 \in \tilde{\Omega}_1$, there is a stable compactum K_i, Depth $\mathcal{V}(K_i) = V_0$, so that $V_D(x_0, K_i) = 0$. This implies $M_D(x_0) \leq M_D(K_i)$ and therefore the estimate (4.4) holds for every $x_0 \in \tilde{\Omega}_1$.

For a stable compactum $K_i \subset \tilde{\Omega}_2 \setminus \tilde{\Omega}_1$ there exist a point $x_1 \in \overline{\mathscr{V}(K_i)} \cap \tilde{\Omega}_1$ and stable compacta $K_{j_0} \subset \tilde{\Omega}_1$, $K_{j_1} \subset \mathscr{V}(K_i)$ such that $V_D(x_1, K_{j_0}) = V_D(x_1, K_{j_1}) = 0$. Note that Lemmas 2.2 and 2.5 imply

$$V_D(K_{j_1}, K_{j_0}) = U_{K_{j_1}}(\partial D). \tag{4.6}$$

Since Depth $\mathscr{V}(K_{j_0}) = V_0$, one can construct $g_0 \in G^{\mathbf{K}_s^*}(K_{j_0} \twoheadrightarrow \{\partial D\})$ from the $\{\partial D\}$-graph g (fixed at the top of the proof) such that

$$\sigma(g) - \sigma(g_0) \geq V_0$$

by the previous methods. Then, define $g_1 \in G^{\mathbf{K}_s^*}(K_i \twoheadrightarrow \{\partial D\})$ from g_0 in the following manner: if $K_{j_1} \xrightarrow{g_0} K_*$, set $g_1 = g_0$; otherwise, g_1 is defined by exchanging m arrows $(K_{j_1} \to K_{j_2})$, $(K_{j_2} \to K_{j_3}), \ldots, (K_{j_{m-1}} \to K_{\mathbf{j}})$, $(K_{\mathbf{j}} \to K_{j_{m+1}})$ in g_0 (also in g), with m arrows $(K_{\mathbf{j}} \to K_{j_{m-1}}), \ldots, (K_{j_2} \to K_{j_1})$, $(K_{j_1} \to K_{j_0})$, where K_* and $K_{\mathbf{j}}$ respectively denote the end compactum of g_0 and the last compactum of g in $\mathscr{V}(K_i)$. Using Lemmas 2.5, 2.7 and (4.6), we have

$$\begin{aligned} \sigma(g_0) - \sigma(g_1) &\geq U(K_{\mathbf{j}}) - U(K_{j_1}) + U_{K_{\mathbf{j}}}(\partial D) - U_{K_{j_1}}(\partial D) \\ &= 0. \end{aligned}$$

With the help of Lemma 4.2, the estimate (4.5) is verified for every stable K_i in $\tilde{\Omega}_2$. For $x_0 \in \tilde{\Omega}_2$, choose a stable compactum K_i in $\tilde{\Omega}_2$ such that $V_D(x_0, K_i) = 0$. Then, we have $M_D(x_0) \leq M_D(K_i)$. Hence, (4.4) is obtained for all $x_0 \in \tilde{\Omega}_2$.

By using the above arguments inductively, one can show the estimate (4.4) for all $x_0 \in \tilde{\Omega}_{k+1}$, $k \geq 1$, which concludes the proof. □

Proof of Theorem 4.1. Fix an arbitrary $x_0 \in D$ and write $K_0 = \{x_0\}$, $K_{l+1} = \{\partial D\}$ and $\mathbf{K}_0^* = \mathbf{K}_s^* \cup \{K_0\} (= \mathbf{K}_s \cup \{x_0, \partial D\})$. We suppose that $g \in G^{\mathbf{K}_0^*}(x_0 \twoheadrightarrow \{\partial D\})$ attains the minimum of $M_D(x_0)$ in the RHS of (3.6) and that $K^* \in \mathbf{K}_0 (= \mathbf{K}_s \cup \{K_0\})$ is the end compactum of g.

