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SECONDARY MULTIPLICATION IN TATE COHOMOLOGY OF GENERALIZED QUATERNION GROUPS

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Abstract

Let k be a field, and let G be a finite group. By a theorem of D. Benson, H. Krause, and S. Schwede, there is a canonical element in the Hochschild cohomology of the Tate cohomology $\gamma_G \in HH^{3,-1}\hat{H}^*(G)$ with the following property: Given any graded $\hat{H}^*(G)$-module X, the image of γ_G in $\mathrm{Ext}^{3,-1}_{\hat{H}^*(G)}(X,X)$ is zero if and only if X is isomorphic to a direct summand of $\hat{H}^*(G,M)$ for some kG-module M. In particular, if $\gamma_G = 0$ then every module is a direct summand of a realizable $\hat{H}^*(G)$-module.

We prove that the converse of that last statement is not true by studying in detail the case of generalized quaternion groups. Suppose that k is a field of characteristic 2 and G is generalized quaternion of order 2^n with $n \geqslant 3$. We show that γ_G is non-trivial for all n, but there is an $\hat{H}^*(G)$-module detecting this non-triviality if and only if $n = 3$.

1. Introduction

Let k be a field, G a finite group, and let $\hat{H}^*(G)$ denote the graded Tate cohomology algebra of G over k. The starting point of this paper is the following theorem of D. Benson, H. Krause, and S. Schwede:

Theorem 1.1. [2] *There exists a canonical element in Hochschild cohomology of $\hat{H}^*(G)$*

$$\gamma_G \in HH^{3,-1}\hat{H}^*(G),$$

such that for any graded $\hat{H}^*(G)$-module X, the following are equivalent:

(i) The image of γ_G in $\mathrm{Ext}^{3,-1}_{\hat{H}^*(G)}(X,X)$ is zero.

(ii) There exists a kG-module M such that X is a direct summand of the graded $\hat{H}^*(G)$-module $\hat{H}^*(G,M)$.

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Let us call an $\hat{H}^*(G)$-module *realizable* if it is isomorphic to a module of the form $\hat{H}^*(G, M)$ for some kG-module M. As an immediate consequence we get the following.

Corollary 1.2. *If $\gamma_G = 0$, then every $\hat{H}^*(G)$-module is a direct summand of a realizable module.*

At this point it is natural to ask for the converse of that statement. That is, given the fact that $\gamma_G \neq 0$, is there some $\hat{H}^*(G)$-module detecting the non-triviality of γ_G? Theorem 1.1 works more generally in the situation of differential graded algebras, and in that setup the converse of the corresponding corollary is known to the false: Benson, Krause, and Schwede provide an example of a dg algebra A such that the canonical class $\gamma_A \in HH^{3,-1}(H^*A)$ is non-trivial, but every H^*A-module is realizable (see [2, Proposition 5.16]). Nevertheless, the author believes that the question whether there is such an example coming from Tate cohomology of groups is still open.

In this paper we will compute γ_G explicitly for the generalized quaternion groups G. In what follows, let $t \geqslant 2$ be a power of 2, and let $G = Q_{4t}$ be the group of generalized quaternions

$$Q_{4t} = \langle g, h \mid g^t = h^2,\, ghg = h \rangle\,.$$

Let k be a field of characteristic 2, and denote by $L = kG$ the group algebra of G over k. Then the Tate cohomology ring $\hat{H}^*(G)$ is well known; it is given by

$$\hat{H}^*(Q_{4t}) = \widehat{\operatorname{Ext}}_L^*(k,k) \cong \begin{cases} k[x,y,s^{\pm 1}]/(x^2+y^2=xy, y^3=0) & \text{if } t=2, \\ k[x,y,s^{\pm 1}]/(x^2=xy, y^3=0) & \text{if } t \geqslant 4, \end{cases}$$

with degrees $|x| = |y| = 1, |s| = 4$ (see, e.g., [4, Chapter XII §11] and [1, IV Lemma 2.10]). Our main goal is to prove the following theorem.

Theorem 1.3. *The element $\gamma_{Q_8} \in HH^{3,-1}\hat{H}^*(Q_8)$ is non-trivial, and the cokernel of the map*

$$\hat{H}^*(Q_8)[-1] \oplus \hat{H}^*(Q_8)[-1] \xrightarrow{\left(\begin{smallmatrix} y & x+y \\ x & y \end{smallmatrix}\right)} \hat{H}^*(Q_8) \oplus \hat{H}^*(Q_8)$$

is a graded $\hat{H}^*(Q_8)$-module which is not a direct summand of a realizable one. For $t \geqslant 4$ the element $\gamma_{Q_{4t}} \in HH^{3,-1}\hat{H}^*(Q_{4t})$ is non-trivial, but every graded $\hat{H}^*(Q_{4t})$-module is a direct summand of a realizable one.

The plan is as follows: In the first section we will briefly recall the definitions needed in Theorem 1.1; most of this part is taken from [2], and the reader interested in details should consult that source. In the second section we turn to the computation of a Hochschild cocycle m representing the canonical class γ_G. In the third section we prove the statements about realizability of modules. Theorem 1.3 will then follow from Theorems 3.6, 3.8, 4.3, and Propositions 4.7 and 4.8.

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2. Prerequisites

2.1. Notation and conventions

All occurring modules will be right modules. We shall often work over a fixed ground field k; then $\otimes$ means tensor product over k. Whenever convenient, we write $(a_1, a_2, \ldots, a_n)$ instead of $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. If G is a group, then k is often considered as a trivial kG-module.

Let R be a ring with unit, and let M be a $\mathbb{Z}$-graded R-module. The degree of every (homogeneous) element $m \in M$ will be denoted by $|m|$. For every integer n the module $M[n]$ is defined by $M[n]^j = M^{n+j}$ for all j. Given two such modules M and L, a morphism $f\colon L \longrightarrow M$ is a family $f^j\colon L^j \longrightarrow M^j$ of R-module homomorphisms. The group of all these morphisms is denoted by $\mathrm{Hom}_R(L, M)$. Furthermore, we have $\mathrm{Hom}_R^m(L, M) = \mathrm{Hom}_R(L, M[m])$, the morphisms of degree m. The graded module $L \otimes M$ is given by $(L \otimes M)^m = \bigoplus_{i+j=m} L^i \otimes M^j$. If M is a differential graded R-module with differential d, then the differential of $M[n]$ is given by $(-1)^n d$.

2.2. Tate Cohomology

Let us recall briefly the definition and basic properties of Tate cohomology. Let k be a field, and let G be a finite group. Then $L = kG$ is a self-injective algebra (i.e., the classes of projective and injective right-L-modules coincide). For any L-module N we get a complete projective resolution P_* of N by splicing together a projective and an injective resolution of N:

$$\begin{tikzcd} \cdots & P_{-2} \arrow[l] & P_{-1} \arrow[l] & & P_0 \arrow[ll] \arrow[dl, two heads] & P_1 \arrow[l] & P_2 \arrow[l] & \cdots \arrow[l] \\ & & & N \arrow[ul, hook] & & & & \end{tikzcd}$$

Given another L-module M, we can apply the functor $\mathrm{Hom}_L(-, M)$ to P_*; then Tate cohomology is defined to be the cohomology groups of the resulting complex:

$$\widehat{\mathrm{Ext}}_L^n(N, M) = H^n(\mathrm{Hom}_L(P_*, M)) \quad \text{for all } n \in \mathbb{Z}.$$

For arbitrary L-modules X, Y, and Z, we have a cup product

$$\widehat{\mathrm{Ext}}_L^m(Y, Z) \otimes \widehat{\mathrm{Ext}}_L^n(X, Y) \longrightarrow \widehat{\mathrm{Ext}}_L^{m+n}(X, Z);$$

see, e.g., [3, §6]. Therefore, $\hat{H}^*(G) = \hat{H}^*(G, k) = \widehat{\mathrm{Ext}}_{kG}^*(k, k)$ is a graded algebra, and $\hat{H}^*(G, M) = \widehat{\mathrm{Ext}}_{kG}^*(k, M)$ is a graded $\hat{H}^*(G)$-module for every kG-module M. We call a graded $\hat{H}^*(G)$-module X *realizable* if it is isomorphic to $\hat{H}^*(G, M)$ for some kG-module M.

There is another way of describing the product of $\hat{H}^*(G)$, in terms of P_*. Consider the differential graded algebra $\mathcal{A} = \mathrm{Hom}_L^*(P_*, P_*)$, which (in degree n) is given by

$$\mathcal{A}^n = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_L(P_{j+n}, P_j),$$

and the differential $d\colon \mathcal{A}^n \longrightarrow \mathcal{A}^{n+1}$ is defined to be

$$(df)_j = \partial \circ f_{j+1} - (-1)^n f_j \circ \partial.$$

Here ∂ denotes the differential of P_*. $\mathcal{A}$ is called the endomorphism dga of P. With this definition, the cocycles of $\mathcal{A}$ (of degree n) are exactly the chain transformations

$P[n] \to P$, and two cocycles differ by a coboundary if and only if they are chain homotopic. Using standard arguments from homological algebra, one shows that the following map is an isomorphism of k-vector spaces:

$$\begin{aligned} H^n\mathcal{A} &\xrightarrow{\cong} \widehat{\operatorname{Ext}}^n_L(k,k), \\ [f] &\mapsto [\epsilon \circ f_0]. \end{aligned} \tag{1}$$

Here $\epsilon\colon P_0 \longrightarrow k$ is the augmentation. This isomorphism is compatible with the multiplicative structures. We will often write $\bar{a}$ for elements of the endomorphism dga; if $\bar{a}$ is a cocycle, then a denotes the corresponding cohomology class.

