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THE STRUCTURE OF POINTWISE RECURRENT MAPS HAVING THE PSEUDO ORBIT TRACING PROPERTY

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Abstract. We show that a continuous map of a metric space is pointwise recurrent and has the pseudo orbit tracing property if and only if the map is uniformly conjugate to an adding-machine-like map restricted to some invariant subset.

§1. Introduction

The pseudo orbit tracing property (abbrev. POTP) comes from the study of Anosov diffeomorphisms: POTP and expansiveness imply the stability of the system, see [AH]. POTP has been studied by many authors. For a manifold M, this property is generic in the space of all homeomorphisms of M with C^0-topology [PP]. For a closed interval I, in [K] the author determines all zero entropy continuous maps of I having POTP. For a positive entropy continuous map of I the situation is more complicated. In fact, for the family of tent maps, almost all maps have POTP and the set of parameters for which the maps do not have POTP is locally uncountable [CKY].

Recurrence is one of the most important subjects in the study of dynamical systems. It is known that a recurrent homeomorphism of the plane or of a compact surface with negative Euler characteristic is periodic ([OT] and [KP]), i.e. some iterates of the homeomorphims are identity. Moreover, a transitive, non-minimal recurrent homeomorphism of a compact metric space does exist [KW].

In this note we shall discuss the structure of pointwise recurrent maps with POTP. In [M], the first author of this note studied some properties of pointwise recurrent maps having POTP. In particular, he proved that

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such maps should be pointwise minimal. Here we completely determine the structure of pointwise recurrent maps having POTP using inverse limit description. To be more precise, we introduce the following notion.

Assume that X is a metric space with metric d and $f : X \to X$ is a continuous map. For $x \in X$, the *orbit* of x, denoted by $O(x, f)$, is the set $\{x, f(x), f^2(x), \dots\}$. $x \in X$ is a *recurrent point* of f, if for each $\epsilon > 0$, there is $n = n(\epsilon) \in \mathbb{N}$ such that $d(f^n(x), x) < \epsilon$. The set of periodic points and recurrent points of f are denoted by $P(f)$ and $R(f)$ respectively. f is said to be *pointwise recurrent* (resp. *pointwise periodic*) if $R(f) = X$ (resp. $P(f) = X$) and f is *recurrent* if there is $n_i \to +\infty$ such that $\sup_{x\in X} d(f^{n_i}(x), x) \to 0$ (in some papers it is called *uniformly rigid*). For a subset X_0 of X, if $f(X_0) \subset X_0$, then we say that X_0 is *invariant* (under f) and $f|X_0 : X_0 \to X_0$ is a *subsystem*. If X_0 is a non-empty, closed, invariant subset of X and each non-empty closed proper subset of X_0 is not invariant, then X_0 is called a *minimal set* of f. If X is a minimal set of f, then f is called a minimal map. It is clear that a minimal map on a space with finitely many points is a cyclic permutation.

Suppose that $(x_n)_{n=0}^{b}$ is a sequence in X with $b \le +\infty$ and $\delta > 0$. If for each $0 \le n < b$ we have $d(x_{n+1}, f(x_n)) < \delta$, then $(x_n)_{n=0}^{b}$ is said to be a δ-pseudo orbit of length b (from x_0 to x_b if b is finite). For a given $\epsilon > 0$, if there is $y \in X$ such that $d(x_0, y) \le \epsilon$ and $d(x_{n+1}, f^{n+1}(y)) \le \epsilon$ for $0 \le n < b$, then we say that the δ-pseudo orbit $(x_n)_{n=0}^{b}$ is ϵ-traced by y or the orbit of y. (X, f) is said to have the *pseudo orbit tracing property* (abbrev. POTP) if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that each δ-pseudo orbit of infinite length can be ϵ-traced by some point in X.

