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Operator Coproduct-Realization of Quantum Group Transformations in Two-Dimensional Gravity I

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Abstract: A simple connection between the universal R matrix of $U_q(sl(2))$ (for spins $\frac{1}{2}$ and J) and the required form of the coproduct action of the Hilbert space generators of the quantum group symmetry is put forward. This leads us to an explicit operator realization of the coproduct action on the covariant operators. It allows us to derive the expected quantum group covariance of the fusion and braiding matrices, although it is of a new type: the generators depend upon worldsheet variables, and obey a new central extension of the $U_q(sl(2))$ algebra realized by (what we call) fixed point commutation relations. This is explained by showing on a general ground that the link between the algebra of field transformations and that of the coproduct generators is much weaker than previously thought. The central charges of our extended $U_q(sl(2))$ algebra, which includes the Liouville zero-mode momentum in a non-trivial way, are related to Virasoro-descendants of unity. We also show how our approach can be used to derive the Hopf algebra structure of the extended quantum-group symmetry $U_q(sl(2)) \odot \dot{U}_{\widehat{q}}(sl(2))$ related to the presence of both of the screening charges of 2D gravity.

1. Introduction

The quantum group structure of two-dimensional gravity in the conformal gauge has led to striking developments [1–13], by allowing us to derive general formulae for the fusion and braiding coefficients of the operator product algebra (OPA) in terms of quantum group symbols of $U_q(sl(2))$. Moreover, there exists [1,2,3,5] a covariant basis of holomorphic operators, where there is a natural quantum group action which is a symmetry of the OPA. However, this characterization of the quantum group symmetry is somewhat implicit, as so far we do not have an explicit construction of the $U_q(sl(2))$ generators as operators on the Hilbert space of

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states, i.e. a Hamiltonian realization of the quantum group symmetry. We would like to stress here that the quantum group symmetry we are talking about is distinct from the so-called "dressing symmetries" [6]. The latter transform solutions of the equations of motion into different ones, while here the "physical observables" (the functionals of the Liouville field) are invariant, and the symmetry is seen only in the enlarged phase space defined by the free field construction of the Liouville field. In fact the situation is very similar for general conformal field theories related to a Coulomb-gas construction, the most basic example being given by the $c < 1$ minimal models, and thus the relevance of this question extends much beyond the specific framework within which we will address it to have a definite setting. Gomez and Sierra [7] showed that there exists a realization of the Borel subalgebra generated by J_+ and J_3 for the $c < 1$ theories[4] in terms of contour-creating operators acting on suitable screened vertex operators. Although the operator product and the braiding of the latter do reproduce the q-Clebsch–Gordan coefficients and braiding matrix of $U_q(sl(2))$, they are not, strictly speaking, conformal objects. The basic message from the work of refs. [1,2,3,5] is that in fact there exists a basis of primary conformal fields which have the desired properties. Furthermore, in analogy to the Gomez–Sierra treatment, the action of J_+ within the new basis is related to multiplication by a suitably defined screening charge [12], though the contour integral realization of this multiplication (if it exists) is not obvious, and probably in any case not very natural. However, as was pointed out in [14], the general realization of quantum group generators, independent of any Coulomb gas picture, should be given in terms of certain operators on the Hilbert space acting on covariant fields by braiding, generalizing the action of "classical" symmetries by commutators. In the present article, we undertake steps towards a concrete realization of this form. Let us first summarize the basic point of the general exposition contained in [14], where the principles of (quasi) Hopf quantum group symmetry in quantum theory have been nicely formulated[5]. First consider a field theory with an ordinary (not q deformed) Lie algebra $\mathcal{G}$ of symmetries. For any element $J^a \in \mathcal{G}$, there exists an operator $\mathcal{O}(J^a)$ such that a typical field Ψ_l transforms as