2.3. Hochschild Cohomology

We now give a short review of Hochschild cohomology. Let Λ be a graded algebra over the field k, and let M be a graded Λ-Λ-bimodule, the elements of k acting symmetrically. Define a cochain complex $C^{\bullet,*}(\Lambda, M)$ by

$$C^{n,m}(\Lambda, M) = \operatorname{Hom}^m_k(\Lambda^{\otimes n}, M),$$

with a differential δ of bidegree $(1,0)$ given by

$$\begin{aligned} (\delta\varphi)(\lambda_1,\dots,\lambda_{n+1}) &= (-1)^{m|\lambda_1|}\lambda_1\varphi(\lambda_2,\dots,\lambda_{n+1}) \\ &\quad + \sum_{i=1}^{n}(-1)^i\varphi(\lambda_1,\dots,\lambda_i\lambda_{i+1},\dots,\lambda_{n+1}) + (-1)^{n+1}\varphi(\lambda_1,\dots,\lambda_n)\lambda_{n+1}. \end{aligned}$$

The Hochschild cohomology groups $HH^{*,*}(\Lambda, M)$ are defined as the cohomology groups of that complex:

$$HH^{s,t}(\Lambda, M) = H^s(C^{*,t}(\Lambda, M)).$$

In particular, we can regard $M = \Lambda$ as a bimodule over itself; we will then write $HH^{s,t}(\Lambda) = HH^{s,t}(\Lambda,\Lambda)$. For example, an element of $HH^{3,-1}(\Lambda)$ is represented by a family of k-linear maps

$$m = \{m_{i,j,l}\colon \Lambda^i \otimes \Lambda^j \otimes \Lambda^l \longrightarrow \Lambda^{i+j+l-1}\}_{i,j,l\in\mathbb{Z}}$$

satisfying the cocycle relation

$$(-1)^{|a|}a\cdot m(b,c,d) - m(ab,c,d) + m(a,bc,d) - m(a,b,cd) + m(a,b,c)\cdot d = 0$$

for all $a,b,c,d \in \Lambda$.

Whenever X and Y are Λ-Λ-bimodules, one has a cup product pairing

$$\cup\colon \operatorname{Hom}_\Lambda(X,Y) \otimes HH^{*,*}\Lambda \longrightarrow \operatorname{Ext}^{*,*}_\Lambda(X,Y).$$

Here $\operatorname{Ext}^{s,t}_\Lambda(X,Y)$ is defined to be $\operatorname{Ext}^s_\Lambda(X,Y[t])$. In particular, we have the map

$$\begin{aligned} HH^{3,-1}\hat{H}^*(G) &\longrightarrow \operatorname{Ext}^{3,-1}_{\hat{H}^*(G)}(X,X) \\ \phi &\mapsto \operatorname{id}_X \cup \phi \end{aligned}$$

for every $\hat{H}^*(G)$-module X. This is the map occurring in Theorem 1.1.

2.4. The canonical element γ

We are now going to describe the construction of the element γ mentioned in Theorem 1.1. More generally, we will construct an element $\gamma_\mathcal{A} \in HH^{3,-1}H^*\mathcal{A}$ for every differential graded algebra $\mathcal{A}$ over k; then we can take $\mathcal{A}$ to be the endomorphism algebra of a complete projective resolution of k as a trivial kG-module to get $\gamma_G \in HH^{3,-1}\hat{H}^*(G)$.

For a dg-algebra $\mathcal{A}$, consider $H^*\mathcal{A}$ as a differential graded k-module with trivial differential. Then choose a morphism of dg-k-modules $f_1\colon H^*\mathcal{A} \longrightarrow \mathcal{A}$ of degree 0 which induces the identity in cohomology. This is the same as choosing a representative in $\mathcal{A}$ for every class in $H^*\mathcal{A}$ in a k-linear way. For every two elements $x, y \in H^*\mathcal{A}$, $f_1(xy) - f_1(x)f_1(y)$ is null-homotopic; therefore, we can choose a morphism of graded modules

$$f_2\colon H^*\mathcal{A} \otimes H^*\mathcal{A} \longrightarrow \mathcal{A}$$

of degree -1 such that for all $x, y \in H^*\mathcal{A}$, we have

$$df_2(x,y) = f_1(xy) - f_1(x)f_1(y).$$

Then for all $a, b, c \in H^*\mathcal{A}$,

$$f_2(a,b)f_1(c) - f_2(a,bc) + f_2(ab,c) - (-1)^{|a|}f_1(a)f_2(b,c)$$

is a cocycle in $\mathcal{A}$, the cohomology class of which will be denoted by $m(a,b,c)$. This defines a map $m\colon (H^*\mathcal{A})^{\otimes 3} \longrightarrow H^*\mathcal{A}$ of degree -1. An explicit computation shows that m is a Hochschild cocycle, thereby representing a class $\gamma_A \in HH^{3,-1}H^*\mathcal{A}$. This class is independent of the choices made.

3. Computation of the canonical element

From now on, let k be a field of characteristic 2. Let $t \geqslant 2$ be a power of 2, and let $G = Q_{4t}$ be the group of generalized quaternions

$$Q_{4t} = \left\langle g, h \mid g^t = h^2,\, ghg = h \right\rangle.$$

We denote by kG the group algebra of G over k, and $F = kG$ denotes the free module of rank 1 over that algebra. In this section, we are going to explicitly compute a Hochschild cochain m representing the canonical class γ_G.

3.1. The class of a map

We begin with an observation that will reduce the subsequent computations somewhat. Let us recall the construction of a representative of γ_G. First of all, we have to construct a projective resolution P, and we will actually find a minimal projective resolution. Then we have to choose a cycle selection homomorphism $f_1\colon \hat{H}^*(G) \to \operatorname{Hom}^*_{kG}(P,P)$ such that any class a is mapped to a representative $f_1(a)$. We can find a k-linear map $f_2\colon \hat{H}^*(G) \otimes \hat{H}^*(G) \to \operatorname{Hom}^*_{kG}(P,P)$ of degree -1 satisfying $df_2(a,b) = f_1(a)f_1(b) - f_1(ab)$ for all a, b. Finally, we are interested in terms of the form

$$f_2(a,b)f_1(c) + f_2(a,bc) + f_2(ab,c) + f_1(a)f_2(b,c); \tag{2}$$

this is a cocycle in $\operatorname{Hom}^*_{kG}(P,P)$. In order to determine the class of this cocycle, it is enough to know the degree 0 map of it (cf. (1)). This observation leads to the following definition.

Definition 3.1. For every $f \in \operatorname{Hom}^n_{kG}(P,P)$, i.e., a family of maps $f_j \colon P_{j+n} \to P_j$ ($j \in \mathbb{Z}$), not necessarily commuting with the differential, we denote by $\mathcal{C}(f)$ the class of the map $\epsilon \circ f_0 \colon P_n \to k$ in $H^n \operatorname{Hom}_{kG}(P_*, k) = \hat{H}^n(G)$.

Note that the complex $\operatorname{Hom}_{kG}(P_*, k)$ has trivial differential; thus, every element in $\operatorname{Hom}_{kG}(P_*, k)$ and in particular $\epsilon \circ f_0$ is a cocycle. The definition above gives a map

$$\boxed{\begin{aligned} \mathcal{C} \colon \operatorname{Hom}^n_{kG}(P,P) &\longrightarrow \hat{H}^n(G) \\ f &\mapsto [\epsilon \circ f_0]. \end{aligned}}$$

Proposition 3.2. *The map $\mathcal{C}$ has the following properties:*

(i) *If $f \in \operatorname{Hom}^n_{kG}(P,P)$ is a cocycle, then $\mathcal{C}(f)$ is the cohomology class of f; in particular, $\mathcal{C} \circ f_1 = \mathrm{id}$.*

(ii) *The map $\mathcal{C}$ is k-linear.*

(iii) *If $\mathcal{C}(f_1) = \mathcal{C}(f_2)$ for some $f_1, f_2 \in \operatorname{Hom}^n_{kG}(P,P)$, then $\mathcal{C}(f_1 g) = \mathcal{C}(f_2 g)$ for all $g \in \operatorname{Hom}^m_{kG}(P,P)$.*

(iv) *If $a \in \operatorname{Hom}^m_{kG}(P,P)$ is a cocycle and $f \in \operatorname{Hom}^n_{kG}(P,P)$ is an arbitrary element, then $\mathcal{C}(fa) = \mathcal{C}(f)\mathcal{C}(a)$.*

Proof. (i) follows from (1). (ii) holds by definition. (iii): If $\mathcal{C}(f_i) = 0$, then $\epsilon \circ f_i = 0$. This implies $\epsilon \circ f_i \circ g = 0$; hence $\mathcal{C}(f_i g) = 0$. For general f_1, f_2, note $\mathcal{C}(f_1 - f_2) = 0$; by what we just proved, $\mathcal{C}((f_1 - f_2)g) = 0$ and therefore $\mathcal{C}(f_1 g) = \mathcal{C}(f_2 g)$. (iv): Choose a cocycle $h \in \operatorname{Hom}^n_{kG}(P,P)$ satisfying $\mathcal{C}(h) = \mathcal{C}(f)$. Then by (iii)

$$\mathcal{C}(fa) = \mathcal{C}(ha) = \mathcal{C}(h)\mathcal{C}(a) = \mathcal{C}(f)\mathcal{C}(a). \qquad \square$$

The following corollary will simplify computations later on.