Given $K = (k_1, k_2, \dots)$ with $k_i \ge 1$, we define $\Sigma_K = \prod_{i=1}^{\infty}\{0, \dots, k_i - 1\}$, where $\{0, \dots, k_i - 1\}$ and Σ_K are equipped with the discrete and the product topology respectively. If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are two elements of Σ_K then their sum $x \oplus y = (z_1, z_2, \dots)$ is defined as follows. If $x_1 + y_1 < k_1$, then $z_1 = x_1 + y_1$; if $x_1 + y_1 \ge k_1$, then $z_1 = x_1 + y_1 - k_1$ and we carry 1 to the next position. The other terms $z_2, \dots$ are successively determined in the same fashion. Let $f_K : \Sigma_K \to \Sigma_K$ be defined by $f_K(z) = z \oplus 1$ for each $z \in \Sigma_K$, where $1 = (1, 0, 0, \dots)$. It is known that f_K is a minimal map, which is called an *adding machine* (abbrev. AM). We note that if $\{i \in \mathbb{N} : k_i > 1\}$ is finite, then f_K is periodic and Σ_K is the unique periodic orbit of f_K.

Assume that $\psi_i : Y_i \to Y_i$ is continuous, Y_i is a metric space, $i = 1, 2$.

ψ_1 is said to be *conjugate* to ψ_2 if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h\psi_1 = \psi_2 h$. If both h and h^{-1} are uniformly continuous, then we say that ψ_1 is *uniformly conjugate* to ψ_2.

Let (X_i, d_i) be a metric space and $\phi_i : X_{i+1} \to X_i$ be continuous onto maps for $i \in \mathbb{N}$. Define $X_\infty = \operatorname{inv}\lim\{X_i, \phi_i\} = \{(x_1, x_2, \dots) : x_i \in X_i, \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$. X_∞ is called the *inverse limit space* of $\{X_i, \phi_i\}$ and ϕ_i is called the *bonding map*. As a subspace of the product space $\prod_{i=1}^\infty X_i$, X_∞ is a metric space with a compatible metric d_∞ defined by $d_\infty(x, y) = \sum_{i=1}^\infty d_i(x_i, y_i)/(2^i(1 + d_i(x_i, y_i)))$ for $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots) \in X_\infty$.

If $f_i : X_i \to X_i$ is continuous and satisfies that $\phi_i f_{i+1} = f_i \phi_i$ for $i \in \mathbb{N}$, then $\{f_i\}$ induces a map f_∞ on X_∞ defined by $f_\infty(x_1, x_2, \dots) = (f_1(x_1), f_2(x_2), \dots)$. We refer [N] for the basic properties of the inverse limit space and the induced map. If for each $i \in \mathbb{N}$, (X_i, d_i) is a *standard discrete metric space*, i.e. $d_i(x, y) = 1$ for all $x \neq y \in X_i$, and f_i is pointwise periodic, then f_∞ is called an *AM-like map*. Obviously, every AM is topologically conjugate to an AM-like map (in this case each f_i has only one periodic orbit). It is easy to see if W is a non-empty invariant subset of an AM-like map, then $f_\infty|W : W \to W$ has POTP.

Now we are ready to state the main result of the paper.

MAIN THEOREM. *Let X be a metric space and $f : X \to X$ be continuous. Then*
(1) *f is pointwise recurrent and has POTP if and only if f is uniformly conjugate to a subsystem of some AM-like map.*
(2) *If X is complete, then f is pointwise recurrent and has POTP if and only if f is uniformly conjugate to some AM-like map.*
(3) *f is a minimal map and has POTP if and only if f is uniformly conjugate to a subsystem of some AM map.*

In [AH] the authors show that minimal homeomorphisms of connected compact metric spaces having more than one point do not have POTP. Theorem A generalizes the result. By [A] distal homeomorphisms of connected compact metric spaces having more than one point do not have POTP. Theorem A also generalizes the result as distal homeomorphisms are pointwise recurrent. We remark that if we replace pointwise recurrent by pointwise non-wandering, the above theorem is false, the full shift serves as such an example.

