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ANALYSIS OF OPTIMAL BOUNDARY CONTROL FOR RADIATIVE HEAT TRANSFER MODELED BY THE SP_1-SYSTEM*

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Abstract. We present an analytic study of an optimal boundary control problem for the diffusive SP_1-system modeling radiative heat transfer. The cost functional is of tracking-type and the control problem is considered as a constrained optimization problem, where the constraint is given by the nonlinear parabolic/elliptic SP_1-system. We prove the existence, uniqueness and regularity of bounded states, which allows for the introduction of the reduced cost functional. Further, we show the existence of an optimal control, derive the first-order optimality system and analyze the adjoint system, for which we prove existence, uniqueness and regularity of adjoint states.

Key words. radiative heat transfer, SP_N-approximation, optimal boundary control, first-order optimality system, analysis, adjoints

AMS subject classifications. 35K55, 49K20, 80A20

1. Introduction

In many industrial high temperature processes and applications radiative heat transfer plays a dominant role, e.g., simulation of gas turbine combustion chambers, combustion in car engines or cooling of a hot glass melt [2]. The appropriate model is given by the radiative heat transfer equations, which are of high numerical complexity. Hence, during the last decade much research was focused on the derivation of approximate models allowing for an accurate description of the important physical phenomena at reasonable numerical costs. Nowadays, a whole hierarchy of approximating equations is available, ranging from half space moment approximations over full space moment systems to the diffusive-type SP_N-systems [5, 9, 14]. Naturally, one is not only interested in the correct simulation of the physical system but also wants to improve processes or operation conditions, which leads directly to optimization problems. During the last years the increased computing power in combination with the usage of the approximate models has allowed for the numerical treatment of such large-scale optimization problems. In particular, optimal boundary control problems for the SP_1-system yielded encouraging results and were successfully employed for many applications [20, 12, 15, 11]. Nevertheless, the mathematical analysis of this optimal boundary control problem is still open. The purpose of this paper is to provide a mathematically sound basis.

In order to model radiative heat transfer we consider for notational simplicity a frequency-independent, gray model without scattering. Stated on a bounded spatial domain $\Omega \subset \mathbb{R}^d$, $d = 1,2$, or 3, the scaled equations read [9]:

$$\varepsilon^2 \partial_t T = \varepsilon^2 \operatorname{div}(k \nabla T) - \int_{S^{d-1}} \kappa (aT^4 - I)\, d\omega \tag{1.1a}$$

$$\forall \omega \in S^{d-1}: \ \varepsilon \omega \cdot \nabla I = \kappa (aT^4 - I), \tag{1.1b}$$

where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. To get a well-posed problem we prescribe the following boundary conditions: Ingoing radiation is prescribed by transparent

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boundary conditions

$$I(t,x,\omega) = au^4, \quad n\cdot\omega < 0, \quad x \in \partial\Omega, \tag{1.1c}$$

and the temperature is assumed to obey Robin-type boundary conditions representing Newton's cooling law

$$n\cdot\nabla T = \frac{h}{\varepsilon k}(u - T), \quad x \in \partial\Omega. \tag{1.1d}$$

At initial time $t = 0$, the temperature is $T(0,x) = T_0(x)$. In these equations, $I(t,x,\omega)$ denotes the specific radiation intensity at point $x \in \Omega$ traveling in direction $\omega \in S^{d-1}$ at time $t \geq 0$. The outside radiation $I_b = au^4$ is assumed to be known for the ingoing directions (i.e., $n\cdot\omega < 0$) on the boundary. We denote the outward normal on $\partial\Omega$ by n. Furthermore, $T(t,x)$ denotes the material temperature and u is the exterior temperature on the boundary, acting as the control variable. The equations contain the parameters opacity κ, heat conductivity k and convective heat transfer coefficient h, which are assumed to be positive constants. The scaled optical thickness is denoted by ε. For notational convenience the constant a is introduced, which is related to the Stefan-Boltzmann constant via $a = \sigma/\pi$. Note that the total thermal radiation is $B(T) = aT^4$ according to Stefan's law.

Since this model has a high dimensional phase space due to the dependence on the direction $\omega \in S^{d-1}$, its numerical complexity is much too high for optimization purposes, where the nonlinear state system has to be solved several times. Here, we use instead the diffusion-type SP_N-approximations [6, 9] to the radiative heat transfer equations. These approximations were developed recently and tested extensively for various radiative transfer problems, where they proved to be sufficiently accurate [16].

The SP_1-approximation to the radiative heat transfer equations is given by the system

$$\partial_t T = k\Delta T + \frac{1}{3\kappa}\Delta\rho, \tag{1.2a}$$

$$0 = -\varepsilon^2 \frac{1}{3\kappa}\Delta\rho + \kappa\rho - \kappa 4\pi a|T|^3 T, \tag{1.2b}$$

with boundary conditions

$$n\cdot\nabla T = \frac{h}{\varepsilon k}(u - T), \tag{1.2c}$$

$$n\cdot\nabla\rho = \frac{3\kappa}{2\varepsilon}\left(4\pi a|u|^3 u - \rho\right), \tag{1.2d}$$

and supplemented with an initial condition $T(0,x) = T_0(x)$ for the temperature. Here, ρ is the radiative flux, and the prescribed temperature at the boundary is denoted by u.

REMARK 1.1. *Note that the radiative flux for the full model* (1.1b) *is given by* $\rho = \int_{S^{d-1}} I \, d\omega$. *Further, we replaced for mathematical reasons the nonlinear function* z^4 *by* $|z|^3 z$ *to ensure its monotonicity also for negative data. For positive data they clearly coincide.*

In [12] an optimal boundary control problem is introduced and studied numerically. There, cost functionals of tracking-type for different norms are considered, e.g.

$$J(T,u) = \frac{1}{2}\|T - T_d\|^2_{L^2(0,1;L^2(\Omega))} + \frac{\delta}{2}\|u - u_d\|^2_{H^1(0,1;\mathbb{R})}, \tag{1.3}$$

where (T,ρ) solves (1.2). Here, $T_d = T_d(t,x)$ is a specified temperature profile and $u_d = u_d(t)$ is a given control of the ambient temperature, which shall be improved. Furthermore, the positive constant δ allows to adjust the weight of the penalty term. The main subject of the analysis in this paper is the following boundary control problem

$$\begin{aligned} &\min\ J(T,u) \text{ w.r.t. } (T,\rho,u), \\ &\text{subject to system (1.2).} \end{aligned} \tag{1.4}$$

This optimal control problem can be considered as a constrained optimization problem and the adjoint variables can be used for the construction of a suitable numerical algorithm [12]. In this paper we provide the analysis for this approach. We prove the existence of an optimal control u and show the unique solvability of the state system, which is essential for the introduction of the reduced cost functional. Then, the unique solvability of the linearized state system is shown and the adjoint equations are identified.

The paper is organized as follows. In Section 2 we study the state system, prove its unique solvability and derive a priori estimates. The existence of an optimal control is shown in Section 3. Further, Section 4 is devoted to the linearized state system. We prove its unique solvability and some regularity estimates. Finally, we investigate the adjoint equations in Section 5 and give concluding remarks in Section 6.

1.1. Notation and auxiliary results. We use the standard notation for Sobolev spaces (see [1]), denoting the norm of $W^{m,p}(\Omega)$ $(m \in \mathbb{N}, p \in [1,\infty])$ by $\|\cdot\|_{W^{m,p}(\Omega)}$. In the special case $p = 2$ we use $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. Further, let $H_0^m(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the $H^m(\Omega)$-norm. Its dual space $(H_0^m(\Omega))^*$ is denoted by $H^{-m}(\Omega)$. The duality pairing of a Banach space X with its dual space X^* is given by $\langle\cdot,\cdot\rangle_{X^*,X}$. For a Hilbert space H the inner product is denoted by $(\cdot,\cdot)_H$; if $H = L^2(0,1;L^2(\Omega))$ we just write $(\cdot,\cdot)$. Moreover, for any Banach space B we define the space $L^p(0,1;B)$ with $p \in [1,\infty]$ consisting of all measurable functions $\varphi : (0,1) \to B$ for which the norm

$$\begin{aligned} \|\varphi\|_{L^p(0,1;B)} &\stackrel{\text{def}}{=} \left(\int_0^1 \|\varphi(t)\|_B^p\, dt\right)^{1/p}, \quad p \in [1,\infty), \\ \|\varphi\|_{L^\infty(0,1;B)} &\stackrel{\text{def}}{=} \sup_{t\in(0,1)} \|\varphi(t)\|_B, \quad p = \infty, \end{aligned}$$

is finite. If the time interval is clear we write shortly $\|\cdot\|_{L^p(B)}$.