First, we consider the case where $K^* = \{x_0\} \notin \mathbf{K}_s$. If there is a stable compactum K_i such that $V_D(x_0, K_i) = 0$, one can suppose that K_i is the last compactum. Indeed, the graph $\tilde{g} \in G^{\mathbf{K}_0^*}(x_0 \twoheadrightarrow \{\partial D\})$ constructed from g by exchanging one arrow starting from K_i with an arrow $(x_0 \to K_i)$ satisfies $\sigma(\tilde{g}) \leq \sigma(g)$ and K_i is the last compactum of $\tilde{g}$. If $V_D(x_0, K_i) > 0$ for all $K_i \in \mathbf{K}_0$, Lemma 2.3 implies $V_D(x_0, \partial D) = 0$. Since one obtains a $\{\partial D\}$-graph $\tilde{g} \in G^{\mathbf{K}_0^*}\{\partial D\}$, which satisfies $\sigma(\tilde{g}) = \sigma(g)$, by adding an arrow $(x_0 \to \partial D)$ to g, one has $W_D \leq M_D(x_0)$ from Theorem 3.1. Combining this with Lemma 5.3 in [FW, Chapter 6], we conclude $W_D - M_D(x_0) = 0$, where we remark $x_0 \notin \tilde{\Omega}$.

Next, we suppose $K^* \in \mathbf{K}_s$ and claim

$$W_D - M_D(x_0) \leq U_{K^*}(\partial D). \tag{4.7}$$

For $\delta > 0$, F_1 denotes the connected component of $\{x \in \bar{D}\,; U(x) \leq U(K^*) + U_{K^*}(\partial D) + \delta\}$ containing K^*. Set

$$F_1^{(1)} = \{x \in F_1\,; \text{there exists } K_i \in \mathbf{K}_s^* \text{ so that } K_i \xrightarrow{g} \partial D \text{ and } V_D(x, K_i) = 0\},$$
$$F_1^{(2)} = \{x \in F_1\,; \text{there exists } K_i \in \mathbf{K}_s^* \text{ so that } K_i \xrightarrow{g} K^* \text{ and } V_D(x, K_i) = 0\}.$$

Since $F_1^{(1)}$ and $F_1^{(2)}$ are non-empty closed subsets of F_1 and satisfy $F_1^{(1)} \cup F_1^{(2)} = F_1$ from Lemma 2.3, one can find $x_1 \in F_1^{(1)} \cap F_1^{(2)}$ and $K_{i_0}, K_{i_1} \in \mathbf{K}_s^*$ such that $K_{i_0} \xrightarrow{g} \partial D$, $K_{i_0} \xrightarrow{g} K^*$ and that $V_D(x_1, K_{i_0}) = V_D(x_1, K_{i_1}) = 0$. Then,

$$V_D(K_{i_1}, K_{i_0}) \leq U(K^*) + U_{K^*}(\partial D) + \delta - U(K_{i_1})$$

by the same methods as (4.3). For $(K_{i_1} \to K_{i_2}), (K_{i_2} \to K_{i_3}), \dots, (K_{i_n} \to K^*) \in g$, $g_0 \in G^{\mathbf{K}_0^*}\{\partial D\}$ denotes a $\{\partial D\}$-graph constructed from g by deleting these n arrows and adding $n+1$ arrows $(K^* \to K_{i_n}), \dots, (K_{i_2} \to K_{i_1}), (K_{i_1} \to K_{i_0})$. From Lemma 2.7, we get

$$\begin{aligned} \sigma(g_0) - \sigma(g) & \\ &= \sum_{k=1}^{n} \{V_D(K_{i_{k+1}}, K_{i_k}) - V_D(K_{i_k}, K_{i_{k+1}})\} + V_D(K_{i_1}, K_{i_0}) \\ &\leq U_{K^*}(\partial D) + \delta, \end{aligned} \tag{4.8}$$

where $K_{i_{n+1}} = K^*$. Hence, (3.5) in Theorem 3.1 verifies (4.7) by letting $\delta \downarrow 0$.