§2. Some properties of pointwise recurrent maps having POTP

Assume that (X, d) is a metric space and $f : X \to X$ is continuous. A *partition* of X is a family of disjoint non-empty subsets of X whose union is all of X. Suppose ϵ and δ are positive numbers. A partition $\{X_\alpha : \alpha \in A\}$ (where A is an index set) is called a *δ-decomposition* of X if $d(X_\alpha, X_\beta) \geq \delta$ for all α and $\beta \in A$ with $\alpha \neq \beta$, and is said to be f-invariant if $f(X_\alpha) \subset X_\alpha$ for all $\alpha \in A$. An δ-decomposition $\{X_\alpha : \alpha \in A\}$ is called an *(ϵ, δ)-decomposition* if $\operatorname{diam}(X_\alpha) \leq \epsilon$ for all $\alpha \in A$. It is obvious that if $\{X_\alpha : \alpha \in A\}$ is an δ-decomposition, then each X_α is open and closed.

For $\delta > 0$ define a relation $\sim_{\delta,f}$ on X as follows: $x \sim_{\delta,f} y$ if and only if there exist $x_0, \ldots, x_k \in X$ such that $x = x_0$, $y = x_k$ and $d(O(x_{i-1}, f), O(x_i, f)) < \delta$ for each $i \in \mathbb{N}_k$, where $\mathbb{N}_k = \{1, \ldots, k\}$. It is easy to verify that $\sim_{\delta,f}$ is an equivalence relation on X. This relation induces a partition $\{X_{\alpha\delta} : \alpha \in A(\delta)\}$, where $A(\delta)$ is an index set depending on δ. It is easy to see

LEMMA 2.1. *The partition $\{X_{\alpha\delta} : \alpha \in A(\delta)\}$ induced by $\sim_{\delta,f}$ is an f-invariant δ-decomposition of X.*

In the sequel we assume that $f : X \to X$ is pointwise recurrent.

LEMMA 2.2. *For $x, y \in X$, $x \sim_{\delta,f} y$ if and only if there exists a δ-pseudo orbit of finite length from x to y.*

For $\alpha \in A(\delta)$ and $x, y \in X_{\alpha\delta}$, let

$$Z(x, y, \delta, f) = \{i \in \mathbb{Z}_+ : \text{there is a } \delta\text{-pseudo orbit of length } i \text{ from } x \text{ to } y\}. \tag{2.1}$$

For convenience, we regard (x) as a δ-pseudo orbit from x to x of length 0. Thus, $0 \in Z(x, y, \delta, f)$ if and only if $x = y$.

For two nonempty subsets M_1 and M_2 of $\mathbb{Z}$, denote $\{i + j : i \in M_1,\ j \in M_2\}$ and $\{-i : i \in M_1\}$ by $M_1 + M_2$ and $-M_1$ respectively. Then we have

$$Z(x, y, \delta, f) + Z(y, x, \delta, f) \subset Z(x, x, \delta, f) \cap Z(y, y, \delta, f), \tag{2.2}$$

and

$$Z(x, y, \delta, f) + Z(y, y, \delta, f) + Z(y, x, \delta, f) \subset Z(x, x, \delta, f). \tag{2.3}$$

From (2.2) and (2.3) we get

$$\begin{aligned} Z(y,y,\delta,f) &\subset (-Z(x,y,\delta,f)) + Z(x,y,\delta,f) + Z(y,y,\delta,f) \\ &\qquad + Z(y,x,\delta,f) + (-Z(y,x,\delta,f)) \\ &\subset Z(x,x,\delta,f) + (-Z(x,y,\delta,f)) + (-Z(y,x,\delta,f)) \\ &\subset Z(x,x,\delta,f) + (-Z(x,x,\delta,f)). \end{aligned} \tag{2.4}$$

For a subset $M \subset \mathbb{Z}$, let $\gcd(M)$ denote the greatest common divisor of numbers in M. It is easy to see that $\gcd(M + (-M)) = \gcd(M)$, and if $M' \subset M$ then $\gcd(M)$ is a divisor of $\gcd(M')$. Thus, it follows from (2.4) that $\gcd(Z(x,x,\delta,f))$ is a divisor of $\gcd(Z(y,y,\delta,f))$. By the same reasoning $\gcd(Z(y,y,\delta,f))$ is a divisor of $\gcd(Z(x,x,\delta,f))$. Thus, we have

$$\gcd(Z(x,x,\delta,f)) = \gcd(Z(y,y,\delta,f)), \quad \text{for all } x,y \in X_{\alpha\delta}. \tag{2.5}$$