Remark 1.2. *Clearly, one can define these spaces on arbitrary time intervals. But due to scaling we assume that the equations are posed on the unit time interval.*

For notational convenience we define

$$\begin{gathered} Q \stackrel{\text{def}}{=} (0,1)\times\Omega, \quad \Sigma \stackrel{\text{def}}{=} (0,1)\times\partial\Omega, \\ V \stackrel{\text{def}}{=} L^2(0,1;H^1(\Omega)), \quad U \stackrel{\text{def}}{=} H^1(0,1;\mathbb{R}), \\ W \stackrel{\text{def}}{=} \{\phi \in V : \phi_t \in V^*\}, \quad X \stackrel{\text{def}}{=} W \times V, \quad Z \stackrel{\text{def}}{=} V \times V \times L^2(\Omega). \end{gathered}$$

Then, we define $X_\infty \stackrel{\text{def}}{=} X \cap [L^\infty(Q)]^2$ as the space of states $x \stackrel{\text{def}}{=} (T,\rho)$, and U is the space of controls. Finally, we set $\alpha = \frac{h}{\varepsilon k}$, $\gamma = \frac{3\kappa}{2\varepsilon}$.

Remark 1.3. *The space W is a Banach space if supplied with the norm $\|\phi\|_W = \|\phi\|_V + \|\phi_t\|_{V^*}$. Note that in one-dimensional space we have $X_\infty = X$. Later, we identify the dual Z^* of Z with $V^* \times V^* \times L^2(\Omega)$.*

For the subsequent considerations we impose the following assumptions:

A.1 Let $\Omega \subset \mathbb{R}^d$, $d = 1,2$, or 3, be a bounded domain with Lipschitz boundary.

A.2 There exists a constant $K = K(\Omega) \in (0,\infty)$ such that for all $f \in L^2(\Omega)$ we have a solution $\Psi \in H^2(\Omega)$ of

$$
\begin{aligned}
-\frac{\varepsilon^2}{3\kappa}\Delta\Psi + \kappa\Psi &= f \quad \text{in } \Omega,\\
n\cdot\nabla\Psi + \gamma\Psi &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

such that

$$
\|\Psi\|_{H^2(\Omega)} \le K\,\|f\|_{L^2(\Omega)}.
$$

Remark 1.4. *Assumption* **A.2** *is essentially a requirement on the smoothness of $\partial\Omega$, which is e.g. fulfilled for $\partial\Omega \in C^{1,\delta}$ for some $\delta \in (0,1)$ (see [21]).*

1.2. The optimal control problem. In this subsection we give the precise mathematical statement of the optimal control problem (1.4). We define the state/control pair $(x,u) \in X_\infty \times U$ and the nonlinear operator $e \stackrel{\text{def}}{=} (e_1,e_2,e_3)$: $X_\infty \times U \to Z^*$ via

$$
\begin{aligned}
\langle e_1(x,u),\phi\rangle_{V^*,V} \stackrel{\text{def}}{=} {}& \langle \partial_t T,\phi\rangle_{V^*,V} + k(\nabla T,\nabla\phi)_{L^2(Q)} + \frac{1}{3\kappa}(\nabla\rho,\nabla\phi)_{L^2(Q)}\\
&+ k\alpha(T-u,\phi)_{L^2(\Sigma)} + \frac{1}{3\kappa}\gamma(\rho - 4\pi a|u|^3u,\phi)_{L^2(\Sigma)}
\end{aligned}
\tag{1.5a}
$$

and

$$
\begin{aligned}
\langle e_2(x,u),\phi\rangle_{V^*,V} \stackrel{\text{def}}{=} {}& \frac{\varepsilon^2}{3\kappa}(\nabla\rho,\nabla\phi)_{L^2(Q)} + \kappa(\rho - 4\pi\kappa a\,|T|^3T,\phi)_{L^2(Q)}\\
&+ \frac{\varepsilon^2}{3\kappa}\gamma(\rho - 4\pi a|u|^3u,\phi)_{L^2(\Sigma)}
\end{aligned}
\tag{1.5b}
$$

for all $\phi \in V$. Further, we define $e_3(x,u) \stackrel{\text{def}}{=} T(0) - T_0$.

Remark 1.5. *Note that for $d \le 2$ it is in fact possible to use X itself as the state space, but for $d = 3$ we cannot guarantee that e_2 is well defined due to the fourth-order nonlinearity in T (compare [7]).*

Then, the minimization problem (1.4) can be shortly written as

$$
\begin{aligned}
&\min J(x,u) \text{ over } (x,u) \in X_\infty \times U,\\
&\text{subject to } e(x,u) = 0 \ \text{ in } \ Z^*.
\end{aligned}
\tag{1.6}
$$

We require standard regularity properties of the cost functional J:

A.3 Let $J: X\times U\to\mathbb{R}$ denote a cost functional which is assumed to be twice continuously Fréchet differentiable with locally Lipschitz continuous second derivatives. Further, let J be of separated type, i.e. $J(x,u)=J_1(x)+J_2(u)$ and radially unbounded w.r.t. u for every $x\in X$, bounded from below and weakly lower semi-continuous.

REMARK 1.6. *Clearly, the cost functional* (1.3) *fits into this setting.*

The existence of an optimal control as well as the introduction of the reduced cost functional depend crucially on the existence, uniqueness, regularity and bounds for the state system, which are studied in the next section.

2. The state system

Now we give a detailed analysis of the state system (1.2) which is essential for the following investigations. Similar results considering the stationary system with a different set of boundary conditions can be found in [3].

2.1. Existence of uniformly bounded states. The solvability of the state system for every control $u\in U$ and the boundedness of the solution is the content of the following result, which is proved by compactness arguments employing the fixed point theorem of Leray-Schauder [4]. The uniform bounds in $L^\infty(Q)$ are derived by Stampacchia's truncation method [18].

THEOREM 2.1. *Assume* **A.1** *and let* $u\in U$ *and* $T_0\in L^\infty(\Omega)$ *be given. Then, the* SP_1 *system* $e(x,u)=0$*, where* e *is defined by* (1.5)*, has at least one solution* $(T,\rho)\in X_\infty$*, and there exists a constant* $c>0$ *such that the following energy estimate holds:*

$$\|T\|_W+\|\rho\|_V\le c\left\{\|T_0\|^4_{L^\infty(\Omega)}+\|u\|^4_U\right\}. \tag{2.1}$$

Further, the solution is uniformly bounded, i.e. $(T,\rho)\in[L^\infty(Q)]^2$*, and we have*

$$\underline{T}\le T\le\overline{T},\quad \underline{\rho}\le\rho\le\overline{\rho}, \tag{2.2}$$

where

$$\underline{T}=\min\left(\inf_{t\in(0,1)}u(t),\inf_{x\in\Omega}T_0(x)\right)\qquad\text{and}\qquad \overline{T}=\max\left(\sup_{t\in(0,1)}u(t),\sup_{x\in\Omega}T_0(x)\right),$$

as well as $\underline{\rho}=4\pi a\,|\underline{T}|^3\,\underline{T}$ *and* $\overline{\rho}=4\pi a\left|\overline{T}\right|^3\overline{T}$.

Proof. For the proof we employ the fixed point theorem of Leray-Schauder [4, Theorem 11.6, page 286]. Let $w\in L^2(L^2(\Omega))$ and $\sigma\in[0,1]$ be given. Consider the auxiliary problem: Find $(T,\rho)\in X$ with $T(0,x)=\sigma T_0$ in $L^2(\Omega)$ such that

$$\partial_t T=k\Delta T+\frac{\sigma}{3\kappa}\Delta\rho, \tag{2.3a}$$

$$-\varepsilon^2\frac{1}{3\kappa}\Delta\rho+\kappa\rho=\kappa 4\pi a\left|[w]_{\underline{T},\overline{T}}\right|^3[w]_{\underline{T},\overline{T}}, \tag{2.3b}$$

with boundary conditions

$$\alpha T+n\cdot\nabla T=\sigma\alpha u, \tag{2.3c}$$

$$\gamma\rho+n\cdot\nabla\rho=\gamma 4\pi a|u|^3u, \tag{2.3d}$$

is fulfilled in the weak sense. Here, the cut-off operator $[\cdot]_{\underline{T},\overline{T}}: L^2(Q) \to L^2(Q)$ is defined as

$$[w]_{\underline{T},\overline{T}} = \begin{cases} \overline{T}, & w \geq \overline{T}, \\ w, & \overline{T} \geq w \geq \underline{T}, \\ \underline{T}, & w \leq \underline{T}. \end{cases}$$

Note, that the two equations decouple. For given $w \in L^2(Q)$, there exists a unique $\rho \in L^\infty(H^1(\Omega))$ using the Lax-Milgram theorem [4, Theorem 5.8, page 83]. Further, it holds that $\Delta\rho \in V^*$, which implies directly the existence of a unique $T \in W$ [13, Theorem 10.3, page 379]. Thus, the fixed point mapping

$$\begin{aligned} G: L^2(Q) \times [0,1] &\to L^2(Q), \\ (w,\sigma) &\mapsto G(w,\sigma) = T \end{aligned}$$

is well defined.

