On the linear maps which are multiplicative on complex *-algebras

By

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1. Introduction

A Jordan *-homomorphism which satisfies the Cauchy-Schwarz inequality is *-homomorphic (E. Størmer [4], M. D. Choi [1] and T. W. Palmer [3]). In this paper, we shall give an elementary proof of this theorem under some weaker assumptions than theirs (Corollary 4). T. W. Palmer [3; Corollary 1] presents a characterization theorem of *-homomorphisms from U*-algebras. We shall give an elementary proof of this theorem (Corollary 6). Finally, we shall show that the linear functional on a Banach algebra which does not take the value 1 on the quasi-invertible elements is multiplicative.

2. Preliminaries

Let A be a *-algebra. We use the following notations:

$$A_H=\{h\in A: h^*=h \text{ (i. e. Hermitian element of } A)\}.$$

$$A_+=\left\{\sum_{j=1}^{n} a_j^* a_j: a_j\in A, n=1, 2, \dots\right\}.$$

For $h, k\in A_H$ we write $h\leqq k$ if $k-h\in A_+$.

$$A_{QI}=\{x\in A: \text{ quasi-invertible element}\}.$$

$$A_{QU}=\{u\in A: u^*u=uu^*=u+u^* \text{ (i. e. quasi-unitary element)}\}.$$

U*-algebra, introduced by T. W. Palmer, is a *-algebra which is the linear span of its quasi-unitary elements. Let A and B be *-algebras. A Jordan *-homomorphism ϕ of A into B is a linear map such that

$$\phi(xy+yx)=\phi(x)\phi(y)+\phi(y)\phi(x) \quad \text{and} \quad \phi(x^*)=\phi(x)^* \quad \text{for all } x, y\in A.$$

All algebras considered in this paper are those over the complex field $\boldsymbol{C}$.

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3. Results

Every Jordan homomorphism ϕ of an algebra A into another one satisfies the following equality:

$$(\phi(xy-yx))^2=(\phi(x)\phi(y)-\phi(y)\phi(x))^2 \qquad \text{for all} \quad x, y\in A.$$

(cf. N. Jacobson and C. E. Rickart [2]). This equality gives the following.

PROPOSITION 1. *Let A be a *-algebra and B be a *-algebra with $\{h\in B_H: h^2=0\}=\{0\}$. Suppose that either A or B is commutative. Then every Jordan *-homomorphism of A into B is automatically *-homomorphic.*

PROOF. Let ϕ be a Jordan *-homomorphism of A into B.

(i) The case where A is commutative. We have

$$(\phi(h)\phi(k)-\phi(k)\phi(h))^2=(\phi(hk-kh))^2=0$$

for $h, k\in A_H$. Hence

$$[i(\phi(h)\phi(k)-\phi(k)\phi(h))]^2=0,$$

and, it follows from the assumption for B that

$$i(\phi(h)\phi(k)-\phi(k)\phi(h))=0$$

i. e. $\qquad \phi(h)\phi(k)=\phi(k)\phi(h).$

It follows from this equality that

$$2\phi(h)\phi(k)=\phi(h)\phi(k)+\phi(k)\phi(h)=\phi(hk+kh)=2\phi(hk).$$

Therefore $\quad \phi(hk)=\phi(h)\phi(k).$

Since $A=A_H+iA_H$, $\phi(xy)=\phi(x)\phi(y)$ holds for every pair x, y of elements of A.

(ii) The case where B is commutative. We have

$$(\phi(hk-kh))^2=(\phi(h)\phi(k)-\phi(k)\phi(h))^2=0$$

for $h, k\in A_H$. So $(\phi(i(hk-kh)))^2=0$, and, from the assumption for B, it follows that

$$\phi(i(hk-kh))=0$$

i. e. $\qquad \phi(hk)=\phi(kh),$

and

$$2\phi(hk)=\phi(hk+kh)=\phi(h)\phi(k)+\phi(k)\phi(h)=2\phi(h)\phi(k).$$

Therefore $\phi(hk)=\phi(h)\phi(k)$. This shows that ϕ is a *-homomorphism.

REMARK 2. Corollary 1 in [5] follows easily from the equality $(\phi(xy-yx))^2=(\phi(x)\phi(y)-\phi(y)\phi(x))^2$ for a Jordan homomorphism ϕ.