Now fix $v \in X_{\alpha\delta}$ and set

$$n_{\alpha\delta} = n(\alpha,\delta) = n(\alpha,\delta,f) = \gcd(Z(v,v,\delta,f)). \tag{2.6}$$

Lemma 2.3. *For any $x,y \in X_{\alpha\delta}$ and $i,j \in Z(x,y,\delta,f)$, we have $n_{\alpha\delta}|(i-j)$.*

Proof. Take $b \in Z(y,x,\delta,f)$. By (2.2) we have $\{i+b, j+b\} \subset Z(x,x,\delta,f)$. By (2.5) and (2.6), it follows that $\{(i+b)/n_{\alpha\delta}, (j+b)/n_{\alpha\delta}\} \subset \mathbb{Z}$. Thus, $n_{\alpha\delta}|(i-j)$. ∎

For $n, j \in \mathbb{Z}$ write $n\mathbb{Z} + j = \{ni + j : i \in \mathbb{Z}\}$. Then by Lemma 2.3 we have

Lemma 2.4. *For $x,y \in X_{\alpha\delta}$ there is a unique $\mu(x,y) \in \{0,1,\dots,n_{\alpha\delta}-1\}$ such that $Z(x,y,\delta,f) \subset n_{\alpha\delta}\mathbb{Z} + \mu(x,y)$.*

It is easy to verify that

$$\begin{cases} \mu(f(x),y) \equiv \mu(x,y) - 1 \pmod{n_{\alpha\delta}}, \\ \mu(x,f(y)) \equiv \mu(x,y) + 1 \pmod{n_{\alpha\delta}}. \end{cases} \tag{2.7}$$

For $i = 0,1,\dots,n_{\alpha\delta}-1$ set

$$W_i = W_{i\alpha\delta} = \{x \in X_{\alpha\delta} : \mu(v,x) = i\}. \tag{2.8}$$

It is easy to see that $\{W_0,\dots,W_{n_{\alpha\delta}-1}\}$ is a partition of $X_{\alpha\delta}$. Moreover, from (2.7), we have

$$f(W_{n_{\alpha\delta}-1}) \subset W_0, \quad \text{and} \quad f(W_i) = W_{i+1} \text{ for } 0 \le i \le n_{\alpha\delta} - 2. \tag{2.9}$$

LEMMA 2.5. *There exists $b \in \mathbb{Z}_+$ such that $n_{\alpha\delta}\mathbb{Z} \cap [b, \infty) \subset Z(v, v, \delta, f)$.*

Proof. From (2.6) there exists $\{m_1, \dots, m_k\} \subset Z(v, v, \delta, f) \setminus \{0\}$ such that $\gcd(m_1, \dots, m_k) = n_{\alpha\delta}$. Set $m_0 = \min(Z(v, v, \delta, f) \setminus \{0\})$ and $\mu_i = m_i/n_{\alpha\delta}$ for $0 \leq i \leq k$. Then $\gcd(\mu_1, \dots, \mu_k) = 1$. Thus, there exists $\{\lambda_1, \dots, \lambda_k\} \subset \mathbb{Z}$ with $\sum_{i=1}^k \lambda_i \mu_i = 1$. This implies that $\sum_{i=1}^k \lambda_i m_i = n_{\alpha\delta}$. Let $b_1 = \sum_{i=1}^k |\lambda_i m_i|$ and $b_2 = \mu_0 b_1$. From (2.2) we know that for $j = 0, 1, \dots, \mu_0$,

$$b_2 - jn_{\alpha\delta} = \sum_{i=1}^k (\mu_0 |\lambda_i| - j\lambda_i) m_i \in Z(v, v, \delta, f). \tag{2.10}$$