Now, let $T \in W$ be given with $G(T,\sigma) = T$. First, we exhibit uniform $L^\infty(Q)$-bounds for the solution. Testing the second equation in (2.3) with $\phi = (\rho - \overline{\rho})^+$, where $(\cdot)^+ \stackrel{\text{def}}{=} \max(0,\cdot)$, for $\overline{\rho} > 0$ yields

$$\begin{aligned} &\frac{\varepsilon^2}{3\kappa} \left\| \nabla(\rho - \overline{\rho})^+(t) \right\|^2_{L^2(\Omega)} + \kappa \left\| (\rho - \overline{\rho})^+(t) \right\|^2_{L^2(\Omega)} \\ &= -\kappa \int_\Omega \left(\overline{\rho} - 4\pi a \left| [T]_{\underline{T},\overline{T}} \right|^3 [T]_{\underline{T},\overline{T}} \right) (\rho - \overline{\rho})^+(t)\, dx \\ &\qquad + \frac{\varepsilon}{2} \int_{\partial\Omega} (4\pi a |u|^3 u - \rho)(\rho - \overline{\rho})^+(t)\, ds \\ &\leq -\kappa \int_\Omega \left(\overline{\rho} - 4\pi a \left| \overline{T} \right|^3 \overline{T} \right) (\rho - \overline{\rho})^+(t)\, dx \\ &\qquad + \frac{\varepsilon}{2} \int_{\partial\Omega} (4\pi a \left| \overline{T} \right|^3 \overline{T} - \rho)(\rho - \overline{\rho})^+(t)\, ds \\ &\leq 0, \qquad \text{for all } t \in (0,1), \end{aligned}$$

if we choose especially $\overline{\rho} = 4\pi a \left|\overline{T}\right|^3 \overline{T}$ with $\overline{T} = \max\left(\sup_{t \in (0,1)} u(t), \sup_{x \in \Omega} T_0(x) \right)$. We deduce $(\rho - \overline{\rho})^+ \equiv 0$ a.e. in Q, i.e. $\rho \leq \overline{\rho}$. One proves the lower bound $\rho \geq \underline{\rho}$ analogously.

To get the upper bound for the temperature T we eliminate the Laplacian of ρ in the first equation of system (2.3) and test with $\phi = (T - \overline{T})^+$, which yields

$$\begin{aligned} &\frac{1}{2} \partial_t \left\| (T - \overline{T})^+(t) \right\|^2_{L^2(\Omega)} + k \left\| \nabla (T - \overline{T})^+(t) \right\|^2_{L^2(\Omega)} \\ &= \sigma \frac{\kappa}{\varepsilon^2} \int_\Omega \left(\rho - 4\pi a \left| [T]_{\underline{T},\overline{T}} \right|^3 [T]_{\underline{T},\overline{T}} \right) (T - \overline{T})^+(t)\, dx + \frac{h}{\varepsilon} \int_{\partial\Omega} (\sigma u - T)(T - \overline{T})^+(t)\, ds \\ &\leq \sigma \frac{\kappa}{\varepsilon^2} \int_\Omega \left(\overline{\rho} - 4\pi a \left| \overline{T} \right|^3 \overline{T} \right) (T - \overline{T})^+(t)\, dx + \frac{h}{\varepsilon} \int_{\partial\Omega} \underbrace{(\sigma \overline{T} - T)(T - \overline{T})^+(t)}_{\leq 0}\, ds \\ &\leq 0. \end{aligned}$$

Now, Gronwall's lemma implies the estimate

$$\int_\Omega \left| (T - \overline{T})^+(t) \right|^2 dx \leq \int_\Omega \left| (T - \overline{T})^+(0) \right|^2 dx = \int_\Omega \left| (\sigma T_0 - \overline{T})^+ \right|^2 dx = 0 \text{ for all } t \in (0,1),$$

and hence $(T-\overline{T})^+ \equiv 0$ a.e. in Q, i.e. $T \leq \overline{T}$. In analogy one proves the lower bound $T \geq \underline{T}$.

From these estimates we deduce that every fixed point of $G(\cdot,1)$ is in fact also a solution of (1.5).

Next, we derive an energy estimate which is sufficient to show the compactness of G. Testing the second equation of system (2.3) with ρ we get

$$\begin{aligned}\frac{\varepsilon^2}{3\kappa}\|\nabla\rho\|^2_{L^2(Q)} + \kappa\|\rho\|^2_{L^2(Q)} &= \kappa 4\pi a\left(|T|^3 T,\rho\right)_{L^2(Q)} + \frac{\varepsilon}{2}(4\pi a|u|^3 u - \rho,\rho)_{L^2(\Sigma)}\\ &\leq c_1\left\{\|T\|^4_{L^\infty(Q)}\|\rho\|_{L^2(Q)} + \|u\|^4_{L^\infty(\Sigma)}\|\rho\|_{L^2(\Sigma)}\right\},\end{aligned}$$

where $c_1 > 0$ depends only on the physical parameters and on the domain, and is especially independent of σ. This implies directly that

$$\|\rho\|_V \leq c_2\|T\|^4_{L^\infty(Q)},$$

for some constant $c_2 > 0$ independent of σ. Further, eliminating the Laplacian of ρ and testing the first equation of system (2.3) with T yields

$$\begin{aligned}&\frac{1}{2}\partial_t\|T(t)\|^2_{L^2(\Omega)} + k\|\nabla T(t)\|^2_{L^2(\Omega)} + k\alpha\|T(t)\|^2_{L^2(\partial\Omega)}\\ &\leq k\alpha\|T(t)\|_{L^2(\partial\Omega)}\|u(t)\|_{L^2(\partial\Omega)} + \frac{\kappa}{\varepsilon^2}\|\rho(t)\|_{L^2(\Omega)}\|T(t)\|_{L^2(\Omega)}.\end{aligned}$$

The estimates derived so far ensure that

$$\|T\|_V \leq c_3\|T\|^4_{L^\infty(Q)},$$

where the constant $c_3 > 0$ is again independent of σ.

To prove the estimate on the time derivative $\partial_t T$ we supply $H^{-1}(\Omega)$ with the norm $\left\|\nabla\Delta^{-1}\cdot\right\|_{L^2(\Omega)}$, where $\Delta^{-1}: H^{-1}(\Omega) \to H^1_0(\Omega)$ is the inverse Laplacian [19]. Using $\phi = -\Delta^{-1}\partial_t T$ as a test function for the first equation in system (2.3) and integrating by parts yields

$$\begin{aligned}\left\|\nabla\Delta^{-1}\partial_t T\right\|^2_{L^2(Q)} &= k\left(\nabla T, \nabla(\Delta^{-1}\partial_t T)\right)_{L^2(Q)} + \frac{\sigma}{3\kappa}\left(\nabla\rho, \nabla(\Delta^{-1}\partial_t T)\right)_{L^2(Q)}\\ &\leq \left[k\|\nabla T\|_{L^2(Q)} + \frac{1}{3\kappa}\|\nabla\rho\|_{L^2(Q)}\right]\left\|\nabla(\Delta^{-1}\partial_t T)\right\|_{L^2(Q)}.\end{aligned}$$

Hence, the estimates derived so far ensure

$$\|\partial_t T\|_{V^*} \leq c_4,$$

with $c_4 > 0$ again independent of σ.

Finally, we deduce that there exists a constant $c_5 > 0$, independent of T and σ, such that each T with $G(T,\sigma) = T$ fulfills

$$\|T\|_W \leq c_5.$$

It is easy to verify that the operator G is continuous. From Aubin's Lemma [17] we deduce the compactness of the embedding $W \hookrightarrow L^2(Q)$, which implies the compactness of the fixed point operator G. Furthermore, $G(w,0) = 0$ for all $w \in L^2(Q)$. Now the existence of at least one solution follows from Leray-Schauder's fixed point theorem. $\square$

2.2. Uniqueness of the state. We prove the uniqueness of the state, which will allow finally for the introduction of the reduced cost functional.