PROPOSITION 3. *Let A and B be *-algebras, and ϕ be a linear map of A into B. Then ϕ is a homomorphism iff $\phi(x^*x)=\phi(x^*)\phi(x)$ for all $x\in A$.*

PROOF. If $\phi(x^*x)=\phi(x^*)\phi(x)$ for all $x\in A$, we have

$$\phi((x+y)^*(x+y))=\phi((x+y)^*)\phi(x+y) \quad \text{for all } x, y\in A.$$

Hence
$$\phi(x^*y)+\phi(y^*x)=\phi(x^*)\phi(y)+\phi(y^*)\phi(x).$$

Replacing y by iy and then multiplying by $-i$, we have

$$\phi(x^*y)-\phi(y^*x)=\phi(x^*)\phi(y)-\phi(y^*)\phi(x).$$

Thus $\phi(x^*y)=\phi(x^*)\phi(y)$. This shows that ϕ is a homomorphism. The convers is evident.

COROLLARY 4. *Let A be a *-algebra, B be a *-algebra with $B_+\cap(-B_+)=\{0\}$, and ϕ be a linear map of A into B. Then ϕ is a *-homomorphism iff ϕ is a Jordan *-homomorphism and satisfies the Cauchy-Schwarz inequality $\phi(x^*x)\geqq\phi(x^*)\phi(x)$ for all $x\in A$.*

PROOF. Let ϕ be a Jordan *-homomorphism and let ϕ satisfy the Cauchy-Schwarz inequality. Then we have

$$\phi(x^*x+xx^*)=\phi(x^*)\phi(x)+\phi(x)\phi(x^*),$$

that is,
$$\phi(x^*x)-\phi(x^*)\phi(x)=-(\phi(xx^*)-\phi(x)\phi(x^*)) \quad \text{for } x\in A.$$

The left hand side of this equality belongs to B_+ and the right hand side belongs to $-B_+$. Hence $\phi(x^*x)=\phi(x^*)\phi(x)$ holds by the assumption for B. Therefore it follows from Proposition 3 that ϕ is a *-homomorphism.

PROPOSITION 5. *Let A be a *-algebra, B be a *-algebra with $B_+\cap(-B_+)=\{0\}$ and ϕ be a linear *-map of A into B. Suppose that there is a subset S of A such that*

(i) *A is the linear span of S,*

(ii) *S is self-adjoint, i. e. $S=S^*$.*

*Then ϕ is a *-homomorphism iff $\phi(x^*x)=\phi(x^*)\phi(x)$ holds for $x\in S$ and the Cauchy-Schwarz inequality $\phi(x^*x)\geqq\phi(x^*)\phi(x)$ holds for $x\in A\backslash S$.*

PROOF. Let $\phi(x^*x)=\phi(x^*)\phi(x)$ hold for $x\in S$ and let $\phi(x^*x)\geqq\phi(x^*)\phi(x)$ hold for $x\in A\backslash S$. Then $\phi(x^*x)=\phi(x^*)\phi(x)$ holds for $x\in \boldsymbol{C}\cdot S=\{\alpha x: \alpha\in \boldsymbol{C}, x\in S\}$.
We have for $x, y\in \boldsymbol{C}\cdot S$,

$$\phi((x+y)^*)\phi(x+y)\leqq\phi((x+y)^*(x+y)),$$

$$\phi((x-y)^*)\phi(x-y)\leqq\phi((x-y)^*(x-y)).$$

It follows from these inequalities that

$$\phi(x^*)\phi(y)+\phi(y^*)\phi(x)\leqq\phi(x^*y)+\phi(y^*x),$$

$$-(\phi(x^*)\phi(y)+\phi(y^*)\phi(x))\leqq-(\phi(x^*y)+\phi(y^*x)).$$

That is

$$(\phi(x^*y)+\phi(y^*x))-(\phi(x^*)\phi(y)+\phi(y^*)\phi(x))\in B_+,$$

$$(\phi(x^*)\phi(y)+\phi(y^*)\phi(x))-(\phi(x^*y)+\phi(y^*x))\in B_+.$$

Therefore the assumption for B induces

$$\phi(x^*y)+\phi(y^*x)=\phi(x^*)\phi(y)+\phi(y^*)\phi(x). \qquad \cdots\cdots(1)$$

For $x,\ iy\in \boldsymbol{C}\cdot S$ this equality implies

$$i\phi(x^*y)-i\phi(y^*x)=i\phi(x^*)\phi(y)-i\phi(y^*)\phi(x),$$

i. e.

$$\phi(x^*y)-\phi(y^*x)=\phi(x^*)\phi(y)-\phi(y^*)\phi(x). \qquad \cdots\cdots(2)$$

From (1) and (2), $\phi(x^*y)=\phi(x^*)\phi(y)$. Hence it follows from (ii) that $\phi(xy)=\phi(x)\phi(y)$ for all $x,\ y\in S$. Therefore it follows from (i) that $\phi(xy)=\phi(x)\phi(y)$ for all $x,\ y\in A$. The converse is evident.