Noting $m_0 = \mu_0 n_{\alpha\delta} \in Z(v, v, \delta, f)$, by (2.10) and (2.2) we have for each $j \in \mathbb{N}$, $b_2 + jn_{\alpha\delta} \in Z(v, v, \delta, f)$. Thus, $[b, \infty) \cap n_{\alpha\delta}\mathbb{Z} \subset Z(v, v, \delta, f)$ if we set $b = b_2 - m_0$. ∎

Assume that $L = (x_0, \dots, x_k)$ and $L_1 = (y_0, \dots, y_m)$ are two finite δ-pseudo orbits of f, and $L_2 = (z_0, z_1, \dots)$ is a δ-pseudo orbit of f. If $x_k = y_0$ denote $(x_0, \dots, x_k, y_1, \dots, y_m)$ by LL_1; if $x_k = z_0$, denote $(x_0, \dots, x_k, z_1, z_2, \dots)$ by LL_2, if $x_k = x_0$ and $k > 0$, denote $L \cdots L$ (j times) by L^j, denote $LL \cdots$ by L^∞.

LEMMA 2.6. *Suppose $x \in W_i$, $0 \leq i \leq n_{\alpha\delta} - 1$ and $y \in X_{\alpha\delta}$. If $d(x, y) < \delta$, then $y \in W_i$.*

Proof. Let L be a δ-pseudo orbit from v to x of length m. Then $m \equiv i \pmod{n_{\alpha\delta}}$. As x is a recurrent point of f, x is also a recurrent point of $f^{n_{\alpha\delta}}$. Thus, there exists $k \in \mathbb{N}$ such that $d(f^{kn_{\alpha\delta}}(x), x) < \delta - d(x, y)$. Set

$$L' = (L, f(x), f^2(x), \dots, f^{kn_{\alpha\delta}-1}(x), y).$$

Then L' is a δ-pseudo orbit from v to y of length $m + kn_{\alpha\delta}$. As $m + kn_{\alpha\delta} \equiv m \equiv i \pmod{n_{\alpha\delta}}$, $y \in W_i$.

A sequence $(W_0, \dots, W_k)$ of subsets of X is said to be *f-cyclic* if $f(W_k) \subset W_0$ and $f(W_{i-1}) \subset W_i$ for $i = 1, \dots, k$.

By Lemma 2.6 and (2.9) we get immediately

LEMMA 2.7. *$(W_{0\alpha\delta}, W_{1\alpha\delta}, \dots, W_{(n_{\alpha\delta}-1)\alpha\delta})$ is a f-cyclic δ-decomposition of $X_{\alpha\delta}$.*

From now on we assume that f is pointwise recurrent and has POTP. Then, there is a function $\eta : (0,\infty) \to (0,\infty)$ such that for each $\epsilon > 0$, each $\eta(\epsilon)$-pseudo orbit of f can be $\epsilon/4$-traced by some orbit of f and $\eta(\epsilon) \le \epsilon$. For any given $\epsilon > 0$, set $\delta = \eta(\epsilon)$.

LEMMA 2.8. *For each* $0 \le i \le n_{\alpha\delta} - 1$ *we have* $\operatorname{diam}(W_{i\alpha\delta}) \le \epsilon$.

Proof. Let $x, y \in W_{i\alpha\delta}$. Assume that L_1 is a δ-pseudo orbit from x to v, L_3 is a δ-pseudo orbit from v to y and L_4 is a δ-pseudo orbit from y to v. Suppose the length of L_j is c_j for $j = 1, 3, 4$. By (2.8) and (2.6), we get $c_3 \equiv i \pmod{n_{\alpha\delta}}$, $c_3 + c_4 \equiv 0 \pmod{n_{\alpha\delta}}$ and $c_1 + c_3 \equiv 0 \pmod{n_{\alpha\delta}}$. By Lemma 2.5, there is a δ-pseudo orbit L_2 from v to v of length c_2 such that $c_1 + c_2 + c_3 \equiv 0 \pmod{m_0}$, where $m_0 = \min(Z(v, v, \delta, f) \setminus \{0\})$ as above.