THEOREM 2.2. *Assume* **A.1** *and let* $u \in U$ *and* $T_0 \in L^\infty(\Omega)$ *be given. Then, the solution* $(T,\rho) \in X_\infty$ *to the* SP_1*-system* (1.5) *is unique.*

Proof. The uniqueness of the solution is shown by contradiction. Assume that there exist two solutions $(T_i,\rho_i) \in X_\infty$, $i=1,2$. Then the difference $(\hat{T},\hat{\rho}) \stackrel{\text{def}}{=} (T_1 - T_2, \rho_1 - \rho_2)$ solves

$$\partial_t \hat{T} = k\Delta \hat{T} + \frac{1}{3\kappa}\Delta\hat{\rho}, \tag{2.4a}$$

$$-\varepsilon^2 \frac{1}{3\kappa}\Delta\hat{\rho} + \kappa\hat{\rho} = \kappa 4\pi a(|T_1|^3 T_1 - |T_2|^3 T_2), \tag{2.4b}$$

with homogeneous Robin data

$$\alpha\hat{T} + n\cdot\nabla\hat{T} = 0, \tag{2.4c}$$

$$\gamma\hat{\rho} + n\cdot\nabla\hat{\rho} = 0, \tag{2.4d}$$

and homogeneous initial data $\hat{T}(0)=0$. Testing the second equation of system (2.4) with $\hat{\rho}$ yields after integration by parts

$$\frac{\varepsilon^2}{3\kappa}\|\nabla\hat{\rho}\|^2_{L^2(Q)} + \kappa\|\hat{\rho}\|^2_{L^2(Q)} \le \kappa 4\pi a(\hat{\rho}, |T_1|^3 T_1 - |T_2|^3 T_2)_{L^2(Q)},$$

from which we get

$$\|\hat{\rho}\|_V \le c_1 \|T\|^3_{L^\infty(Q)} \left\|\hat{T}\right\|_{L^2(Q)}$$

for some constant $c_1 > 0$. Now we eliminate the Laplacian of $\hat{\rho}$ in the first equation of (2.4) and use $\hat{T}$ as a test function. Employing the monotonicity of the nonlinearity we deduce for all $t \in (0,1)$ that it holds that

$$\begin{aligned}\frac{1}{2}\partial_t \left\|\hat{T}(t)\right\|^2_{L^2(\Omega)} + k\left\|\nabla\hat{T}(t)\right\|^2_{L^2(\Omega)} &\le \frac{\kappa}{\varepsilon^2}\int_\Omega (\hat{\rho}(t) - 4\pi a(|T_1|^3 T_1 - |T_2|^3 T_2)(t))\hat{T}(t)\,dx \\ &\le c_2\|\hat{\rho}(t)\|_{L^2(\Omega)}\left\|\hat{T}(t)\right\|_{L^2(\Omega)} \\ &\le c_3\left\|\hat{T}(t)\right\|^2_{L^2(\Omega)}\end{aligned}$$

for some positive constants c_2, c_3. Making use of Gronwall's Lemma, the homogeneous initial condition implies that

$$\left\|\hat{T}(t)\right\|_{L^2(\Omega)} = 0 \quad \text{for all } t \in (0,1),$$

which directly yields $\hat{T}=0$ a.e. in Q as well as $\hat{\rho}=0$ a.e. in Q. Hence, the solution is unique. ∎

REMARK 2.3. *Note that due to Theorem 2.1 we know that the unique solution in fact fulfills the desired a priori bounds, i.e., for positive boundary data* u *and initial data* T_0 *we can deduce that the temperature is always positive.*

Remark 2.4. *Due to Theorem 2.1 and Theorem 2.2 we can rewrite the minimization problem* (1.6) *equivalently introducing the* reduced cost functional $\hat{J}(u) \stackrel{\text{def}}{=} J(x(u),u)$ *as*

$$\min \hat{J}(u) \text{ over } u \in U, \qquad \text{where } x(u) \in X_\infty \text{ satisfies } e(x(u),u) = 0. \tag{2.5}$$

3. Existence of an optimal control

In this section we establish the existence of a solution to the optimal control problem (1.6).

Theorem 3.1. *Assume* **A.1** *and* **A.3**. *Then, there exists a minimizer* $(x^*,u^*) \in X_\infty \times U$ *of the constrained minimization problem* (1.6).

Proof. By **A.3** we have $J_0 \stackrel{\text{def}}{=} \inf_{X_\infty \times U} J(x,u) > -\infty$. We choose a minimizing sequence $(x_k,u_k)_{k\in\mathbb{N}} \in X_\infty \times U$. Then, the radial unboundedness of J with respect to u implies that $(u_k)_{k\in\mathbb{N}}$ is bounded in U. Hence, there exists a weakly convergent subsequence, again denoted by $(u_k)_{k\in\mathbb{N}}$ such that

$$u_k \rightharpoonup u^* \quad \text{weakly in } U$$

for $k \to \infty$. From Sobolev's embedding theorem [1] we deduce that up to a subsequence we also have $u_k \to u^*$ strongly in $C^0(0,1;\mathbb{R})$ for $k \to \infty$. Now, the bounds stated in Theorem 2.1 imply the boundedness of $(\|x_k\|_X)_{k\in\mathbb{N}}$. Hence, there exist subsequences such that

$$\begin{aligned} T_k &\rightharpoonup T^* && \text{weakly in } V, \\ \partial_t T_k &\rightharpoonup \partial_t T^* && \text{weakly in } V^*, \\ \rho_k &\rightharpoonup \rho^* && \text{weakly in } V, \end{aligned}$$

for $k \to \infty$, i.e. $x_k = (T_k,\rho_k) \rightharpoonup (T^*,\rho^*) = x^*$ weakly in $W \times V$. The weak lower semi-continuity of J implies

$$J(x^*,u^*) = J_0.$$

Finally, we have to show the constraint $e(x^*,u^*) = 0$. Aubin's Lemma [17] implies the strong convergence of $(T_k)_{k\in\mathbb{N}}$ in $L^2(0,1;L^2(\Omega))$. Further, note the uniform boundedness of the solution, which yields

$$(T_k,\rho_k) \rightharpoonup (T^*,\rho^*), \quad \text{weakly-* in } L^\infty(Q),$$

for $k \to \infty$. These convergences are by far sufficient to pass to the limit in (1.5), yielding

$$e(x^*,u^*) = 0 \quad \text{in } Z^*,$$

which finally proves the assertion. □

Remark 3.2. *In general, we cannot expect the uniqueness of an optimal control u, since the set of states given by the constraint e is not convex. Only for cases where δ is large we can overcome this problem.*

4. The linearized state system

This section is devoted to the study of the linearization of the state system (1.2). Let $x=(T,\rho)\in X_\infty$ be given. We define the linear operator $\tilde{A}(x)\in\mathcal{L}(X_\infty,Z^*)$ by

$$\tilde{A}(x)v \stackrel{\text{def}}{=} \begin{pmatrix} \partial_t v_T - k\Delta v_T - \frac{1}{3\kappa}\Delta v_\rho \\ -\frac{\varepsilon^2}{3\kappa}\Delta v_\rho + \kappa v_\rho - \kappa 16\pi a |T|^3 v_T \\ v_T(0) \end{pmatrix}, \qquad \text{for } v=(v_T,v_\rho)\in X_\infty,$$

as well as its natural extension $A(x)\in\mathcal{L}(X,Z^*)$ for a given $x\in X_\infty$. Given $g=(g_T,g_\rho,v_0)^T\in Z^*$, we say that $v\in X$ solves

$$A(x)v = \begin{pmatrix} g_T \\ g_\rho \\ v_0 \end{pmatrix} \qquad \text{in } Z^*,$$

iff v is a variational solution of the linear system

$$\partial_t v_T - k\Delta v_T - \frac{1}{3\kappa}\Delta v_\rho = g_T, \tag{4.1a}$$

$$-\frac{\varepsilon^2}{3\kappa}\Delta v_\rho + \kappa v_\rho - \kappa 16\pi a |T|^3 v_T = g_\rho, \tag{4.1b}$$

supplemented with boundary conditions

$$\alpha v_T + n\cdot\nabla v_T = 0, \tag{4.1c}$$

$$\gamma v_\rho + n\cdot\nabla v_\rho = 0, \tag{4.1d}$$

and initial condition

$$v_T(0) = v_0. \tag{4.1e}$$

4.1. Existence and uniqueness. The existence of a unique solution to (4.1) is the content of the following result.

THEOREM 4.1. *Assume* **A.1** *and* **A.2**. *Let $x\in X_\infty$, $v_0\in L^2(\Omega)$ and $(g_T,g_\rho)\in V^*\times V^*$ be given. Then, there exists a unique $v\in X$ fulfilling* (4.1). *Further, there exists a constant $C>0$ such that*

$$\|v\|_X + \|v\|_{L^\infty(L^2)} \le C\left\{\|v_0\|_{L^2(\Omega)} + \|g_T\|_{V^*} + \|g_\rho\|_{V^*}\right\}.$$

The proof of Theorem 4.1 relies on the reformulation of (4.1) as one linear parabolic equation and the derivation of a Gårding inequality. We write (4.1) in weak form: Find $v\in X$ with $v_T(0)=v_0$ in $L^2(\Omega)$ such that

$$\begin{aligned}\langle \partial_t v_T, \phi_T\rangle_{V^*,V} + k(\nabla v_T,\nabla\phi_T)_{L^2(Q)} + \frac{1}{3\kappa}(\nabla v_\rho,\nabla\phi_T)_{L^2(Q)} \\ + k\alpha(v_T,\phi_T)_{L^2(\Sigma)} + \frac{\gamma}{3\kappa}(v_\rho,\phi_T)_{L^2(\Sigma)} = \langle g_T,\phi_T\rangle_{V^*,V}\end{aligned}$$

and

$$\frac{\varepsilon^2}{3\kappa}(\nabla v_\rho,\nabla\phi_\rho)_{L^2(Q)} + \kappa\left(v_\rho - 16\pi a|T|^3 v_T,\phi_\rho\right)_{L^2(Q)} + \frac{\varepsilon^2\gamma}{3\kappa}(v_\rho,\phi_\rho)_{L^2(\Sigma)} = \langle g_\rho,\phi_\rho\rangle_{V^*,V}$$

for all $\phi = (\phi_T, \phi_\rho) \in V^2$. We define the operator $\Psi : H^{-1}(\Omega) \to H^1(\Omega)$, where $\Psi = \Psi[f]$ solves

$$-\frac{\varepsilon^2}{3\kappa}\Delta\Psi + \kappa\Psi = f \quad \text{in } \Omega,$$
$$\gamma\Psi + n\cdot\nabla\Psi = 0 \quad \text{on } \partial\Omega.$$

Due to standard results [4, Theorem 5.8, page 83] this operator is well defined and there exists a positive constant $c = c(\Omega)$ such that we have the estimate $\|\Psi\|_{H^1(\Omega)} \le c\|f\|_{H^{-1}(\Omega)}$.