Finally, we show the assertions in Theorem 4.1. Since $U_{K^*}(\partial D) \leq V_0$, (4.1) is obtained immediately by combining (4.7) with Proposition 4.3. We move to the second assertion (4.2). To do this, let $x_0 \notin \tilde{\Omega}$ and suppose $V_D(x_0, K_j) = 0$ for some $K_j \in \mathbf{K}_s^*$ since the other cases have been shown. Furthermore, from (4.7) one can suppose $U_{K^*}(\partial D) = V_0$ and, in particular, $K^* \subset \bar{\Omega}$. It suffices to construct $\tilde{g} \in G^{\mathbf{K}_0^*}(K^* \nrightarrow \{\partial D\})$ satisfying

$$\sigma(\tilde{g}) < \sigma(g). \tag{4.9}$$

Indeed, for $\tilde{g} \in G^{\mathbf{K}_0^*}(K^* \nrightarrow \{\partial D\})$, there is a $\{\partial D\}$-graph $\tilde{g}_0 \in G^{\mathbf{K}_0^*}\{\partial D\}$ such that $\sigma(\tilde{g}_0) - \sigma(\tilde{g}) \leq V_0$ by using a parallel argument to the proof of (4.8). Together with (4.9), we get $\sigma(\tilde{g}_0) - \sigma(g) < V_0$ and therefore (4.2) from Theorem 3.1. For a sequence of arrows $(x_0 \to K_{j_1}), (K_{j_1} \to K_{j_2}), \dots, (K_{j_m} \to K^*) \in g$, set

$$\mathbf{m} = \min\{p \geq 0\,; K_{j_{p+1}} \subset \tilde{\Omega}\},$$

where $K_{j_0} = \{x_0\}$ and $K_{j_{m+1}} = K^*$. If $\mathbf{m} = 0$, we know $V_D(x_0, K_{j_1}) > 0$ from $x_0 \notin \tilde{\Omega}$ and $K_{j_1} \subset \tilde{\Omega}$, and $V_D(x_0, K_j) = 0$ for some $K_j \in \mathbf{K}_s^*$. Then, the graph $\tilde{g} \in$

$G^{\mathbf{K}_\delta^*}(K^* \not\to \{\partial D\})$ derived from g by exchanging an arrow $(x_0 \to K_{j_1})$ with $(x_0 \to K_j)$ satisfies (4.9). Let $\mathbf{m} \geq 1$. Note that from Lemma 2.2 one has

$$\begin{aligned} V_D(K_{j_\mathrm{m}}, K_{j_{\mathrm{m}+1}}) &= \inf_{y \in \partial \mathscr{V}(K_{j_\mathrm{m}})} \{V_D(K_{j_\mathrm{m}}, y) + V_D(y, K_{j_{\mathrm{m}+1}})\} \\ &> U_{K_{j_\mathrm{m}}}(\partial D), \end{aligned}$$

since Lemma 2.5 and $\overline{\mathscr{V}(K_{j_\mathrm{m}})} \cap \tilde{\Omega} = \emptyset$ imply $\inf_{y \in \partial \mathscr{V}(K_{j_\mathrm{m}})} V_D(K_{j_\mathrm{m}}, y) \geq U_{K_{j_\mathrm{m}}}(\partial D)$ and $\inf_{y \in \partial \mathscr{V}(K_{j_\mathrm{m}})} V_D(y, K_{j_{\mathrm{m}+1}}) > 0$ respectively. For $0 < \delta' < V_D(K_{j_\mathrm{m}}, K_{j_{\mathrm{m}+1}}) - U_{K_{j_\mathrm{m}}}(\partial D)$, we denote by F_2 the connected component of $\{x \in \bar{D}\,; U(x) \leq U(K_{j_\mathrm{m}}) + U_{K_{j_\mathrm{m}}}(\partial D) + \delta'\}$ which contains K_{j_m} and set

$$F_2^{(1)} = \{x \in F_2\,; \text{there exists } K_i \in \mathbf{K}_\mathrm{s}^* \text{ so that } K_i \overset{g}{\not\to} K_{j_\mathrm{m}} \text{ and } V_D(x, K_i) = 0\},$$

$$F_2^{(2)} = \{x \in F_2\,; \text{there exists } K_i \in \mathbf{K}_\mathrm{s}^* \text{ so that } K_i \overset{g}{\to} K_{j_\mathrm{m}} \text{ and } V_D(x, K_i) = 0\}.$$