Let $L_5 = (v_0, \ldots, v_{m_0})$ be a δ-pseudo orbit from v to v of length m_0. Set $L = L_1 L_2 L_3 L_4 L_5^\infty$. Then L is a δ-pseudo orbit of f. As $\delta = \eta(\epsilon)$, L can be $\epsilon/4$-traced by some point $w \in X$. Let $w_j = f^j(w)$, $c = c_1 + c_2 + c_3 + c_4$ and $Y_j = O(w_{c+j}, f^{m_0})$, $(j \in \mathbb{Z}_+)$. As w is a recurrent point of f, we have

$$O(w, f) \subset \overline{O(w_c, f)} = \bigcup_{k=0}^{m_0 - 1} \overline{Y_k}, \tag{2.11}$$

and

$$f(Y_j) = Y_{j+1}, \quad \text{and} \quad \overline{Y_{j+m_0}} \subset \overline{Y_j}, \ j \in \mathbb{Z}_+. \tag{2.12}$$

From (2.11), there exists $q \in \{0, 1, \ldots, m_0 - 1\}$ such that $w \in \overline{Y_q}$. Let $a = c_1 + c_2 + c_3$. As $a \in m_0\mathbb{Z}$, by (2.12) we have $w_a \in \overline{Y_q}$. Since L_5^∞ is $\epsilon/4$-traced by $O(w_c, f)$, we have $\overline{Y_k} \subset \overline{B(v_k, \epsilon/4)}$ for $0 \le k < m_0$. As $L_1 L_2 L_3$ is $\epsilon/4$-traced by $(w, f(w), \ldots, f^a(w))$, we have $d(x, \overline{Y_q}) \le d(x, w) < \epsilon/4$ and $d(y, \overline{Y_q}) \le d(y, w_a) < \epsilon/4$. Thus,

$$d(x, y) < \epsilon/4 + \operatorname{diam}(\overline{Y_q}) + \epsilon/4 \le \epsilon$$

for any $x, y \in W_{i\alpha\delta}$. Hence, $\operatorname{diam}(W_{i\alpha\delta}) \le \epsilon$. ∎

By Lemmas 2.8 and 2.7 we obtain

LEMMA 2.9. $\{W_{i\alpha\delta} : 0 \le i \le n_{\alpha\delta} - 1\}$ *is an* (ϵ, δ)*-decomposition of* $X_{\alpha\delta}$.

Remark 2.10. By Lemmas 2.9 and 2.6, if $\operatorname{diam}(X_{\alpha\delta}) > \epsilon$ then $n_{\alpha\delta} > 1$, and if $\operatorname{diam}(X_{\alpha\delta}) < \delta$ then $n_{\alpha\delta} = 1$.

§3. Proof of Main Theorem

Proof of Main Theorem. The sufficiency of the theorem is obvious and it remains to show the necessity of the theorem. Let $\eta : (0,\infty) \to (0,\infty)$ be the function defined in Section 2. Set $\epsilon_0 = 1$. For $j = 0, 1, \dots$, let $\delta_j = \eta(\epsilon_j)$ and $\epsilon_{j+1} = \delta_j/2$. Set

$$S_j = \{W_{i\alpha\delta_j} : i = 0, 1, \dots, n_{\alpha\delta_j - 1}, \text{ and } \alpha \in A(\delta_j)\}.$$

Each $W_{i\alpha\delta_j}$ can be regarded as a "point" of S_j. Let ρ_j be the standard discrete metric on S_j. Then (S_j, ρ_j) is a standard discrete metric space. Define $\phi_j : (S_{j+1}, \rho_{j+1}) \to (S_j, \rho_j)$ such that

$$\begin{aligned} &\phi_j(W_{k\beta\delta_{j+1}}) = W_{i\alpha\delta_j} \text{ if and only if,} \\ &\text{as subsets of } X,\ W_{k\beta\delta_{j+1}} \subset W_{i\alpha\delta_j}. \end{aligned} \tag{3.1}$$