Next, we define the bilinear form $a : V\times V \to \mathbb{R}$ via

$$\begin{aligned} a(r,\phi) = k(\nabla r, \nabla\phi)_{L^2(Q)} &+ \frac{1}{3\kappa}\left(\nabla\Psi[\kappa 16\pi a|T|^3 r], \nabla\phi\right)_{L^2(Q)} \\ &+ k\alpha(r,\phi)_{L^2(\Sigma)} + \frac{\gamma}{3\kappa}\left(\Psi[\kappa 16\pi a|T|^3 r], \phi\right)_{L^2(\Sigma)}. \end{aligned} \tag{4.2}$$

This bilinear form is well defined, bounded and fulfills a Gårding inequality.

LEMMA 4.2. *The bilinear form a defined by* (4.2) *is bounded on $V\times V$, i.e., there exists a constant $C > 0$ such that*

$$|a(r,s)| \le C\|r\|_V \|s\|_V \quad \text{for all } r,s \in V.$$

Moreover, there exist constants $\mu,\eta > 0$ such that

$$a(r,r) \ge \mu\|r\|_V^2 - \eta\|r\|_{L^2(Q)}^2 \quad \text{for all } r \in V.$$

Proof. First, we prove the boundedness of the bilinear form a employing the Cauchy-Schwarz inequality

$$\begin{aligned} |a(r,s)| \le k\|\nabla r\|_{L^2(Q)}\|\nabla s\|_{L^2(Q)} &+ \frac{1}{3\kappa}\|\nabla\Psi\|_{L^2(Q)}\|\nabla s\|_{L^2(Q)} \\ &+ k\alpha\|r\|_{L^2(\Sigma)}\|s\|_{L^2(\Sigma)} + \frac{\gamma}{3\kappa}\|\Psi\|_{L^2(\Sigma)}\|s\|_{L^2(\Sigma)} \\ \le C\|r\|_V\|s\|_V, \end{aligned}$$

for some positive constant C depending only on the data, where $\Psi = \Psi[\kappa 16\pi a|T|^3 r]$ for notational convenience. Here, we used the continuity of the trace operator $tr : V \to L^2(\Sigma)$ to estimate $\|\Psi\|_{L^2(\Sigma)}$ by $\|\nabla\Psi\|_{L^2(Q)}$, as well as **A.2**, which yields

$$\|\nabla\Psi\|_{L^2(Q)} \le \|\Psi\|_{L^2(H^2)} \le K\|r\|_{L^2(L^2)},$$

where $K = K(T) > 0$. The Gårding inequality is derived using Young's inequality and **A.2**, yielding

$$\begin{aligned} a(r,r) &\ge k\|\nabla r\|_{L^2(Q)}^2 - \frac{1}{3\kappa}\|\nabla\Psi\|_{L^2(Q)}\|\nabla r\|_{L^2(Q)} + k\alpha\|r\|_{L^2(\Sigma)}^2 - \frac{\gamma}{3\kappa}\|\Psi\|_{L^2(\Sigma)}\|r\|_{L^2(\Sigma)} \\ &\ge \frac{k}{2}\|\nabla r\|_{L^2(Q)}^2 - \frac{1}{18k\kappa^2}\|\nabla\Psi\|_{L^2(Q)}^2 + \frac{k\alpha}{2}\|r\|_{L^2(\Sigma)}^2 - \frac{\gamma^2}{18k\kappa^2\alpha}\|\Psi\|_{L^2(\Sigma)}^2 \\ &\ge \frac{k}{2}\|\nabla r\|_{L^2(Q)}^2 - c(\Omega,a,k,\alpha,\gamma,T)\|r\|_{L^2(Q)}^2 \\ &\ge \mu\|r\|_V^2 - \eta\|r\|_{L^2(Q)}^2, \end{aligned}$$

with $\mu = k/2$ and $\eta = c(\Omega, a, k, \alpha, \gamma, T) + k/2$. ∎

Now, we are in the position to prove the main theorem of this section.

Proof of Theorem 4.1.

Proof. We rewrite (4.1) as one equation for v_T using the bilinear form a which yields: Find $v_T \in V$ such that $v_T(0) = v_0$ in the sense of $L^2(\Omega)$ and

$$\langle \partial_t v_T, \phi \rangle_{V^*,V} + a(v_T, \phi) = \left\langle g_T + \frac{1}{\varepsilon^2}\left(\kappa \Psi[g_\rho] - g_\rho\right), \phi \right\rangle_{V^*,V} \qquad \text{for all } \phi \in V. \tag{4.3}$$

Due to Lemma 4.2 we have the boundedness and weak coercivity of a, and the continuity of the right hand side is immediate, such that standard results for linear parabolic equations [13, Theorem 10.3, page 379] imply that there exists a unique solution $v_T \in W$ with $v_T(0) = v_0$ in $L^2(\Omega)$. Hence, also (4.1) is uniquely solvable and the solution is given by $v = (v_T, v_\rho) \stackrel{\text{def}}{=} (v_T, \Psi[16\kappa\pi a |T|^3 v_T + g_\rho]) \in X$.

Finally, we derive the energy estimate. In the following let $c_i > 0$, $i = 1, \ldots, 9$, denote constants depending only on the data. Testing (4.3) with $\phi = v_T$, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t \|v_T(t)\|^2_{L^2(\Omega)} + k\|\nabla v_T(t)\|^2_{L^2(\Omega)} + k\alpha \|\nabla v_T(t)\|^2_{L^2(\Sigma)} \\
&- \frac{\kappa}{\varepsilon^2}\Big(\Psi[16\kappa\pi a|T|^3 v_T(t) + g_\rho(t)], v_T(t)\Big)_{L^2(\Omega)} + \frac{1}{\varepsilon^2}\Big(g_\rho + 16\kappa\pi a|T|^3 v_T(t), v_T(t)\Big)_{L^2(Q)} \\
&= \langle g_T(t), v_T(t)\rangle_{H^{-1}(\Omega), H^1(\Omega)}.
\end{aligned}$$

Employing **A.2** and Young's inequality we have the estimates

$$\begin{aligned}
&\Big(\Psi[16\kappa\pi a|T|^3 v_T(t) + g_\rho], v_T(t)\Big)_{L^2(Q)} \\
&\le c_1 \Big\{ \|v_T(t)\|^2_{L^2(\Omega)} + \|g_\rho(t)\|_{H^{-1}(\Omega)} \|v_T(t)\|_{H^1(\Omega)} \Big\} \\
&\le c_2 \|v_T(t)\|^2_{L^2(\Omega)} + \frac{k}{4}\|\nabla v_T(t)\|^2_{L^2(\Omega)} + c_3(k)\|g_\rho(t)\|^2_{H^{-1}(\Omega)}.
\end{aligned}$$

Employing an analogous estimate for the right hand side, this yields

$$\begin{aligned}
&\frac{1}{2}\partial_t \|v_T(t)\|^2_{L^2(\Omega)} + \frac{k}{2}\|\nabla v_T(t)\|^2_{L^2(\Omega)} \\
&\le c_4 \|v_T(t)\|^2_{L^2(\Omega)} + c_5 \Big\{ \|g_T(t)\|^2_{H^{-1}(\Omega)} + \|g_\rho(t)\|^2_{H^{-1}(\Omega)} \Big\}.
\end{aligned}$$

From Gronwall's Lemma we get immediately

$$\|v_T\|_{L^\infty(L^2(\Omega))} \le c_6 \Big\{ \|v_0\|_{L^2(\Omega)} + \|g_T\|_{V^*} + \|g_\rho\|_{V^*} \Big\}$$

and further

$$\|v_T\|_V \le c_7 \Big\{ \|v_0\|_{L^2(\Omega)} + \|g_T\|_{V^*} + \|g_\rho\|_{V^*} \Big\}.$$

Finally, using $\phi = \nabla\Delta^{-1} v_T$ as a test function and following the argument in the proof of Theorem 2.1, we get

$$\|\partial_t v_T\|_{V^*} \le c_8,$$

which altogether yields

$$\|v\|_X \le c_9 \left\{ \|v_0\|_{L^2(\Omega)} + \|g_T\|_{V^*} + \|g_\rho\|_{V^*} \right\}. \qquad \square$$

In view of Theorem 4.1 we have

COROLLARY 4.3. *Let $(x,u)\in X_\infty \times U$ be given. Then $e_x(x,u): X \to Z^*$ is a homeomorphism.*

4.2. Regularity. For more regular data we expect that the solution of the linearized system has also a higher regularity. We show that uniformly bounded data implies that also the linearized solution is bounded.