Since $F_2^{(1)}$ and $F_2^{(2)}$ are non-empty closed subsets of F_2 and satisfy $F_2^{(1)} \cup F_2^{(2)} = F_2$ from Lemma 2.3, one can find $x_2 \in F_2^{(1)} \cap F_2^{(2)}$ and $K_{i_0'}, K_{i_1'} \in \mathbf{K}_\mathrm{s}^*$ so that $K_{i_0'} \overset{g}{\not\to} K_{j_\mathrm{m}}$, $K_{i_1'} \overset{g}{\to} K_{j_\mathrm{m}}$ and that $V_D(x_2, K_{i_0'}) = V_D(x_2, K_{i_1'}) = 0$. Then, we get

$$V_D(K_{i_1'}, K_{i_0'}) \leq U(K_{j_\mathrm{m}}) + U_{K_{j_\mathrm{m}}}(\partial D) + \delta' - U(K_{i_1'})$$

in a similar manner to (4.3). For $(K_{i_1'} \to K_{i_2'}), (K_{i_2'} \to K_{i_3'}), \ldots, (K_{i_{n'}'} \to K_{j_\mathrm{m}}) \in g$, $\tilde{g}$ denotes a graph obtained from g by replacing these $n' + 1$ arrows with $n' + 1$ arrows $(K_{j_\mathrm{m}} \to K_{i_{n'}'}), \ldots, (K_{i_2'} \to K_{i_1'}), (K_{i_1'} \to K_{i_0'})$. By Lemma 2.7, one can easily prove that $\tilde{g} \in G^{\mathbf{K}_\delta^*}(K^* \not\to \{\partial D\})$ satisfies (4.9). □

5. Further results

One can easily know that $W_D - M_D(x)$, $x \in D \setminus \Omega$, is not determined generally by $U_{\{x\}}(\partial D)$, $U_{\{x\}}(K_i)$, Depth $\mathscr{V}(K_i)$, etc., even if x belongs to some valley $\mathscr{V} \not\subset \Omega$. However, for the bottom valley $\mathscr{V}(K_i)$, one can show that $W_D - M_D(x)$ coincides with the depth of valley $\mathscr{V}(K_i)$ for $x \in \mathscr{V}(K_i)$ in a similar manner to Theorem 1. Namely, we have the next proposition.

PROPOSITION 5.1. *Let a stable compactum K_i satisfy $U(K_i) = \min\{U(K_j)\,; K_j \in \mathbf{K}_\mathrm{s}\}$. Then,*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log E_x[\tau^\varepsilon] = \text{Depth } \mathscr{V}(K_i)$$

for all $x \in \mathscr{V}(K_i)$.

Proof. From Theorem 3.2 and Lemma 4.2, it is sufficient to show $W_D - M_D(K_i) = \mathrm{Depth}\,\mathcal{V}(K_i)$. However, we have $W_D - M_D(K_i) \geq \mathrm{Depth}\,\mathcal{V}(K_i)$ by using the same arguments as Proposition 4.3. Let $g \in G^{\mathbf{K}_s \cup \{\partial D\}}(K_i \twoheadrightarrow \{\partial D\})$ attain the minimum of $M_D(K_i)$ in the RHS of (3.7). We denote the end compactum of g by $K^* \in \mathbf{K}_s$. Then, one can easily show that $U_{K_i}(K^*) \leq U_{K_i}(\partial D)$ in a similar manner to Lemma 4.2. In particular, $U_{K^*}(\partial D) \leq \mathrm{Depth}\,\mathcal{V}(K_i)$. Hence, the proof is completed immediately from (4.7). □