As $\epsilon_{j+1} < \delta_j$, by Lemma 2.9 each point of S_{j+1} (as a subset of X) is contained in a unique point of S_j. Thus, ϕ_j is well defined. It is easy to see that ϕ_j is surjective and continuous. Hence, we get an inverse limit space

$$(S_\infty, \rho_\infty) = \operatorname{inv}\lim\{(S_j, \rho_j), \phi_j\}. \tag{3.2}$$

Define $g_j : (S_j, \rho_j) \to (S_j, \rho_j)$ such that for each $\alpha \in A(\delta_j)$

$$g_j(W_{i\alpha\delta_j}) = \begin{cases} W_{0\alpha\delta_j}, & \text{if } i = n_{\alpha\delta_j} - 1, \\ W_{(i+1)\alpha\delta_j}, & \text{if } 0 \le i \le n_{\alpha\delta_j} - 2. \end{cases} \tag{3.3}$$

By (2.9),

$$g_j(W_{i\alpha\delta_j}) = W_{k\beta\delta_j} \text{ if and only if } f(W_{i\alpha\delta_j}) \subset W_{k\beta\delta_j}. \tag{3.4}$$

By (3.1) and (3.4) we have $g_j\phi_j = \phi_j g_{j+1}$. Thus, $(g_0, g_1, \dots)$ induces a map $g_\infty : (S_\infty, \rho_\infty) \to (S_\infty, \rho_\infty)$. By (3.3), g_j is pointwise periodic and hence g_∞ is an AM-like map.

Each $\sigma = (W_{i_0\alpha_0\delta_0}, W_{i_1\alpha_1\delta_1}, \dots) \in S_\infty$ is a sequence of subsets of X with $W_{i_0\alpha_0\delta_0} \supset W_{i_1\alpha_1\delta_1} \supset \cdots$. Set $X_\sigma = \bigcap_{j=0}^\infty W_{i_j\alpha_j\delta_j}$. As $\lim_{j\to\infty} \epsilon_j = 0$, by Lemma 2.8 X_σ contains at most one point. Since each $W_{i_j\alpha_j\delta_j}$ is a closed and open subset of X, X_σ contains exact one point if X is complete. If $X_\sigma \ne \emptyset$, set $X_\sigma = \{x_\sigma\}$.

As each S_j is a partition of X, for each $y \in X$, there is a unique $\sigma(y) \in S_\infty$ such that $y = x_{\sigma(y)}$. Thus, we may define $h : X \to S_\infty$ by

$h(y) = \sigma(y)$ for each $y \in X$. As $\lim_{j\to\infty} \epsilon_j = 0$, h is injective. Moreover, if X is complete, h is surjective.

Define $p_j : S_\infty \to S_0 \times S_1 \times \cdots \times S_j$ such that

$$p_j(W_{i_0\alpha_0\delta_0}, W_{i_1\alpha_1\delta_1}, W_{i_2\alpha_2\delta_2}, \dots) = (W_{i_0\alpha_0\delta_0}, \dots, W_{i_j\alpha_j\delta_j}).$$

By Lemma 2.9, for $x, y \in X$ with $d(x,y) < \delta_j$, $p_j h(x) = p_j h(y)$. Thus, h is uniformly continuous. At the same time, for $\sigma, \sigma' \in h(X)$, if $p_j(\sigma) = p_j(\sigma')$, by Lemma 2.8, $d(h^{-1}(\sigma), h^{-1}(\sigma')) \leq \epsilon_j$. Thus, $h^{-1}|h(X)$ is uniformly continuous. Hence $h : X \to h(X)$ is a uniform homeomorphism.

It is easy to see from (3.4) that $g_\infty h = hf$. Hence f is uniformly conjugate to $g_\infty|h(X)$. This proves (1) and (2) of Theorem A.

If f is minimal, for each $\delta > 0$, $A(\delta)$ is a singleton. Thus for $j \geq 0$, by Lemma 2.7, $g_j : S_j \to S_j$ is a cyclic permutation. This proves (3) of Theorem A. ∎

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