Proposition 3.3. *The map f_2 can be chosen in such a way that $\mathcal{C} \circ f_2 = 0$.*

Proof. Choose any $\tilde{f}_2$ (satisfying $d\tilde{f}_2(a,b) = f_1(a)f_1(b) - f_1(ab)$). Put $f_2 = \tilde{f}_2 - f_1 \circ \mathcal{C} \circ \tilde{f}_2$. Since $df_1 = 0$, we get

$$df_2(a,b) = d\tilde{f}_2(a,b) = f_1(a)f_1(b) - f_1(ab),$$

and from $\mathcal{C} \circ f_1 = \mathrm{id}$, it follows that

$$\mathcal{C} \circ f_2 = \mathcal{C} \circ \tilde{f}_2 - \mathcal{C} \circ f_1 \circ \mathcal{C} \circ \tilde{f}_2 = 0. \qquad \square$$

Consider (2) with this simplified version of f_2. By applying $\mathcal{C}$, we get the term

$$\mathcal{C}(f_2(a,b)f_1(c)) + \mathcal{C}(f_2(a,bc)) + \mathcal{C}(f_2(ab,c)) + \mathcal{C}(f_1(a)f_2(b,c)).$$

This is the cohomology class of (2). Note that the individual terms $f_2(a,b)f_1(c)$, $f_2(a,bc)\ldots$ will not be cocycles in general, but the map $\mathcal{C}$ assigns cohomology classes to them in such a way that the sum will be the class we are looking for.

By our choice of f_2 (such that $\mathcal{C} \circ f_2 = 0$), the first three terms in the sum vanish (note that $\mathcal{C}(f_2(a,b)f_1(c)) = \mathcal{C}(f_2(a,b))c$ by Proposition 3.2.(iv)). Thus we are interested in terms of the form $\mathcal{C}(f_1(a)f_2(b,c))$, where a, b, c run through all elements of a k-basis of $\hat{H}^*(G)$.

3.2. Generating cocycles and homotopies

Now we start the actual computation of γ. We begin with the construction of a minimal projective resolution P and some cocycles in the endomorphism dga of P. Let us define some elements of the group algebra kG as follows. Put $a = g + 1$, $b = h + 1$ and $c = hg + 1$. Furthermore, we write $N = \sum_{j \in G} j$ for the norm element. Here are some formulae we will frequently use:

$$
\begin{aligned}
a^t &= b^2 = c^2, & a^{2t} &= b^4 = 0,\\
ba &= ac = a + b + c, & N &= a^{2t-1}b,\\
c &= a + bg, & gc &= a + b,\\
N &= ca^{2t-2}b = ca^{2t-1}, & N &= a^{2t-1} + a^{2t-2}b + ca^{2t-2},\\
ca^{t-1}b &= ca^{t-1} + a^{t-1}b. & &
\end{aligned}
$$

Also note that a^{2t-1}, a^{2t-2}, and a^{2t-4} lie in the center of kQ_{4t}. Now a 4-periodic complete projective resolution of the trivial kG-module k is given as follows (see [4, Chapter XII §7]):

$$
\cdots \xleftarrow{N} P_0 = F \xleftarrow{(a\ b)} P_1 = F^2 \xleftarrow{\left(\begin{smallmatrix} a^{t-1} & c \\ b & a \end{smallmatrix}\right)} P_2 = F^2 \xleftarrow{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)} P_3 = F \xleftarrow{N} P_4 = F \longleftarrow \cdots
$$

Since the resolution is minimal, the differential of the complex $\operatorname{Hom}_{kG}(P_*, k)$ vanishes; therefore, we immediately get the well-known additive structure of $\hat{H}^*(G)$:

$$
\hat{H}^{4n}(G) \cong \hat{H}^{4n+3}(G) \cong k, \qquad \hat{H}^{4n+1}(G) \cong \hat{H}^{4n+2}(G) \cong k^2.
$$

Let us write $\bar{s}\colon P \to P[4]$ for the shift map, given by the identity map in every degree. This is an invertible cocycle; thus, multiplication by a suitable power of s yields an isomorphism $\hat{H}^{4n+u}(G) \cong \hat{H}^u(G)$ for $u = 0, 1, 2, 3$ and $n \in \mathbb{Z}$. Now we are heading for explicit generators x, y of $\hat{H}^1(G) \cong H^1 \operatorname{Hom}^*_{kG}(P, P)$, which are represented by chain maps $\bar{x}, \bar{y}\colon P[1] \to P$. By construction, we have $P_1 = F^2$ and $P_0 = F$. We extend the two projections $P_1 \to P_0$ to chain transformations $P[1] \to P$ as follows: For $\bar{x}\colon P \to P[1]$ we take

$$
\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "a^{2t-2}b"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "(1\ 0)"] & F^2 \arrow[l, "\left(\begin{smallmatrix} a^{t-1} & c \\ b & a \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} a^{t-2} & 1 \\ 0 & g \end{smallmatrix}\right)"] & F \arrow[l, "\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)"] & F \arrow[l, "N"'] \arrow[d, "a^{2t-2}b"] & \cdots \arrow[l] \\
\cdots & F \arrow[l] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\left(\begin{smallmatrix} a^{t-1} & c \\ b & a \end{smallmatrix}\right)"] & F \arrow[l, "\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)"] & \cdots \arrow[l]
\end{tikzcd}
$$

and extend this 4-periodically. The 4-periodic chain map $\bar{y}\colon P \to P[1]$ is defined as follows:

$$
\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "a^{2t-1}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "(0\ 1)"] & F^2 \arrow[l, "\left(\begin{smallmatrix} a^{t-1} & c \\ b & a \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & F \arrow[l, "\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & F \arrow[l, "N"'] \arrow[d, "a^{2t-1}"] & \cdots \arrow[l] \\
\cdots & F \arrow[l] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\left(\begin{smallmatrix} a^{t-1} & c \\ b & a \end{smallmatrix}\right)"] & F \arrow[l, "\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)"] & \cdots \arrow[l]
\end{tikzcd}
$$

Since these cocycles are 4-periodic, they commute with $\bar{s}$. Let us determine the pairwise products of these maps. We start with $\bar{x}\bar{y}$:

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "\begin{psmallmatrix} a^{2t-1} \\ a^{2t-1} \end{psmallmatrix}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "(0\ a^{2t-2}b)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"'] \arrow[d, "(0\ 1)"] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"'] \arrow[d, "\begin{psmallmatrix} 1 \\ g \end{psmallmatrix}"] & F \arrow[l, "N"'] \arrow[d, "\begin{psmallmatrix} a^{2t-1} \\ a^{2t-1} \end{psmallmatrix}"] & \cdots \arrow[l] \\
\cdots & F^2 \arrow[l] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

The product $\bar{y}\bar{x}$ is given as follows:

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "\begin{psmallmatrix} 0 \\ a^{2t-2}b \end{psmallmatrix}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "(a^{2t-1}\ 0)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"'] \arrow[d, "(0\ g)"] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"'] \arrow[d, "\begin{psmallmatrix} 1 \\ 1 \end{psmallmatrix}"] & F \arrow[l, "N"'] \arrow[d, "\begin{psmallmatrix} 0 \\ a^{2t-2}b \end{psmallmatrix}"] & \cdots \arrow[l] \\
\cdots & F^2 \arrow[l] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

Next, we compute $\bar{x}^2$:

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "\begin{psmallmatrix} a^{2t-2}b \\ a^{2t-2}b \end{psmallmatrix}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "(a^{2t-2}b\ 0)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"'] \arrow[d, "(a^{t-2}\ 1)"] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"'] \arrow[d, "\begin{psmallmatrix} a^{t-2}+1 \\ g \end{psmallmatrix}"] & F \arrow[l, "N"'] \arrow[d, "\begin{psmallmatrix} a^{2t-2}b \\ a^{2t-2}b \end{psmallmatrix}"] & \cdots \arrow[l] \\
\cdots & F^2 \arrow[l] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

And now $\bar{y}^2$:

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "\begin{psmallmatrix} 0 \\ a^{2t-1} \end{psmallmatrix}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "(0\ a^{2t-1})"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"'] \arrow[d, "(1\ 0)"] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"'] \arrow[d, "\begin{psmallmatrix} 1 \\ 0 \end{psmallmatrix}"] & F \arrow[l, "N"'] \arrow[d, "\begin{psmallmatrix} 0 \\ a^{2t-1} \end{psmallmatrix}"] & \cdots \arrow[l] \\
\cdots & F^2 \arrow[l] & F \arrow[l, "\begin{psmallmatrix} a \\ c \end{psmallmatrix}"] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{psmallmatrix} a^{t-1} & c \\ b & a \end{psmallmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

In each of these cocycles, the map $P_2 \to P_0$ determines the cohomology class by the isomorphism (1); in k^2, they correspond to $(0\,1), (0\,1), (\epsilon(a^{t-2})\,1)$, and $(1\,0)$, respectively. Hence, $\hat{H}^2(G)$ is generated by x^2 and y^2, and we have $xy = yx$. Furthermore, we also see from this description that

$$xy = \begin{cases} x^2 + y^2 & \text{if } t = 2, \\ x^2 & \text{otherwise.} \end{cases}$$

But we will need explicit chain homotopies for all these relations later on, so let us start with the commutator relation $xy = yx$. Let $\bar{p}$ be the 4-periodic null-homotopy for $\bar{x}\bar{y} + \bar{y}\bar{x}$ defined as follows:

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[dr, "a^{2t-2}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[dr, "0"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & F \arrow[l, "N"'] & \cdots \arrow[l] \\
\cdots & F^2 \arrow[l] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

Now let us compute $\bar{y}^3$:

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[d, "0"] & F \arrow[l, "N"'] \arrow[d, "\begin{pmatrix} a^{2t-1} \\ 0 \end{pmatrix}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[d, "\begin{pmatrix} 0 & 0 \\ 0 & a^{2t-1} \end{pmatrix}"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"'] \arrow[d, "(a^{2t-1}\ 0)"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"'] \arrow[d, "0"] & \cdots \arrow[l] \\
\cdots & F \arrow[l] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"] & F \arrow[l, "N"] & \cdots \arrow[l]
\end{tikzcd}$$

Then we find a null-homotopy for that map in two steps: First, consider the 4-periodic extension of the map

$$\begin{tikzcd}
\cdots & F \arrow[l] \arrow[dr, "0"] & F \arrow[l, "N"'] \arrow[dr, "\begin{pmatrix} bh^{-1} \\ a^{t-1}h^{-1} \end{pmatrix}"] & F^2 \arrow[l, "(a\ b)"'] \arrow[dr, "\begin{pmatrix} cg^{-1} \\ a^{t-1} \end{pmatrix}^T h^{-1}"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"'] \arrow[dr, "0"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"'] & \cdots \arrow[l] \\
\cdots & F \arrow[l] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"] & F \arrow[l, "N"] & \cdots \arrow[l]
\end{tikzcd}$$

and call it $\bar{w}'$. Note that this will not quite be a homotopy for $\bar{y}^3$, because it yields the wrong result in degrees $P_{4n+2} \to P_{4n-1}$ for all $n \in \mathbb{Z}$. But if we put

$$P_{8n+j+3} \to P_{8n+j} \colon \bar{w}_{8n+j} = \begin{cases} \bar{w}'_{8n+j} & \text{if } j = 0, 1, 2, 3, \\ (\bar{w}' + \bar{y}^2)_{8n+j} & \text{if } j = 4, 5, 6, 7, \end{cases}$$

then we get an 8-periodic null-homotopy for $\bar{y}^3$ which will be called $\bar{w}$ and satisfies $\bar{s}\bar{w} + \bar{w}\bar{s} = \bar{y}^2\bar{s}$.

3.3. Computation for the quaternion group

Due to the different multiplicative relation in $\hat{H}^*(G)$, we need to consider the cases $t = 2$ and $t \geqslant 4$ separately. We start with $t = 2$. In this case, the map

$$\begin{tikzcd}
\cdots & F^2 \arrow[l] \arrow[dr, "0"] & F^2 \arrow[l, "\begin{pmatrix} a & c \\ b & a \end{pmatrix}"'] \arrow[dr, "0"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"'] \arrow[dr, "0"] & F \arrow[l, "N"'] \arrow[dr, "a^3+a^2+ab"] & F^2 \arrow[l, "(a\ b)"'] & \cdots \arrow[l] \\
\cdots & F \arrow[l] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{pmatrix} a & c \\ b & a \end{pmatrix}"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

can be extended (as we did with $\bar{w}$ above) to an 8-periodic null-homotopy $\bar{r}$ for $\bar{x}^2 + \bar{x}\bar{y} + \bar{y}^2$ satisfying $\bar{s}\bar{r} + \bar{r}\bar{s} = (\bar{x} + \bar{y})\bar{s}$. Notice that $\bar{x}\bar{y}^2 \colon P_3 \to P_0$ is the identity map, which implies that $xy^2 \neq 0 \in \hat{H}^3(G)$. Gathering the results we obtained so far, we recover the known fact that

$$\hat{H}^*(G) \cong k[x, y, s^{\pm 1}]/(x^2 + y^2 = xy, y^3 = 0).$$

Let us remark here that all monomials in x and y of degree bigger than 3 vanish in this ring.

Proposition 3.4. *Let α, β, γ be monomials in the (non-commutative) variables $\bar{x}, \bar{y}$, and assume that the degree $|\beta| \geqslant 3$. Then we have the following formulae:*

$$\begin{aligned}
\mathcal{C}(\bar{p}\alpha) &= 0, & \mathcal{C}(\bar{r}\alpha) &= 0, & \mathcal{C}(\bar{w}\alpha) &= 0,\\
\mathcal{C}(\bar{x}\bar{p}\alpha) &= xy\mathcal{C}(\alpha), & \mathcal{C}(\gamma\bar{r}\alpha) &= 0, & \mathcal{C}(\gamma\bar{w}\alpha) &= 0,\\
\mathcal{C}(\bar{y}\bar{p}\alpha) &= 0, &&&&\\
\mathcal{C}(\bar{x}^2\bar{p}\alpha) &= x^2y\mathcal{C}(\alpha), &&&&\\
\mathcal{C}(\bar{y}^2\bar{p}\alpha) &= 0, &&&&\\
\mathcal{C}(\beta\bar{p}\alpha) &= 0. &&&&
\end{aligned}$$

Proof. By Proposition 3.2.(iii) we can assume that the degree of β is at most 3. Furthermore, we can assume $\alpha = 1$ by Proposition 3.2.(iv). In order to determine $\mathcal{C}(\bar{a}\bar{w})$ for any given cocycle $\bar{a}$ of degree n, we consider the composition

$$P_{n+2} \xrightarrow{\bar{w}_n} P_n \xrightarrow{\bar{a}_0} P_0 \xrightarrow{\epsilon} k$$

as an element of $H^{n+2}\operatorname{Hom}_{kG}(P_*, k)$. Notice $\operatorname{im}(\bar{w}_n) \subset \ker(\epsilon) \cdot P_n$. Therefore, $\operatorname{im}(\bar{a}_0 \circ \bar{w}_n) \subset \ker(\epsilon) \cdot P_0 = \ker(\epsilon)$, and hence $\epsilon \circ \bar{a}_0 \circ \bar{w}_n = 0$. The same proof works for $\bar{r}$ instead of $\bar{w}$, so we are left with $\bar{p}$. For $\mathcal{C}(\bar{x}\bar{p})$, consider $\bar{x}\bar{p}$ in degree 0; i.e.,

$$P_2 \ \underset{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}{\xrightarrow{\bar{p}_1}} \ P_1 \ \underset{\left(\begin{smallmatrix}1&0\end{smallmatrix}\right)}{\xrightarrow{\bar{x}_0}} \ P_0.$$

This equals $\left(\begin{smallmatrix}0&1\end{smallmatrix}\right) : P_2 \longrightarrow P_0$, which corresponds to xy. The remaining cases can be shown analogously. □

Remark 3.5. Using $\mathcal{C}$, we can prove that there is no 4-periodic null-homotopy for $\bar{x}^2 + \bar{x}\bar{y} + \bar{y}^2$ as follows: Suppose there is a 4-periodic null-homotopy; call it $\hat{r}$. Since $d(\hat{r} - \bar{r}) = 0$, $\bar{q} = \hat{r} - \bar{r}$ is a cocycle, representing some class q. By construction, $\bar{s}\bar{r} = (\bar{r} + \bar{x} + \bar{y})\bar{s}$. Since $\hat{r}$ is 4-periodic, we have $\mathcal{C}(\bar{s}\bar{q}) = \mathcal{C}(\bar{q}\bar{s}) - \mathcal{C}((\bar{x} + \bar{y})\bar{s}) = qs - (x + y)s$ by Proposition 3.2. On the other hand, $\mathcal{C}(\bar{s}\bar{q}) = sq$, and hence $(x + y)s = 0$, a contradiction. In a similar way, one shows that there is no 4-periodic null-homotopy for $\bar{x}^3$.

As a next step, we are going to define the functions f_1 and f_2. A k-basis of $\hat{H}^*(G)$ is given by $\mathfrak{C} = \{s^i, xs^i, ys^i, x^2s^i, y^2s^i, x^2ys^i \mid i \in \mathbb{Z}\}$. Define the k-linear map f_1 on the basis $\mathfrak{C}$ by

$$\begin{aligned} f_1 \colon \hat{H}^*(G) &\to \operatorname{Hom}^*_{kG}(P, P)\\ x^\varepsilon y^\delta s^i &\mapsto \bar{x}^\varepsilon \bar{y}^\delta \bar{s}^i \end{aligned}$$

for all $i, \varepsilon, \delta \in \mathbb{Z}$ for which the expression on the left-hand side lies in $\mathfrak{C}$. Let us define the set $\mathcal{B} = \{1, x, y, x^2, y^2, x^2y\}$. For all $b, c \in \mathcal{B}$ and $i, j \in \mathbb{Z}$, we have $f_1(bs^icsj) =$

$f_1(bc)\bar{s}^{i+j}$ and $f_1(bs^i)f_1(cs^i) = f_1(b)f_1(c)\bar{s}^{i+j}$, since $\bar{s}$ commutes with both $\bar{x}$ and $\bar{y}$. This implies that we can define f_2 on $\mathcal{B} \times \mathcal{B}$ and then extend it to $\mathfrak{C} \times \mathfrak{C}$ via $f_2(bs^i, cs^j) = f_2(b,c)\bar{s}^{i+j}$. Now define f_2 on $\mathcal{B} \times \mathcal{B}$ as follows:

$f_2(b,c)$		c: 1	x	y
b	1	0	0	0
	x	0	0	$\bar{r}$
	y	0	$\bar{p}+\bar{r}$	0
	x^2	0	$\bar{x}\bar{r}+\bar{r}\bar{y}+\bar{w}$	0
	y^2	0	$\bar{y}\bar{p}+\bar{y}\bar{r}+\bar{w}+\bar{p}\bar{x}+\bar{x}\bar{p}+\bar{x}\bar{y}$	$\bar{w}$
	x^2y	0	$\bar{x}^2\bar{p}+\bar{x}\bar{r}\bar{y}+\bar{r}\bar{y}^2+\bar{w}\bar{y}+\bar{x}^2\bar{y}$	$\bar{r}\bar{y}^2+\bar{x}\bar{w}+\bar{y}\bar{w}$

		x^2	y^2	x^2y
b	1	0	0	0
	x	$\bar{x}\bar{r}+\bar{r}y+\bar{w}$	$\bar{r}\bar{y}+\bar{w}$	$\bar{x}\bar{r}\bar{y}+\bar{r}\bar{y}^2+\bar{w}\bar{y}$
	y	$\bar{p}\bar{x}+\bar{x}\bar{p}+\bar{x}\bar{y}$	$\bar{w}$	$\bar{y}\bar{r}\bar{y}+\bar{p}\bar{y}^2+\bar{x}\bar{w}+\bar{y}\bar{w}$
	x^2	$\bar{x}\bar{r}\bar{x}+\bar{r}\bar{y}\bar{x}+\bar{w}\bar{x}$	$\bar{r}\bar{y}^2+\bar{x}\bar{w}+\bar{y}\bar{w}$	$*$
	y^2	$\bar{y}^2\bar{r}+\bar{y}^2\bar{p}+\bar{w}\bar{x}+\bar{w}\bar{y}$	$\bar{w}\bar{y}$	$*$
	x^2y	$*$	$*$	$*$

Direct verification shows that $df_2(b,c) = f_1(bc) - f_1(b)f_1(c)$ for all b, c for which f_2 is defined. Each $*$ can be replaced by a suitable polynomial expression in $\bar{x}, \bar{y}, \bar{p}, \bar{r}, \bar{w}$ such that $df_2(b,c) = f_1(bc) - f_1(b)f_1(c)$ holds for all b, c; as we will see, it does not matter which choice we make here. Our f_2 will then already be simplified in the sense of Proposition 3.3, which is why some apparently unnecessary terms occur (e.g., the $\bar{x}\bar{y}$ in $f_2(y, x^2)$). Indeed, $\mathcal{C} \circ f_2 = 0$, as one can check using Proposition 3.4.

As a final step, we need to investigate the term

$$m(a,b,c) = \mathcal{C}(f_1(a)f_2(b,c))$$

for all $a, b, c \in \mathfrak{C}$. Since $f_2(b,c)$ is 8-periodic, we have

$$m(as^{2h}, bs^i, cs^j) = m(a,b,c)s^{2h+i+j}$$

for all integers h, i, j and $a, b, c \in \mathfrak{C}$. Therefore, it is enough to consider all triples $(a,b,c) \in \big(\mathcal{B} \cup \mathcal{B}s\big) \times \mathcal{B} \times \mathcal{B}$.

Consider the case $a \in \mathcal{B}$. If $a = 1$, then $\mathcal{C}(f_1(a)f_2(b,c)) = \mathcal{C}(f_2(b,c)) = 0$. If $a \in \{y^2, x^2y\}$, then $f_1(a)f_2(b,c)$ is a sum of terms $\beta\bar{p}\alpha$, $\beta\bar{r}\alpha$, $\beta\bar{w}\alpha$, and $\beta\bar{x}\bar{y}\alpha$, where α and β are monomials in $\bar{x}$ and $\bar{y}$, the degree of β is at least 2, and $\beta \neq \bar{x}^2$. Hence, $\mathcal{C}(f_1(a)f_2(b,c)) = 0$ by Proposition 3.4.

Next, consider $a = x$. By Proposition 3.4 we get $\mathcal{C}(\bar{x}f_2(b,c))$ from $f_2(b,c)$ by the following rule: Put an $\bar{x}$ in front of all monomials in $\bar{x}$ and $\bar{y}$. Then remove all summands containing $\bar{p}$, $\bar{r}$, or $\bar{w}$, except those beginning with $\bar{p}$, $\bar{x}\bar{p}$, or $\bar{y}\bar{p}$, where we replace the $\bar{p}$ by xy, and $\bar{x}\bar{p}$ and $\bar{y}\bar{p}$ by x^2y. Finally, replace all $\bar{x}$ and $\bar{y}$ by x and y, respectively. Using this procedure, we get the following table for $\mathcal{C}(\bar{x}f_2(b,c))$:

$\mathcal{C}(\bar{x}f_2(b,c))$; b \ c	1	x	y	x^2	y^2	x^2y
1	0	0	0	0	0	0
x	0	0	0	0	0	0
y	0	xy	0	$xyx + x^2y + x^2y$	0	$*$
x^2	0	0	0	$*$	$*$	$*$
y^2	0	$x^2y + xyx + x^2y + x^2y$	0	$*$	$*$	$*$
x^2y	0	$*$	$*$	$*$	$*$	$*$

Here each $*$ stands for some homogeneous polynomial in x, y of degree at least 4. Almost all these expressions vanish, and the only remaining terms are

$$m(x, y, x) = xy,$$
$$m(x, y, x^2) = x^2y.$$

For the case $a = y$ we use a similar method resulting from Proposition 3.4, and we end up with $m(y, b, c) = 0$ for all $b, c \in \mathcal{B}$. Finally, for $a = x^2$ we find that the only non-zero term is $m(x^2, y, x) = x^2y$.

The case $a \in \mathcal{B}s$ is slightly more difficult. Consider the map

$$h(b, c) = \bar{s} f_2(b, c)\bar{s}^{-1} - f_2(b, c),$$

measuring how far away f_2 is from 4-periodicity. From the equations

$$\bar{s}\bar{p}\bar{s}^{-1} = \bar{p},$$
$$\bar{s}\bar{r}\bar{s}^{-1} = \bar{r} + \bar{x} + \bar{y},$$
$$\bar{s}\bar{w}\bar{s}^{-1} = \bar{w} + \bar{y}^2,$$

we get the following table for h:

$h(b,c)$; b \ c	1	x	y	x^2	y^2	x^2y
1	0	0	0	0	0	0
x	0	0	$\bar{x} + \bar{y}$	$\bar{x}^2$	$\bar{x}\bar{y}$	$\bar{x}^2\bar{y}$
y	0	$\bar{x} + \bar{y}$	0	0	$\bar{y}^2$	$\bar{y}\bar{x}\bar{y} + \bar{x}\bar{y}^2$
x^2	0	$\bar{x}^2$	0	$\bar{x}^3$	0	$*$
y^2	0	$\bar{y}\bar{x}$	$\bar{y}^2$	0	$\bar{y}^3$	$*$
x^2y	0	$\bar{x}^2\bar{y}$	0	$*$	$*$	$*$

where $*$ denotes certain homogeneous polynomials in $\bar{x}$ and $\bar{y}$ of degree at least 4. Applying $\mathcal{C}$ to this table and using relations in $\hat{H}^*(G)$, we get

$\mathcal{C}(h(b,c))$; b \ c	1	x	y	x^2	y^2	x^2y
1	0	0	0	0	0	0
x	0	0	$x + y$	x^2	$x^2 + y^2$	x^2y
y	0	$x + y$	0	0	y^2	0
x^2	0	x^2	0	0	0	0
y^2	0	$x^2 + y^2$	y^2	0	0	0
x^2y	0	x^2y	0	0	0	0

(3)

By definition of h, we have $h(b,c)\bar{s} = \bar{s}f_2(b,c) - f_2(b,c)\bar{s}$; hence

$$\mathcal{C}(h(b,c))s = \mathcal{C}(\bar{s}f_2(b,c)) - \underbrace{\mathcal{C}(f_2(b,c))}_{0}s = m(s,b,c).$$

Therefore, this table shows the values $m(s,b,c)$ with $b,c \in \mathcal{B}$. On the other hand, we know that m is a Hochschild-cocycle; in particular, for all $a,b,c \in \mathcal{B}$,

$$a\,m(s,b,c) + m(as,b,c) + m(a,sb,c) + m(a,s,bc) + m(a,s,b)c = 0.$$

Using $m(a,s,b)c = m(a,1,b)sc = 0$, $m(a,s,bc) = m(a,1,bc)s = 0$, and $m(a,sb,c) = m(a,b,c)s$, we get

$$m(as,b,c) = a\,m(s,b,c) + m(a,b,c)s. \tag{4}$$

We know the right-hand side for all $a,b,c \in \mathcal{B}$. Gathering all results, we get the following theorem.

Theorem 3.6. *The canonical element γ_G is represented by the Hochschild cocycle m which is given by the formulae*

$$\begin{aligned}
m(x,y,x) &= xy, \\
m(x,y,x^2) &= x^2y, \\
m(x^2,y,x) &= x^2y, \\
m(a,b,c) &= 0 && \textit{for all other } a,b,c \in \mathcal{B}, \\
m(sa,b,c) &= sm(a,b,c) + sa\,\mathcal{C}(h(b,c)), && \textit{where } \mathcal{C}(h(b,c)) \textit{ is given by } (3), \\
m(s^{2i}a, s^jb, s^lc) &= s^{2i+j+l}m(a,b,c).
\end{aligned}$$

The element $\gamma \in HH^{3,-1}\hat{H}^*(G)$ represented by m is non-trivial.