For one-dimensional Euclidean space, one can calculate $W_D - M_D(x)$ for all $x \in D$. Set $D = (d_1, d_2)$ and define $U^*(x)$, $x \in D$, by

$$U^*(x) = \begin{cases} \max_{x \in \overline{\mathcal{V}(K_i)}} U(x), & x \in \mathcal{V}(K_i), \\ U(x), & x \notin \cup_{i=1}^{l} \mathcal{V}(K_i). \end{cases}$$

Recall Ω defined in Section 1. Let $\mathcal{V}_k^+$ be the deepest valley at the right side of $\mathcal{V}_{k-1}^+$ satisfying $\min_{x \in \mathcal{V}_k^+} U(x) < \min_{x \in \mathcal{V}_{k-1}^+} U(x)$ and let $\mathcal{V}_k^-$ be that at the left side of $\mathcal{V}_{k-1}^-$ satisfying $\min_{x \in \mathcal{V}_k^-} U(x) < \min_{x \in \mathcal{V}_{k-1}^-} U(x)$ for $k = 1,2,\ldots$, where we write $\mathcal{V}_0^+ = \mathcal{V}_0^- = \Omega$ simply. Set $V_k^+ = \mathrm{Depth}\,\mathcal{V}_k^+$, $V_k^- = \mathrm{Depth}\,\mathcal{V}_k^-$ and $\sigma_k^+ = \sup\{x\,; x \in \mathcal{V}_k^+\}$, $\sigma_k^- = \inf\{x\,; x \in \mathcal{V}_k^-\}$ for $k = 0,1,\ldots$. We remark that they satisfy the following:

$$\begin{gathered} d_1 \leq \sigma_{k_1}^- < \sigma_{k_1-1}^- < \cdots < \sigma_0^- < \sigma_0^+ < \cdots < \sigma_{k_2-1}^+ < \sigma_{k_2}^+ \leq d_2, \\ V_{k_1}^- \leq \cdots \leq V_1^- < V_0,\ V_0 > V_1^+ \geq \cdots \geq V_{k_2}^+, \end{gathered}$$

where k_1, $k_2 \geq 0$ are chosen so that, respectively,

$$\min_{x \in \mathcal{V}_{k_1}^-} U(x) = \min_{x \in \cup_{K_i \in \mathbf{K}_s} K_i \cap (d_1, \sigma_0^-)} U(x), \quad \min_{x \in \mathcal{V}_{k_2}^+} U(x) = \min_{x \in \cup_{K_i \in \mathbf{K}_s} K_i \cap (\sigma_0^+, d_2)} U(x).$$

With these notations in mind, we define $V^*(x)$, $x \in D$, by

$$V^*(x) = \begin{cases} [V_{k_1}^- + U^*(x) - U(\sigma_{k_1}^-)] \vee 0, & x \in (d_1, \sigma_{k_1}^-], \\ [V_{k-1}^- + U^*(x) - U(\sigma_{k-1}^-)] \vee V_k^-, & x \in (\sigma_k^-, \sigma_{k-1}^-],\ k = 1,\ldots, k_1, \\ V_0 & x \in (\sigma_0^-, \sigma_0^+), \\ [V_{k-1}^+ + U^*(x) - U(\sigma_{k-1}^+)] \vee V_k^+, & x \in [\sigma_{k-1}^+, \sigma_k^+),\ k = 1,\ldots, k_2, \\ [V_{k_2}^+ + U^*(x) - U(\sigma_{k_2}^+)] \vee 0, & x \in [\sigma_{k_2}^+, d_2). \end{cases}$$

Now we state the next proposition without proof since it is quite simple.

PROPOSITION 5.2. *We have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log E_x[\tau^\varepsilon] = V^*(x)$$

uniformly in x belonging to every compact subset of D.

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