Corollary 6. *Let A be a U^*-algebra and B be a $*$-algebra with $B_+\cap(-B_+)=\{0\}$. Let ϕ be a linear map of A into B. Then ϕ is a $*$-homomorphism iff $\phi(A_{qU})\subset B_{qU}$ and the Cauchy-Schwarz inequality $\phi(x^*x)\geqq\phi(x^*)\phi(x)$ holds for all $x\in A$.*

Proof. Let $\phi(A_{qU})\subset B_{qU}$ and $\phi(x^*x)\geqq\phi(x^*)\phi(x)$ hold for all $x\in A$. Then, we know, making use of the argument in the proof of [3; Corollary 1], that ϕ is a linear $*$-map. For any $u\in A_{qU}$

$$\phi(u^*u)=\phi(u^*+u)=\phi(u^*)+\phi(u)=\phi(u)^*+\phi(u)=\phi(u^*)\phi(u).$$

Therefore it follows from Proposition 5 that ϕ is a $*$-homomorphism. The converse is evident.

W. Żelazko [5] gives a characterization of the multiplicative linear functionals on complex Banach algebras. Making use of this characteization we have the following.

Proposition 7. *Let A be a Banach algebra and f be a linear functional on A. Then f is multiplicative iff $f(A_{qU})\subset \boldsymbol{C}\setminus\{1\}$.*

Proof. If f is multiplicative, then

$$f(x)+f(y)=f(x)f(y)$$

for $x\in A_{qI}$ and its quasi-inverse y. So $f(x)\neq 1$,

i. e. $\quad f(A_{qI})\subset \boldsymbol{C}\setminus\{1\}$.

Conversely assume $f(A_{qI})\subset \boldsymbol{C}\setminus\{1\}$. Whether A has a unit element or not, we make the unitization $A_1=A+\boldsymbol{C}$ and extend f onto A_1 by putting $f(1)=1$. If (x,α) is an invertible element of A_1, then there exists an element y of A such that

$$(x,\alpha)\left(y,\frac{1}{\alpha}\right)=(0,\ 1),$$

i. e.
$$\left(-\frac{1}{\alpha}x\right)+(-\alpha y)-\left(-\frac{1}{\alpha}x\right)(-\alpha y)=0.$$

So
$$-\frac{1}{\alpha}x\in A_{qI}\quad\text{and}\quad f\left(-\frac{1}{\alpha}x\right)\neq 1,$$

from which $f((x,\alpha))\neq 0$. Therefore $f((x,\alpha))(0,1)-(x,\alpha)$ is a singular element for any $(x,\alpha)\in A_1$. Hence $f((x,\alpha))$ belongs to the spectrum of (x,α). It is now clear from [5; Theorem 2] that f is multiplicative on A.

PROPOSITION 8. *Let A be a Banach algebra, B be a commutative semi-simple Banach algebra and ϕ be a linear map of A into B. Then ϕ is multiplicative iff $\phi(A_{qI})\subset B_{qI}$.*

PROOF. Denote the set of all multiplicative linear functionals on A (or on B) by $M(A)$ (or by $M(B)$).
Let $\phi(A_{qI})\subset B_{qI}$. Then

$$(f\cdot\phi)(A_{qI})=f(\phi(A_{qI}))\subset f(B_{qI})\in C\backslash\{1\}\quad\text{for any }f\in M(B).$$

So it follows from Proposition 7 that $f\circ\phi\in M(A)$. Therefore

$$f(\phi(xy))=f(\phi(x))\,f(\phi(y))=f(\phi(x)\,\phi(y))$$

for $x, y\in S$ and $f\in M(B)$.
Now, the assumption for B implies that $\phi(x)\phi(y)=\phi(xy)$.

REMARK 9. Proposition 8 is not true if B fails to be semi-simple.

References

[1] M. D. CHOI, *A Schwarz inequality for positive linear maps on C*-algebras,* Illinois J. Math., 18 (1974), 565–574.

[2] N. JACOBRON, and C. E. RICKART, *Jordan homomorphisms of rings,* Trans. Amer. Math. Soc., 69 (1950), 479–502.

[3] T. W. PALMER, *Characterizations of *-homomorphisms and expectations,* Proc. Amer. Math. Soc., 46 (1974), 265–272.

[4] E. STØRMER, *On the Jordan Structure of C*-algebras,* Trans. Amer. Math. Soc., 120 (1965), 438–447.

[5] W. ŻELAZKO, *A characterization of multiplicative linear functionals in complex Banach algebras,* Studia Math., 30 (1968), 83–85.