$$[\mathcal{O}(J^a), \Psi_l] = \sum_m \Psi_m [J^a]_{ml} \,. \tag{1.1}$$

In this equation $[J^a]_{ml}$ is the matrix of the particular representation of $\mathcal{G}$ under which Ψ_l transforms. By text-book calculations, one of course verifies that the group law is satisfied since

$$[\mathcal{O}(J^b), [\mathcal{O}(J^a), \Psi_l]] = \sum_{nm} \Psi_n [J^b]_{nm} [J^b]_{ml} \,,$$

together with the Jacobi identity, which implies

$$[[\mathcal{O}(J^a), \mathcal{O}(J^b)], \Psi_l] = \sum_n \Psi_n [J^a, J^b]_{nl} \,. \tag{1.2}$$

Products of fields obey

$$[\mathcal{O}(J^a), \Psi_{l_1} \Psi_{l_1}] = \sum_{m_1 m_2} \Psi_{m_1} \Psi_{m_2} ([J^a]_{m_1 l_1} \delta_{m_2 l_2} + \delta_{m_1 l_1} [J^a]_{m_2 l_2}) \,, \tag{1.3}$$

so that they transform by a tensor product of representations as expected. This last point clearly shows that such generators cannot exist for quantum groups for which

[4] This work was generalized to the WZNW theories in [8].

[5] See also [15] for a general discussion with emphasis on affine quantum group and Yangian symmetry.

simple tensor products of representations do not form representations. In order to introduce the remedy, let us rewrite Eq. (1.1) in the form

$$\mathcal{O}(J^a)\Psi_l = \Psi_m([J^a]_{ml}\mathcal{I} + \delta_{m,l}\mathcal{O}(J^a)), \tag{1.4}$$

where we introduced the identity operator $\mathcal{I}$ in the Hilbert space of states. As is well known, standard Lie groups may be regarded as particular cases of Hopf algebras endowed with a (trivial) coproduct; namely, given an element $a \in \mathcal{G}$, one lets[6] $\Lambda_0(J^a) = J^a \otimes 1 + 1 \otimes J^a$. Clearly, this coproduct appears in formulae (1.3) and (1.4), the first with the coproduct between two matrix representations, the second with the coproduct between a matrix representation and the operator realization within the Hilbert space of states.

From there, the generalization to quantum Hopf algebras becomes natural, following ref. [14]. Consider a quantum deformation of an enveloping algebra with generators J^a, and coproduct[6]

$$\Lambda(J^a) := \sum_{cd} \Lambda^a_{cd} J^c \otimes J^d \tag{1.5}$$

which is co-associative

$$\sum_e \Lambda^e_{bc}\Lambda^a_{ed} = \sum_e \Lambda^e_{cd}\Lambda^a_{be}\,. \tag{1.6}$$

Then Eq. (1.4) is to be replaced by

$$\mathcal{O}(J^a)\Psi_l = \sum_m \sum_{b,c} \Psi_m \Lambda^a_{bc}[J^b]_{ml}\mathcal{O}(J^c)\,. \tag{1.7}$$

The present action is now consistent. Indeed, an easy calculation using the co-associativity shows that Eq. (1.3) is replaced by

$$\mathcal{O}(J^a)\Psi_{l_1}\Psi_{l_2} = \Psi_{m_1}\Psi_{m_2}\Lambda^a_{bc}\left\{\Lambda^b_{de}[J^d]_{m_1 l_1}[J^e]_{m_2 l_2}\right\}\mathcal{O}(J^c)\,. \tag{1.8}$$

Summations over repeated indices are understood from now on. Now the products of fields transform by action of the coproduct of the individual representations, and thus do span a representation.