Proof. It remains to prove the non-triviality of γ. Assume $m = \delta g$ for some Hochschild $(2,-1)$-cochain g. Then,

$$m(a,b,c) = (\delta g)(a,b,c) = a\,g(b,c) + g(ab,c) + g(a,bc) + g(a,b)c$$

for all a,b,c. In particular,

$$\begin{aligned}
0 = m(y,x,y) &= yg(x,y) + g(yx,y) + g(y,xy) + g(y,x)y, \\
0 = m(x,y,y) &= xg(y,y) + g(xy,y) + g(x,y^2) + g(x,y)y, \\
0 = m(y,y,x) &= yg(y,x) + g(y^2,x) + g(y,yx) + g(y,y)x, \\
0 = m(x,x,x) &= xg(x,x) + g(x^2,x) + g(x,x^2) + g(x,x)x, \\
xy = m(x,y,x) &= xg(y,x) + g(xy,x) + g(x,yx) + g(x,y)x.
\end{aligned}$$

Adding up these equations, we get (using $x^2 + y^2 = xy$)

$$xy = x \cdot (g(x,y) + g(y,x)).$$

This implies $g(x,y) + g(y,x) = y$. On the other hand, interchanging the roles of x and y we get $g(x,y) + g(y,x) = x$, a contradiction. ∎

3.4. Computation for the generalized quaternion group

From now on, we assume that $t \geqslant 4$. Then there is an 8-periodic null-homotopy $\bar{v}$ for $\bar{x}^2 + \bar{x}\bar{y}$, partially given by

$$\begin{tikzcd}
\cdots & F^2 \arrow[l] \arrow[dr, "0"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} a^{t-3} & 0 \\ 0 & 0 \end{pmatrix}"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"'] \arrow[dr, "0"] & F \arrow[l, "N"'] \arrow[dr, "u"] & F^2 \arrow[l, "(a\ b)"'] & \cdots \arrow[l] \\
\cdots & F \arrow[l] & F \arrow[l, "N"] & F^2 \arrow[l, "(a\ b)"] & F^2 \arrow[l, "\begin{pmatrix} a^{t-1} & c \\ b & a \end{pmatrix}"] & F \arrow[l, "\begin{pmatrix} a \\ c \end{pmatrix}"] & \cdots \arrow[l]
\end{tikzcd}$$

satisfying $\bar{s}\bar{v} + \bar{v}\bar{s} = \bar{x}$. Here we write $u = ca^{2t-2} + ba^{2t-3}$ and need to prove

$$au = a^{2t-2}b + a^{2t-1}, \qquad cu = a^{2t-2}b + a^{2t-1},$$
$$ua = a^{2t-2}b + N, \qquad ub = a^{2t-2}b.$$

For instance, to prove the first formula, note that

$$au + aca^{2t-2} = aba^{2t-3} = a^{2t-3}ba = a^{2t-3}ac = ca^{2t-2} = (a + b + ac)a^{2t-2}.$$

The other formulae can be proved similarly.

Again one verifies that $x^2y \neq 0$, so that we recover the well-known structure of $\hat{H}^*(G)$ to be

$$\hat{H}^*(G) \cong k[x, y, s^{\pm 1}]/(y^3, x^2 + xy).$$

Using the variable $z = x + y$, we obtain the isomorphism

$$\hat{H}^*(G) \cong k[x, z, s^{\pm 1}]/(xz, x^3 + z^3).$$

In the following, we will frequently switch between these two descriptions.

Proposition 3.7. *We have the following formulae:*

$$\begin{aligned}
\mathcal{C}(\bar{p}\alpha) &= 0, & \mathcal{C}(\bar{v}\alpha) &= 0, & \mathcal{C}(\bar{w}\alpha) &= 0, \\
\mathcal{C}(\bar{x}\bar{p}\alpha) &= x^2\mathcal{C}(\alpha), & \mathcal{C}(\gamma\bar{v}\alpha) &= 0, & \mathcal{C}(\gamma\bar{w}\alpha) &= 0, \\
\mathcal{C}(\bar{y}\bar{p}\alpha) &= 0, \\
\mathcal{C}(\bar{x}^2\bar{p}\alpha) &= x^2y\mathcal{C}(\alpha), \\
\mathcal{C}(\bar{y}^2\bar{p}\alpha) &= 0, \\
\mathcal{C}(\beta\bar{p}\alpha) &= 0,
\end{aligned}$$

for any α, β, γ monomials in $\bar{x}, \bar{y}$ with $|\beta| \geqslant 3$.

We omit the straightforward proof and turn to the definition of the maps f_1 and f_2. As before, let $\mathcal{B} = \{1, x, y, x^2, y^2, x^2y\}$; we define f_1 as

$$f_1(s^i x^a y^b) = \bar{s}^i \bar{x}^a \bar{y}^b$$

for all $a, b, i \in \mathbb{Z}$ for which $x^a y^b$ lies in $\mathcal{B}$. Now we define f_2 on $\mathcal{B} \times \mathcal{B}$ as follows:

$f_2(b,c)$	c: 1	x	y
b: 1	0	0	0
x	0	0	$\bar{v}$
y	0	$\bar{p}+\bar{v}$	0
x^2	0	$\bar{x}\bar{v}$	0
y^2	0	$\bar{y}\bar{p}+\bar{p}\bar{y}+\bar{v}\bar{y}$	$\bar{w}$
x^2y	0	$\bar{x}^2\bar{p}+\bar{x}\bar{v}\bar{y}+\bar{v}\bar{y}^2+\bar{x}\bar{w}+\bar{x}^2\bar{y}$	$\bar{v}\bar{y}^2+\bar{x}\bar{w}$

	c: x^2	y^2	x^2y
b: 1	0	0	0
x	$\bar{x}\bar{v}$	$\bar{v}\bar{y}$	$\bar{x}\bar{v}\bar{y}+\bar{v}\bar{y}^2+\bar{x}\bar{w}$
y	$\bar{p}\bar{x}+\bar{x}\bar{p}+\bar{x}^2$	$\bar{w}$	$\bar{y}\bar{v}\bar{y}+\bar{p}\bar{y}^2+\bar{x}\bar{w}$
x^2	$\bar{x}^2\bar{v}+\bar{x}\bar{v}\bar{y}+\bar{v}\bar{y}^2+\bar{x}\bar{w}$	$\bar{v}\bar{y}^2+\bar{x}\bar{w}$	$\bar{x}^2\bar{v}\bar{y}+\bar{x}\bar{v}\bar{y}^2+\bar{x}^2\bar{w}$
y^2	$\bar{y}^2\bar{v}+\bar{y}^2\bar{p}+\bar{w}\bar{x}$	$\bar{w}\bar{y}$	$\bar{y}^2\bar{v}\bar{y}+\bar{y}^2\bar{p}\bar{y}+\bar{w}\bar{x}\bar{y}$
x^2y	$\bar{x}^2\bar{p}\bar{x}+\bar{x}\bar{v}\bar{y}\bar{x}+\bar{v}\bar{y}^2\bar{x}+\bar{x}\bar{w}\bar{x}$	$\bar{x}^2\bar{w}$	$\bar{x}^2\bar{y}\bar{v}\bar{y}+\bar{x}^2\bar{p}\bar{y}^2+\bar{x}^3\bar{w}$

Also put $f_2(s^ia, s^jb) = f_2(a,b)\bar{s}^{i+j}$ for all $i,j \in \mathbb{Z}$ and $a,b \in \mathcal{B}$. This function is chosen in such a way that $\mathcal{C}(f_2(a,b)) = 0$ for all $a,b \in \mathcal{B}$. One verifies that

$$m(x,y,x) = x^2,$$
$$m(x^2,y,x) = x^2y,$$
$$m(x,y,x^2) = x^2y,$$

and m vanishes on all other triples $(a,b,c) \in \mathcal{B}^{\times 3}$. Let us define m' as follows:

$$m'(s^ia, s^jb, s^kc) = s^{i+j+k}m(a,b,c) \qquad \text{for all } a,b,c \in \mathcal{B}, \tag{5}$$

and define $h(a,b) = \bar{s}f_2(a,b)\bar{s}^{-1} - f_2(a,b)$. Then $\mathcal{C}(h(b,c))$ is given by the following table:

$\mathcal{C}(h(b,c))$	c: 1	x	y	x^2	y^2	x^2y
b: 1	0	0	0	0	0	0
x	0	0	x	x^2	x^2	x^2y
y	0	x	0	0	y^2	0
x^2	0	x^2	0	0	0	0
y^2	0	x^2	y^2	0	0	0
x^2y	0	x^2y	0	0	0	0

(6)

So we get the following explicit description of m:

Theorem 3.8. *The canonical element γ_G is represented by the Hochschild cocycle m*

which is given by the formulae:

$$\begin{aligned}
m(x,y,x) &= x^2, \\
m(x^2,y,x) &= x^2y, \\
m(x,y,x^2) &= x^2y, \\
m(a,b,c) &= 0 && \textit{for all other } a,b,c\in\mathcal{B}, \\
m(sa,b,c) &= sm(a,b,c) + sa\,\mathcal{C}(h(b,c)), && \textit{where } \mathcal{C}(h(b,c)) \textit{ is given by } (6), \\
m(s^{2i}a,s^jb,s^lc) &= s^{2i+j+l}m(a,b,c).
\end{aligned}$$

The element $\gamma \in HH^{3,-1}\hat{H}^*(G)$ represented by m is non-trivial.