THEOREM 4.4. *Assume* **A.1** *and* **A.2**. *Let $x\in X_\infty$, $v_0 \in L^\infty(\Omega)$ and $(g_T,g_\rho)\in [L^\infty(Q)]^2$ be given. Then, the unique solution $v\in X$ of (4.1) is in fact uniformly bounded, i.e. $v\in[L^\infty(Q)]^2$.*

For the proof we use Moser's iteration technique [4, page 188].

Proof. For $l\in\mathbb{N}$ and $p>1$ we define $[s]_l = \min(l,\max(-l,s))$ and $\Phi_l(s) = |[s]_l|^{p-2}[s]_l$. Note that it holds that $\Phi_l \in H^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $\Phi_l'(s)\ge 0$ a.e. in $\mathbb{R}$, and

$$\int_0^s \Phi_l(z)\,dz \ge \frac{1}{p}|[s]_l|^p.$$

We use $\Phi_l(v_T)\in V$ as a test function in (4.1) and get

$$\begin{aligned}
&\langle \partial_t v_T, \Phi_l(v_T)\rangle_{H^{-1}(\Omega),H^1(\Omega)} + k\int_\Omega \Phi_l'(v_T)\,|\nabla v_T|^2\,dx \\
&-\frac{1}{3\kappa}\int_\Omega \Delta\Psi[\kappa 16\pi a|T|^3 v_T + g_\rho]\Phi_l(v_T)\,dx + k\alpha\int_{\partial\Omega} v_T\Phi_l(v_T)\,ds = \int_\Omega g_T\,\Phi_l(v_T)\,dx
\end{aligned}$$

Further, we have

$$\begin{aligned}
&-\frac{1}{3\kappa}\int_\Omega \Delta\Psi[\kappa 16\pi a|T|^3 v_T + g_\rho]\Phi_l(v_T)\,dx \\
&= \frac{1}{\varepsilon^2}\int_\Omega \kappa 16\pi a|T|^3 v_T\Phi_l(v_T) + (g_\rho - \kappa\Psi)\Phi_l(v_T)\,dx \\
&\ge -\frac{1}{\varepsilon^2}\int_\Omega |\kappa\Psi + g_\rho|\,|[v_T]_l|^{p-1}\,dx.
\end{aligned}$$

Using Young's inequality

$$a^{p-1}b \le \frac{p-1}{p}a^p + \frac{1}{p}b^p, \quad a,b\ge 0, \quad p>1$$

and due to Assumption **A.2**, we get

$$\begin{aligned}
\int_\Omega |\kappa\Psi + g_\rho|\,|[v_T]_l|^{p-1}\,dx &\le \frac{\kappa}{p}\int_\Omega |\Psi|^p\,dx + \frac{1}{p}\int_\Omega |g_\rho|^p\,dx + \frac{p-1}{p}\int_\Omega |[v_T]_l|^p\,dx \\
&\le \frac{\kappa K(\Omega)^p}{p}\|v_T\|^p_{L^2(\Omega)} + \frac{1}{p}\|g_\rho\|^p_{L^p(\Omega)} + \|[v_T]_l\|^p_{L^p(\Omega)}.
\end{aligned}$$

In analogy we have

$$\int_\Omega g_T\,\Phi_l(v_T)\,dx \le \frac{1}{p}\,\|g_T\|^p_{L^p(\Omega)} + \frac{p-1}{p}\,\|[v_T]_l\|^p_{L^p(\Omega)}.$$

Combining all these estimates and integration with respect to t yields

$$\begin{aligned}\frac{1}{p}\,\|[v_T(t)]_l\|^p_{L^p(\Omega)} &\le \frac{1}{p}\,\|[v_0]_l\|^p_{L^p(\Omega)} + \left(1+\frac{1}{\varepsilon^2}\right)\int_0^t \|[v_T(\tau)]_l\|^p_{L^p(\Omega)}\,d\tau\\ &+\frac{1}{p}\left\{\int_0^t \|g_T(\tau)\|^p_{L^p(\Omega)}\,d\tau + \frac{1}{\varepsilon^2}\int_0^t \|g_\rho(\tau)\|^p_{L^p(\Omega)}\,d\tau + \frac{\kappa K(\Omega)^p}{\varepsilon^2}\int_0^t \|v_T(\tau)\|^p_{L^2(\Omega)}\,d\tau\right\},\end{aligned}$$

or

$$\begin{aligned}\frac{1}{p}\,\|[v_T(t)]_l\|^p_{L^p(\Omega)} &\le \left(1+\frac{1}{\varepsilon^2}\right)\int_0^t \|[v_T(\tau)]_l\|^p_{L^p(\Omega)}\,d\tau\\ &\quad+\frac{K_2^p}{p}\left\{\|v_0\|^p_{L^\infty(\Omega)} + \|g_T\|^p_{L^\infty(Q)} + \|g_\rho\|^p_{L^\infty(Q)} + \|v_T\|^p_{L^\infty(L^2)}\right\},\end{aligned}$$

for some $K_2 = K_2(\Omega,\kappa,\varepsilon) > 0$. Now, Gronwall's Lemma and Theorem 4.1 imply that

$$\|[v_T(t)]_l\|^p_{L^p(\Omega)} \le K_2^p\left\{\|v_0\|^p_{L^\infty(\Omega)} + \|g_T\|^p_{L^\infty(Q)} + \|g_\rho\|^p_{L^\infty(Q)}\right\} e^{(1+1/\varepsilon^2)pt},$$

for all $t\in[0,1]$ and $p\ge 1$. Finally, we can go to the limit $l\to\infty$ and get

$$\|v_T(t)\|_{L^p(\Omega)} \le K_2\left\{\|v_0\|_{L^\infty(\Omega)} + \|g_T\|_{L^\infty(Q)} + \|g_\rho\|_{L^\infty(Q)}\right\} e^{(1+1/\varepsilon^2)t},$$

for some constant K_2, independent of p. Now, we let $p\to\infty$ and get $v_T\in L^\infty(Q)$. The boundedness of v_ρ follows now from standard results. ∎

5. Adjoints and derivatives

In this section we want to identify the adjoint system and prove the existence and uniqueness of the adjoint states.

THEOREM 5.1. *Assume* **A.1**-**A.3** *and let $x\in X_\infty$ be given. Then, for every $f=(f_T,f_\rho)\in X^*$ the adjoint equation*

$$A(x)^*\xi = f \qquad \text{in } X^*$$

possesses a unique variational solution $\xi=(\xi_T,\xi_\rho,\xi_0)\in Z$. Furthermore, if $f\in V^*\times V^*$, then we have that $(\xi_T,\xi_\rho)\in X$, and ξ can be characterized as the variational solution of

$$-\partial_t\xi_T - k\,\Delta\xi_T - 16\pi a\kappa\,|T|^3\,\xi_\rho = f_T, \tag{5.1a}$$

$$-\frac{\varepsilon^2}{3\kappa}\,\Delta\xi_\rho + \kappa\xi_\rho - \frac{1}{3\kappa}\,\Delta\xi_T = f_\rho \qquad \text{in } Q \tag{5.1b}$$

with boundary conditions

$$k(n\cdot\nabla\xi_T + \alpha\xi_T) = 0, \tag{5.1c}$$

$$n\cdot\nabla\xi_T + \gamma\xi_T + \varepsilon^2(n\cdot\nabla\xi_\rho + \gamma\xi_\rho) = 0 \qquad \text{on } \Sigma \tag{5.1d}$$

and terminal condition

$$\xi_T(1)=0 \quad \text{in } \Omega. \tag{5.1e}$$

Moreover, $\xi_T(0)=\xi_0$, *and we have the following a priori estimate:*

$$\|\xi_T\|_V+\|\xi_\rho\|_V\le C\|f\|_{X^*}.$$

For $f\in V^*\times V^*$ *it even holds that*

$$\|\xi\|_X\le C\|f\|_{V^*\times V^*}.$$

Proof. From Theorem 4.1 we learn that, given $x\in X_\infty$, the linear operator $A(x)$ possesses a bounded inverse $A(x)^{-1}\in\mathcal{L}(Z^*,X)$. A direct calculation leads to the adjoint operator

$$\begin{aligned}
&\langle A(x)v,\xi\rangle_{Z^*,Z}\\
&=\langle v,A(x)^*\xi\rangle_{X,X^*}\\
&=\langle\partial_t v_T,\xi_T\rangle_{V^*,V}\\
&\quad+\left\langle v_T,-k\Delta\xi_T-\kappa 16\pi a|T|^3\xi_\rho\right\rangle_{V,V^*}+\left\langle v_\rho,-\frac{\varepsilon^2}{3\kappa}\Delta\xi_\rho+\kappa\xi_\rho-\frac{1}{3\kappa}\Delta\xi_T\right\rangle_{V,V^*}\\
&\quad+k\alpha(v_T,\xi_T)_{L^2(\Sigma)}+\frac{\gamma}{3\kappa}(v_\rho,\varepsilon^2\xi_\rho)_{L^2(\Sigma)}+\frac{\gamma}{3\kappa}(v_\rho,\xi_T)_{L^2(\Sigma)}\\
&\quad+k\langle v_T,n\cdot\nabla\xi_T\rangle_{L^2(H^{1/2}(\partial\Omega)),L^2(H^{-1/2}(\partial\Omega))}\\
&\quad+\frac{\varepsilon^2}{3\kappa}\langle v_\rho,n\cdot\nabla\xi_\rho\rangle_{L^2(H^{1/2}(\partial\Omega)),L^2(H^{-1/2}(\partial\Omega))}\\
&\quad+\frac{1}{3\kappa}\langle v_\rho,n\cdot\nabla\xi_T\rangle_{L^2(H^{1/2}(\partial\Omega)),L^2(H^{-1/2}(\partial\Omega))}\\
&\quad+(v_T(0),\xi_0)_{L^2(\Omega)}
\end{aligned}$$

for every $v\in X$.

Since $A^{-*}(x)\in\mathcal{L}(X^*,Z)$, we find for every $f=(f_T,f_\rho)\in X^*$ a unique solution $\xi=(\xi_T,\xi_\rho,\xi_0)\in Z$ of

$$\langle v,A(x)^*\xi\rangle_{X,X^*}=\langle v,f\rangle_{X,X^*}$$

for all $v\in X$.

Combining the bounded invertibility of $A(x)$ with the norm identity $\|A^{-1}(x)\|_{\mathcal{L}(Z^*,X)}=\|A^{-*}(x)\|_{\mathcal{L}(X^*,Z)}$, we get

$$\|\xi\|_Z\le c\|f\|_{X^*} \tag{5.2}$$

for some constant $c>0$.

Now, assume that the right hand side fulfills $f\in V^*\times V^*$. Then, the function

$$t\mapsto B(t)\overset{\text{def}}{=}(k\Delta\xi_T+\kappa 16\pi a|T|^3\xi_\rho+f_T)(t)$$

is in V^*. Let $\partial_t\xi_T$ be the distributional derivative of ξ_T and extend the inner product $(\cdot,\cdot)_{L^2(\Omega)}$ continuously to $H^{-1}(\Omega)\times H^1(\Omega)$. Then $\partial_t\xi_T\in V^*$, which can be seen as follows. Testing appropriately yields

$$-\left(\int_0^1\partial_t\xi_T\chi\,dt,h\right)_{L^2(\Omega)}=\left(\int_0^1 B(t)\chi\,dt,h\right)_{L^2(\Omega)},\quad \text{for all } \chi\in C_0^\infty(0,1),h\in H^1(\Omega),$$

and using a density argument we get $\partial_t \xi_T \in V^*$. Due to (5.2) we finally have $\xi \in X \times L^2(\Omega)$, and standard regularity theory implies $\xi_T \in C^0([0,1], H^{-1}(\Omega))$. Note that ξ_T is well defined in $H^{-1}(\Omega)$. Hence, by means of the Gelfand triple, we have $\xi_T \in C^0([0,1], L^2(\Omega))$, which leads to the terminal condition $\xi_T(1) = 0$ as well as $\xi_T(0) = \xi_0$. ∎

5.1. Derivatives. In this section we study the differentiability properties of the mapping e defined in Section 2, which are necessary for superlinear numerical algorithms, like SQP or Newton-like methods [11]. Further, we introduce the reduced cost functional $\hat{J}(u) \stackrel{\text{def}}{=} J(x(u), u)$ and derive a representations for its first variation, which is necessary for an appropriate numerical treatment [12, 15].

THEOREM 5.2. *The mapping $e = (e_1, e_2, e_3) : X_\infty \times U \to Z^*$ is twice continuously Fréchet-differentiable with locally Lipschitz-continuous second derivative. The action of the first two derivatives at $(x,u) \in X_\infty \times U$ in the direction $\tilde{x} \stackrel{\text{def}}{=} (\tilde{T}, \tilde{\rho})$ or $(\tilde{x}, \hat{x}) \stackrel{\text{def}}{=} ((\tilde{T}, \tilde{\rho}), (\hat{T}, \hat{\rho})) \in X_\infty^2$, respectively, is given by*

$$\begin{aligned}
\langle e_{1x}(x,u)\tilde{x}, \phi_T \rangle_{V^*,V} &= \left\langle \partial_t \tilde{T}, \phi_T \right\rangle_{V^*,V} + k \left(\nabla \tilde{T}, \nabla \phi_T \right)_{L^2(Q)} + \frac{1}{3\kappa} (\nabla \tilde{\rho}, \nabla \phi_T)_{L^2(Q)} \\
&\quad + k\alpha \left(\tilde{T}, \phi_T \right)_{L^2(\Sigma)} + \frac{\gamma}{3\kappa} (\tilde{\rho}, \phi_T)_{L^2(\Sigma)}, \\
\langle e_{2x}(x,u)\tilde{x}, \phi_\rho \rangle_{V^*,V} &= \frac{\varepsilon^2}{3\kappa} (\nabla \tilde{\rho}, \nabla \phi_\rho)_{L^2(Q)} + \kappa \left(\tilde{\rho} - 16\pi a \, |T|^3 \tilde{T}, \phi_\rho \right)_{L^2(Q)} \\
&\quad + \frac{\varepsilon^2}{3\kappa} \gamma (\tilde{\rho}, \phi_\rho)_{L^2(\Sigma)}
\end{aligned}$$

and $e_{1xx} = 0$, as well as

$$\langle e_{2xx}(x,u)[\tilde{x}, \hat{x}], \phi_\rho \rangle = -\kappa 48\pi a \left(T|T| \, \tilde{T}\hat{T}, \phi_\rho \right)_{L^2(Q)},$$

for all $\phi = (\phi_T, \phi_\rho) \in X^2$. Further, we have for $\tilde{u} \in U$ that

$$\langle e_{1u}(x,u)\tilde{u}, \phi_T \rangle_{V^*,V} = -k\alpha (\tilde{u}, \phi_T)_{L^2(\Sigma)} - \frac{\gamma}{3\kappa} \left(16\pi a \, |u|^3 \, \tilde{u}, \phi_T \right)_{L^2(\Sigma)}$$

and

$$\langle e_{2u}(x,u)\tilde{u}, \phi_\rho \rangle_{V^*,V} = -\frac{\varepsilon^2 \gamma}{3\kappa} \left(16\pi a \, |u|^3 \, \tilde{u}, \phi_\rho \right)_{L^2(\Sigma)}.$$

Next, we compute the derivative of the reduced functional $\hat{J}$. For this we need the differentiability of the mapping $u \mapsto x(u)$, which is the content of the following theorem.

THEOREM 5.3. *Assume **A.1**, **A.2** and let $T_0 \in L^\infty(\Omega)$. Then, the mapping $u \mapsto x(u)$ is Fréchet-differentiable as a mapping from U to X_∞ and its derivative is given by*

$$x'(u) = -e_x^{-1}(x(u), u) e_u(x(u), u).$$

Proof. We split the operator $e(x,u)$ into its linear part D and its nonlinear part N acting on x, as well as a nonlinear operator B acting on u, i.e.,

$$e(x,u)=Dx+N(x)+B(u),$$

where $D:X\to Z^*$, $N:X_\infty\to Y_Q:=[L^\infty(Q)]^3$ and $B:U\to Y_\Sigma:=[L^\infty(\Sigma)]^3$ are defined by

$$\begin{aligned}
\langle D_1(x),\phi\rangle_{V^*,V} &\overset{\text{def}}{=} \langle\partial_t T,\phi\rangle_{V^*,V}+k(\nabla T,\nabla\phi)_{L^2(Q)}-\frac{\kappa}{\varepsilon^2}(\rho,\phi)_{L^2(Q)}+k\alpha(T,\phi)_{L^2(\Sigma)},\\
\langle D_2(x),\phi\rangle_{V^*,V} &\overset{\text{def}}{=} \frac{\varepsilon^2}{3\kappa}(\nabla\rho,\nabla\phi)_{L^2(Q)}+\kappa(\rho,\phi)_{L^2(Q)}+\frac{\varepsilon^2}{3\kappa}\gamma(\rho,\phi)_{L^2(\Sigma)},\\
\langle N_1(x),\phi\rangle_{V^*,V} &\overset{\text{def}}{=} \frac{\kappa}{\varepsilon^2}(4\pi\kappa a\,|T|^3T,\phi)_{L^2(Q)},\\
\langle N_2(x),\phi\rangle_{V^*,V} &\overset{\text{def}}{=} -\kappa(4\pi\kappa a\,|T|^3T,\phi)_{L^2(Q)},\\
\langle B_1(u),\phi\rangle_{V^*,V} &\overset{\text{def}}{=} -k\alpha(u,\phi)_{L^2(\Sigma)},\\
\langle B_2(u),\phi\rangle_{V^*,V} &\overset{\text{def}}{=} -\frac{\varepsilon^2}{3\kappa}\gamma(4\pi a|u|^3u,\phi)_{L^2(\Sigma)},
\end{aligned}$$

for $\phi\in V$, as well as $D_3(x)\overset{\text{def}}{=}T(0)-T_0$, $N_3(x)=B_3(u)=0$. Note that we used the definition of e_2 to rewrite e_1, as in the proof of Theorem 2.1.