Proof. It remains to prove the non-triviality of γ. Suppose that m is a Hochschild coboundary; then $m = \delta g$ for some $g\colon \Lambda^{\otimes 2} \to \Lambda[-1]$. Adding up the equations

$$\begin{aligned}
x^3 = m(x,z,x^2) &= xg(z,x^2) + g(x,z)x^2 \\
0 = m(x^2,x,z) &= x^2g(x,z) + g(x^3,z) + g(x^2,x)z \\
0 = m(z,x^2,x) &= zg(x^2,x) + g(z,x^3) + g(z,x^2)x \\
0 = m(z,z^2,z) &= zg(z^2,z) + g(z^3,z) + g(z,z^3) + g(z,z^2)z \\
0 = zm(z,z,z) &= z^2g(z,z) + zg(z^2,z) + zg(z,z^2) + zg(z,z)z
\end{aligned}$$

and simplifying, we get the contradiction $x^3 = 0$. □

4. Realizability of modules

4.1. Massey products

There is a strong connection between the canonical class γ and triple Massey products over $\hat{H}^*(G)$. This has already been noted in [2, Lemma 5.14], and we will generalize this fact to Massey products of matrices (as introduced by May [5]). We start with some notation. Let Λ be a graded k-algebra, and suppose that I is a graded set; i.e., a set together with a function $|\cdot|\colon I \to \mathbb{Z}$. For every such set, we define $I[n]$ to be the shifted graded set given by the same set with new grading $|i|_{[n]} = |i| + n$ for all $i \in I$. We denote by Λ^I the shifted free Λ-module

$$\Lambda^I = \bigoplus_{i\in I} \Lambda[|i|].$$

Then $\Lambda^I[n] = \Lambda^{I[n]}$. If J is another graded set, we can consider morphisms $f\colon \Lambda^J \to \Lambda^I$. Every such map can be represented by a (possibly infinite) matrix $(f_{i,j})_{i\in I, j\in J}$ with $|f_{i,j}| = |i| - |j|$. Such a matrix is column-finite; i.e., for every j there are only finitely many non-zero $f_{i,j}$'s. Let us denote by $\Lambda^{I,J}$ the set of such matrices. Every such yields a map $f\colon \Lambda^J \to \Lambda^I$.

A triple of matrices (A,B,C) will be called *composable* if there are graded sets I,J,K,L with $A \in \Lambda^{I,J}, B \in \Lambda^{J,K}, C \in \Lambda^{K,L}$. Every morphism $m\colon \Lambda^{\otimes 3} \to \Lambda[-1]$ can be extended to the module of all composable triples by putting

$$m(A,B,C) \in \Lambda^{I[-1],L}: \quad m(A,B,C)_{i[-1],l} = \sum_{j\in J}\sum_{k\in K} m(a_{ij}, b_{jk}, c_{kl}).$$

From now on we assume $\Lambda = H^*\mathcal{A} \cong \hat{H}^*(G)$, where $\mathcal{A}$ is the endomorphism-dgA of some projective resolution of the trivial kG-module k. Also, let $m\colon \Lambda^{\otimes 3} \to \Lambda[-1]$ be some Hochschild cocycle representing the canonical element $\gamma \in HH^{3,-1}\hat{H}^*(G)$. Recall that (see, e.g., [5]) for every composable triple of matrices (A, B, C) with $AB = 0$ and $BC = 0$ the triple matric Massey product $\langle A, B, C\rangle$ is defined and a coset of $A \cdot \Lambda^{J[-1],L} + \Lambda^{I[-1],K} \cdot C$. Notice that there is no obstruction to generalizing May's definition to infinite matrices.

Proposition 4.1. *For every composable triple (A, B, C) with $AB = 0$ and $BC = 0$, we have that $m(A, B, C) \in \langle A, B, C\rangle$.*

Proof. We have

$$\begin{aligned} m(A, B, C) &= f_1(A)f_2(B, C) + f_2(AB, C) + f_2(A, BC) + f_2(A, B)f_1(C) \\ &= f_1(A)f_2(B, C) + f_2(A, B)f_1(C), \end{aligned}$$

and the last term represents one element of the Massey product. □

A triple (A, B, C) will be called *exact* if it is composable and the sequence

$$\Lambda^I \xleftarrow{A} \Lambda^J \xleftarrow{B} \Lambda^K \xleftarrow{C} \Lambda^L$$

is exact.

Proposition 4.2. *Let $A \in \Lambda^{I,J}$ be any matrix, and define $M = \operatorname{coker} A$. Then the following are equivalent:*

(i) The module M is a direct summand of a realizable module.

(ii) For every composable triple (A, B, C) with $AB = 0$ and $BC = 0$, we have that $0 \in \langle A, B, C\rangle$.

(iii) For some exact triple (A, B, C), we have $0 \in \langle A, B, C\rangle$.

Proof. For (i) ⇒ (ii), let M be a direct summand of H^*N, where N is some dg-$\mathcal{A}$-module. Then there are maps $M \xrightarrow{i} H^*N \xrightarrow{r} M$ with $ri = \mathrm{id}_M$. Let $\pi\colon \Lambda^I \to M$ be the projection map, and put $W = i\pi$. Then $WA = 0$, so that $\langle W, A, B\rangle$ is defined, and the juggling formula (see [5, Corollary 3.2.(iii)]) yields $W\langle A, B, C\rangle = \langle W, A, B\rangle C$ as cosets of $W\Lambda^{I[-1],K}C$. Let $E\colon \Lambda^K \to H^*N[-1]$ be some element in $\langle W, A, B\rangle$. Since Λ^K is free, we know that the composition $r \circ E$ lifts as $\Lambda^K \xrightarrow{S} \Lambda^{I[-1]} \xrightarrow{\pi} M[-1]$ for some matrix S. But then

$$\pi SC = rEC \in r\langle W, A, B\rangle C = rW\langle A, B, C\rangle = \pi\langle A, B, C\rangle.$$

This means that there is some matrix T such that $AT + SC \in \langle A, B, C\rangle$, which implies $0 \in \langle A, B, C\rangle$.

The implication (ii) ⇒ (iii) is obvious. For (iii) ⇒ (i), note that

$$M \leftarrow \Lambda^I \xleftarrow{A} \Lambda^J \xleftarrow{B} \Lambda^K \xleftarrow{C} \Lambda^L$$

is the beginning of a (shifted) free resolution of M. We have $m(A, B, C) \in \Lambda^{I[-1],L}$, and a representative of $\gamma \cup \mathrm{id}_M \in \widehat{\mathrm{Ext}}_\Lambda^{3,-1}(M, M)$ is given by the composition

$$g\colon \Lambda^L \xrightarrow{m(A,B,C)} \Lambda^{I[-1]} \to (\operatorname{coker} A)[-1] = M[-1].$$

By assumption and Proposition 4.1, $m(A, B, C) = AX + YC$ for some matrices X

and Y, so that this composition equals

$$\Lambda^L \xrightarrow{C} \Lambda^K \xrightarrow{Y} \Lambda^{I[-1]} \to M[-1],$$

which in turn says that g is the coboundary of $\Lambda^K \xrightarrow{Y} \Lambda^{I[-1]} \to M[-1]$; hence $\gamma \cup \mathrm{id}_M = 0$. By Theorem 1.1 of [2], M is a direct summand of some realizable module. □

4.2. The group of quaternions

Let $G = Q_8$. We shall make use of one of the implications of Proposition 4.2 to prove the existence of a $\hat{H}^*G$-module which detects the non-triviality of γ_G:

Theorem 4.3. *The cokernel of the map*

$$\Lambda[-1] \oplus \Lambda[-1] \xrightarrow{\begin{pmatrix} y & x+y \\ x & y \end{pmatrix}} \Lambda \oplus \Lambda$$

is not a direct summand of a realizable $\hat{H}^*G$-module.

Proof. Let $A = \left(\begin{smallmatrix} y & x+y \\ x & y \end{smallmatrix}\right)$; then $A^2 = 0$ and therefore the Massey product $\langle A, A, A\rangle$ is defined. We claim that it does not contain 0. An explicit calculation using the description of m given in Theorem 3.6 yields

$$m(A, A, A) = \begin{pmatrix} x^2 & 0 \\ x^2 & x^2 \end{pmatrix}.$$

Let us denote the latter matrix by B; then by Proposition 4.2 we need to prove that B is not of the form $B = AQ + RA$ for some 2×2-matrices Q and R. To do so, define $D = \left(\begin{smallmatrix} x & y \\ x+y & x \end{smallmatrix}\right)$; then $AD = DA = 0$. If we denote by tr the trace of a matrix, then we have

$$\mathrm{tr}(BD) = \mathrm{tr}(AQD) + \mathrm{tr}(RAD) = \mathrm{tr}(QDA) + \mathrm{tr}(RAD) = 0$$

(note that these computations take place in a commutative ring). But

$$\mathrm{tr}(BD) = \mathrm{tr}\begin{pmatrix} 0 & * \\ * & x^2y \end{pmatrix} = x^2y \neq 0,$$

a contradiction. □

Remark 4.4. The triple (A, A, A) is actually exact, but we do not need this.