The linear operator D is boundedly invertible by linear elliptic/parabolic theory [13, Theorem 10.3, page 379], since the solution of $Dx=z$ requires just the solution of two decoupled linear problems. By the weak maximum principle we even get that $D^{-1}\in\mathcal{L}(Y_Q,X_\infty)$ and $D^{-1}\in\mathcal{L}(Y_\Sigma,X_\infty)$. We define the operator $R:X_\infty\times U\to X_\infty$ by

$$R(x,u)=x+D^{-1}N(x)+D^{-1}B(u).$$

Then, $e(x(u),u)=0$ is equivalent with $R(x(u),u)=0$. To show the Fréchet-differentiability of $u\mapsto x(u)$, we apply the implicit function theorem to R. First note that R is continuously Fréchet-differentiable, since $N:X_\infty\to Y_Q$ and $B:U\to Y_\Sigma$ are continuously Fréchet-differentiable. The linear operator D^{-1} is clearly also continuously Fréchet-differentiable, so we can apply the chain rule (see also [10]).

Next, we need to show the invertibility of $R_x(x,u)$ for given $(x,u)\in X_\infty\times U$. Let $g\in X_\infty$ be given. We will show that there exists a unique $w\in X_\infty$ with $R_x(x,u)w=g$. This equation is equivalent to

$$w+D^{-1}N_x(x)w=g \quad \text{in } X_\infty,$$

or, setting $v=w-g$, we get

$$v+D^{-1}N_x(x)(v+g)=0 \quad \text{in } X_\infty.$$

This can be written as

$$Dv+N_x(x)v=-N_x(x)g \quad \text{in } Z^*,$$

which just corresponds with the linearized state system, i.e. $A(x)v=-N_x(x)g$. Since $N_x(x)g\in Y_Q$, we get from Theorem 4.1 and Theorem 4.4 that there exists a unique solution $v\in X_\infty$ and thus also a unique $w=v+g\in X_\infty$. This verifies all assumptions of the implicit function theorem, which we apply now to deduce that the Fréchet-derivative of $u \mapsto x(u)$ exists and is given by

$$x'(u) = -e_x^{-1}(x(u),u)e_u(x(u),u).$$

□

5.2. The first-order optimality condition. The necessary first-order optimality condition is given by

$$\hat{J}'(u) = 0.$$

Using the chain rule one obtains for $\tilde{u} \in U$ that

$$\begin{aligned}
\left\langle \hat{J}'(u), \tilde{u} \right\rangle_{U^*,U} &= \langle J_x(x(u),u), x'(u)\tilde{u} \rangle_{X^*,X} + \langle J_u(x(u),u), \tilde{u} \rangle_{U^*,U} \\
&= \left\langle J_x(x(u),u), -e_x^{-1}(x(u),u)e_u(x(u),u)\tilde{u} \right\rangle_{X^*,X} + \langle J_u(x(u),u), \tilde{u} \rangle_{U^*,U} \\
&= \left\langle -e_u^*(x(u),u)e_x^{-*}(x(u),u)J_x(x(u),u), \tilde{u} \right\rangle_{U^*,U} + \langle J_u(x(u),u), \tilde{u} \rangle_{U^*,U}.
\end{aligned}$$

Introducing the variable

$$\xi = -e_x^{-*}(x(u),u)J_x(x(u),u) \in Z,$$

we get

$$\hat{J}'(u) = J_u(x(u),u) + e_u^*(x(u),u)\xi.$$

The above representation of the derivative and the adjoint variable $\xi \in Z$ yields the following theorem.

THEOREM 5.4. *Let $(x^*,u^*) \in X_\infty \times U$ be a solution of the constrained minimization problem* (1.6). *Then, there exists a unique Lagrange multiplier $\xi^* \in Z$ which together with the optimal solution (x^*,u^*) satisfies the first-order optimality system*

$$\begin{aligned}
e(x^*,u^*) &= 0 \quad \textit{in } Z^*, \\
e_x^*(x^*,u^*)\xi^* + J_x(x^*,u^*) &= 0 \quad \textit{in } X^*, \\
e_u^*(x^*,u^*)\xi^* + J_u(x^*,u^*) &= 0 \quad \textit{in } U^*.
\end{aligned}$$

Proof. Since we have $e_x(x^*,u^*) = A(x^*)$ and $J_x(x^*,u^*) \in X^*$ as well as $(x^*,u^*) \in X_\infty \times U$, the assertion directly follows from Theorem 5.1. □

6. Conclusions

We have studied an optimal boundary control problem for radiative heat transfer modeled by the SP_1-system from the analytical point of view, derived the first-order optimality system and proved existence, uniqueness and regularity for the adjoint state. It is easily possible to generalize the presented results to frequency-dependent models, and one can also employ spatially non-constant controls along the boundary, if one adjusts the penalty term in the cost functional. Future work will concentrate on more sophisticated models of the SP_N hierarchy and the investigation of so-called frequency-averaged equations [8].

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REFERENCES

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, first edition, 1975.

[2] M.K. Choudhary and N.T. Huff, *Mathematical modeling in the glass industry: an overview of status and needs*, Glastech. Ber. Glass Sci. Technol., 70, 363–370, 1997.

[3] L. Gergó and G. Stoyan, *On a mathematical model of a radiating, viscous, heatconducting fluid: remarks on a paper by J. Förster*, ZAMM, 77(5), 367–375, 1997.

[4] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, first edition, 1983.

[5] D. Levermore, *Moment closure hierachies for kinetic theories*, J. Stat. Phys., 83, 1996.

[6] E. Larsen, G. Pomraning and V.C. Badham, *Asymptotic analysis of radiative transfer problems*, J. Quant. Spectr. and Radiative Transfer, 29, 285–310, 1983.

[7] M.T. Laitinen and T. Tiihonen, *Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials*, Math. Meth. Appl. Sci., 21, 375–392, 1998.

[8] E.W. Larsen, G. Thömmes and A. Klar, *Frequency-averaged approximations to the equations of radiative heat transfer in glass*, SIAM J. Appl. Math., 64(2), 565–582, 2003.

[9] E.W. Larsen, G. Thömmes, M. Seaid, Th. Götz and A. Klar, *Simplified P_N approximations to the equations of radiative heat transfer and applicatons to glass manufacturing*, J. Comp. Phys., 183(2), 652–675, 2002.

[10] C. Meyer, P. Philip and F. Tröltzsch, *Optimal control of a semilinear PDE with nonlocal radiation interface conditions*, SIAM J. Control Optim., 45, 699–721, 2006.

[11] R. Pinnau and A. Schulze, *Newton's method for optimal temperature-tracking of glass cooling processes*, Inverse Probl. Sci. Eng., 15(4), 303–323, 2007.

[12] R. Pinnau and G. Thömmes, *Optimal boundary control of glass cooling processes*, M2AS, 120, 1261–1281, 2004.

[13] M. Renardy and R.C. Rogers, *An Introduction to Partial Differential Equations*, Springer-Verlag, New York, 1993.

[14] M. Schäfer, M. Frank and R. Pinnau, *A hierarchy of approximations to the radiative heat transfer equations: modelling, analysis and simulation*, Math. Mod. Meth. Appl. Sci., 15, 643–665, 2005.

[15] A. Schulze, *Minimizing thermal stresses in glass cooling processes*, PhD thesis, TU Kaiserslautern, 2006.

[16] M. Seaid, M. Frank, A. Klar, R. Pinnau and G. Thömmes, *Efficient numerical methods for radiation in gas turbines*, J. Comput. Appl. Math., 170(1), 217–239, 2004.

[17] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Math. Pura Appl., 146, 65–96, 1987.

[18] G. Stampaccia, *Contributi alla regolarizzazione delle soluzioni dei problemi al contorno per secondo ordine ellittiche*, Ann. Scuola Norm. Suo. Pisa, 12, 223–245, 1958.

[19] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, second edition, 1997.

[20] G. Thömmes, R. Pinnau, M. Seaid, T. Götz and A. Klar, *Numerical methods and optimal control for glass cooling processes*, TTSP, 31(4–6), 513–529, 2002.

[21] G.M. Troianiello, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, New York, first edition, 1987.

[22] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, volume II/A and II/B, Springer-Verlag, Berlin, first edition, 1990.