In order to construct a module which is not a direct summand of a realizable one, it is often enough to consider "ordinary" Massey products, i.e., the case of 1×1-matrices; this is true for example in the cases $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ([2, Example 7.7]) and $G = \mathbb{Z}/3\mathbb{Z}$ (characteristic 3, [2, Example 7.6]). In our present case, it is not that easy:

Proposition 4.5. *Let $k = \mathbb{F}_2$ be the field with 2 elements. For all $a, b, c \in \hat{H}^*(Q_8)$ satisfying $ab = 0$ and $bc = 0$, we have $0 \in \langle a, b, c\rangle$.*

Proof. By [2, Lemma 5.14], the class $m(a,b,c)$ is contained in the Massey product $\langle a,b,c\rangle$. Therefore, it is enough to show that $m(a,b,c)$ is an element of the indeterminacy

$$a\cdot \hat{H}^{|b|+|c|-1}(G)+\hat{H}^{|a|+|b|-1}(G)\cdot c$$

for all a,b,c. By construction of m it is enough to do so for those triples (a,b,c) and (sa,b,c) with $a,b,c\in\{1,x,y,x+y,x^2,y^2,x^2+y^2,x^2y\}$ which satisfy $ab=0$ and $bc=0$.

If $|a|,|b|\leqslant 1$, then $ab=0$ implies $a=0$ or $b=0$ (here we use that $k=\mathbb{F}_2$). If $|b|\geqslant 2$, then $m(a,b,c)=0$ unless $b\in\{y^2,y^2+x^2\}$ and $a,c\in\{x,x+y\}$, in which case $m(a,b,c)=x^2y$ is divisible by a. So we can assume that $|b|=1$ and therefore $|a|\geqslant 2$ and $|c|\geqslant 2$, which implies $m(a,b,c)=0$ by Theorem 3.6.

For $m(sa,b,c)$, we have by (4)

$$m(sa,b,c)=a\,m(s,b,c)+m(a,b,c)s.$$

We have already seen that the second summand lies in the indeterminacy; the first summand is contained in

$$a\cdot\hat{H}^{|s|+|b|+|c|-1}(G)=sa\cdot\hat{H}^{|b|+|c|-1}(G)$$

and therefore in the indeterminacy. □

Remark 4.6. Note that the proposition is not true for arbitrary fields of characteristic 2: If the field k contains an element $\alpha\in k$ satisfying $\alpha^2+\alpha+1=0$, then the Massey product

$$\left\langle \alpha x+y,\alpha^2x+y,\alpha x+y\right\rangle$$

is defined and does not contain 0.

4.3. Generalized quaternions

The picture changes as soon as we consider generalized quaternion groups $G=Q_{4t}$ with $t\geqslant 4$. It turns out that there is no module detecting the non-triviality of the canonical element γ_G.

Let m be as in Theorem 3.8, and write $m=m'+m''$, where m' is defined in (5). Notice that m' is a Hochschild cocycle, because it is defined to be s-periodic, so it is enough to check the cocycle condition on elements in $\mathcal{B}$. But on these elements, m' agrees with m. Hence, m' is a cocycle, and so is m''. Let γ' and γ'' be the corresponding elements in $HH^{3,-1}\hat{H}^*(G)$. In the next two propositions we will show that, for every module M, $\gamma'\cup\mathrm{id}_M=0$ and $\gamma''\cup\mathrm{id}_M=0$ in $\mathrm{Ext}^{3,-1}(M,M)$, respectively. It will then follow that M is a direct summand of a realizable module.

Proposition 4.7. *For every Λ-module M we have $\gamma'\cup\mathrm{id}_M=0$.*

Proof. Notice that every matrix $A\in\Lambda^{I,J}$ can be uniquely written as a sum

$$A=A_1+A_xx+A_yy+A_{x^2}x^2+A_{y^2}y^2+A_{x^2y}x^2y,$$

where the six matrices on the right-hand side lie in $k[s^{\pm1}]^{I,J[?]}$. The first step in our proof will be to find a suitable free resolution

$$M\leftarrow\Lambda^I\xleftarrow{A}\Lambda^J\xleftarrow{B}\Lambda^K\xleftarrow{C}\Lambda^L$$

of M. We begin with the definition of A. Let I be a minimal set of generators of the right Λ-module M; i.e., I generates M but any proper subset of I does not generate M (in the case where M is not finitely generated, one has to use Zorn's lemma to prove the existence of I). The inclusion $I \subseteq M$ induces a surjection $\Lambda^I \to M$. Let J be a minimal set of generators for the kernel of that map; then we obtain an exact sequence $\Lambda^J \xrightarrow{A} \Lambda^I \to M$. Taking K to be a minimal set of generators for the kernel of A, we get a map $\Lambda^K \xrightarrow{B} \Lambda^J$ onto that kernel, and finally we let L be a minimal set of generators for the kernel of B to obtain an exact sequence

$$M \leftarrow \Lambda^I \xleftarrow{A} \Lambda^J \xleftarrow{B} \Lambda^K \xleftarrow{C} \Lambda^L.$$

We claim that $A_1 = 0$. Assume the contrary and let $i \in I$, $j \in J$ be such that $(A_1)_{i,j} \neq 0$. Then $I - \{i\}$ generates M, which contradicts the choice of I. Similarly one shows that $B_1 = 0$ and $C_1 = 0$, and therefore $B_y C_y = (BC)_{y^2} = 0$.

Now define $W = A_x B_y x + A_{x^2} B_y x^2$ and $V = B_y C_{y^2} y^2$. Then

$$\begin{aligned} AV &= A_x B_y C_{y^2} x^3, \\ WC &= A_x B_y C_x x^2 + A_x \underbrace{B_y C_y}_{0} x^2 + A_x B_y C_{x^2} x^3 + A_x B_y C_{y^2} x^3 \\ &\quad + A_{x^2} B_y C_x x^3 + A_{x^2} \underbrace{B_y C_y}_{0} x^3. \end{aligned}$$

Therefore, $m'(A, B, C) = AV + WC$, and by Proposition 4.2 we get $\gamma' \cup \mathrm{id}_M = 0$. $\square$

Proposition 4.8. *For every Λ-module M, we have $\gamma'' \cup \mathrm{id}_M = 0$.*

Proof. We start with a slight modification of the representative m''. Let us put $\mathcal{B} = \{1, x, z, x^2, z^2, x^3\}$, and define the function g as follows: For all integers i, put

$$\begin{aligned} g(s^{-1}x^2, s^i x) &= s^{i-1} z^2, \\ g(s^{-1}x^2, s^i z) &= s^{i-1} x^2, \end{aligned}$$

and $g(a, b) = 0$ on all other elements a, b in $\{s^i c \mid c \in \mathcal{B}\}$. Then $\tilde{m} = m'' + \partial g$ defines a new representative for the element γ''. For all $a, b, c \in \mathcal{B}$ and $i, j \geqslant 1$, we have

$$\begin{aligned} \tilde{m}(a, s^i b, s^j c) = m''(a, s^i b, s^j c) + a g(s^i b, s^j c) + g(s^i ab, s^j c) \\ + g(a, s^{i+j} bc) + g(a, s^i b) s^j c, \end{aligned}$$

and by definition of m'' and g each summand on the right-hand side vanishes. We also have that

$$\begin{aligned} \tilde{m}(s^{-1}a, s^i b, s^j c) = m''(s^{-1}a, s^i b, s^j c) + s^{-1}a \underbrace{g(s^i b, s^j c)}_{0} + \underbrace{g(s^{i-1}ab, s^j c)}_{0} \\ + g(s^{-1}a, s^{i+j} bc) + g(s^{-1}a, s^i b) s^j c. \end{aligned}$$

We claim that this is zero if $|a| \geqslant 2$, $|b| \geqslant 1$, and $|c| \geqslant 1$. In that case, we have $|bc| \geqslant 2$ and therefore $g(s^{-1}a, s^{i+j} bc) = 0$, so that it remains to show $m''(s^{-1}a, s^i b, s^j c) =$

$g(s^{-1}a, s^i b)s^j c$, or equivalently,

$$m''(s^{-1}a, b, c) = g(s^{-1}a, b)c.$$

To see this, we consider the several cases for a separately. If $a = x^3$, then

$$m''(s^{-1}a, b, c) = s^{-1}x^3\mathcal{C}(h(b, c)),$$

where h is as in Theorem 3.8. But $|h(b, c)| \geqslant 1$, so the last expression vanishes, as does $g(s^{-1}a, b)c$. For $a = z^2$ we get

$$m''(s^{-1}a, b, c) = s^{-1}z^2\mathcal{C}(h(b, c)),$$

but $|h(b, c)| \geqslant 2$ or $\mathcal{C}(h(b, c))$ is divisible by x, and therefore again the right-hand side vanishes. The last case is $a = x^2$, where we need to show

$$s^{-1}x^2\mathcal{C}(h(b, c)) = g(s^{-1}x^2, b)c.$$

Both sides vanish for degree reasons unless $|b| = |c| = 1$, and in that case both sides will equal $s^{-1}x^3$ if $b \neq c$, and 0 otherwise.

The rest is easy. We start with a free resolution of M as in the proof of Proposition 4.7. We can (and do) assume that the degree $|i|$ of every element $i \in I$ lies in $\{0, 1, 2, 3\}$. Also, we assume that the degree of every element of J lies in $\{-1, 0, 1, 2\}$, the degree of every element of K belongs to $\{-8, -7, -6, -5\}$, and the degree of every element of L is in $\{-15, -14, -13, -12\}$. Then we know that every non-zero entry of B and C is a linear combination of terms of the form $s^i b$ with $i \geqslant 1$ and $b \in \mathcal{B}$, $|b| \geqslant 1$. Furthermore, every non-zero entry of A is a linear combination of elements in $\mathcal{B} \cup \{s^{-1}x^2, s^{-1}z^2, s^{-1}x^3\}$. By what we have shown above, $\tilde{m}(A, B, C) = 0$, and we are done. □

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