Next two general remarks are in order which will be useful below. First, in the same way as for ordinary symmetries (see Eq. (1.2)), one may verify that the transformation law just recalled is consistent with the assumption that the generators $\mathcal{O}(J^a)$ and the matrices $[J^a]_{ml}$ satisfy the same algebra; this is expressed by the equality

$$[\mathcal{O}(J^a), \mathcal{O}(J^b)] = \mathcal{O}([J^a, J^b])\,, \tag{1.9}$$

which uses the fact that the algebra preserves the coproduct. However, this is not necessarily true. As we will see below for the case of $U_q(sl(2))$, consistency of the present coproduct action does not require that the algebra of the generators coincides with the one of the matrices: the former may be a suitable deformation of the latter, containing additional "central terms" that commute with all the Ψ_l fields. This will be the subject of Sect. 2. Indeed, we will see later on that it is an algebra of this type that will come out from our discussion, realized in a somewhat non-standard way. Thus at this point we depart from the general scheme of ref. [14], where it

[6] We denote the coproducts by the letter Λ instead of the more usual letter Δ, since the latter is used for conformal weights.

is assumed that the generators form a representation of the algebra; we will make further comments on this below.

Second, clearly the two sides of Eq. (1.7) are not on the same footing: one multiples by $\mathcal{O}(J^a)$ on the left, and reads the transformation law on the right. What would happen if we reversed the roles of left and right? This brings in the antipode map S_a^b, which is such that (from now on, summation over repeated indices is understood)

$$\Lambda^a_{de} S^{-1}{}^d_b S^{-1}{}^e_c = \Lambda^f_{cb} S^{-1}{}^a_f ,$$

$$\Lambda^a_{lc} S^{-1}{}^l_b [J^b J^c]_{nm} = \Lambda^a_{bl} S^{-1}{}^l_c [J^b J^c]_{nm} = \delta_{nm} \varepsilon(J^a) , \tag{1.10}$$

where $[J^b J^c]_{nm}$ is the matrix element in any representation, and ε is the co-unit. For completeness, let us recall that the latter is a complex number such that

$$\Lambda^a_{bc} \varepsilon(J^c) = \delta_{a,b} . \tag{1.11}$$

Using the formulae just summarized, it is easy to verify that Eq. (1.7) is equivalent to

$$\Psi_l \mathcal{O}(J^a) = \bar{\Lambda}^a_{bc} \mathcal{O}(J^b) \Psi_m [J^c_{(S)}]_{ml} , \tag{1.12}$$

where

$$J^c_{(S)} = S^{-1}{}^c_d J^d , \tag{1.13}$$

and

$$\bar{\Lambda}^a_{bc} = \Lambda^a_{cb} \tag{1.14}$$

is the other coproduct (transpose of the previous one). In our discussion both possibilities will be useful. We will refer to Eq. (1.7) as describing the right-action (the generator acts to its right), and to Eq. (1.12) as describing the left-action. Thus going from the left-action to the right-action corresponds to the antipode map. For products of fields, we obtain an equation similar to Eq. (1.8):

$$\Psi_{l_1} \Psi_{l_2} \mathcal{O}(J^a) = \bar{\Lambda}^a_{bd} \mathcal{O}(J^b) \Psi_{m_1} \Psi_{m_2} \bar{\Lambda}^d_{ec} [J^c_{(S)}]_{m_1 l_1} [J^e_{(S)}]_{m_2 l_2} , \tag{1.15}$$

which shows that $\bar{\Lambda}$ appears as in Eq. (1.12).

In ref. [14] the general properties of the generators $\mathcal{O}$ were characterized, but no attempt at an explicit construction was made. On the other hand, the complete study of the operator algebra of Liouville (2D gravity) has revealed [2, 3, 5] that a particular basis of chiral operators noted $\xi_M^{(J)}$ exists whose OPA is quantum group symmetric, with products of operators transforming by the coproduct of the individual representations of each. These ξ should be the operators to which the general construction just recalled applies. It is the purpose of the present paper to show how this is realized or, to be more precise, how a suitable redefinition of the formulae just given is directly realized by the OPA of the ξ fields. In the present quantum group picture, M is a magnetic quantum number, like the indices displayed in Eq. (2.3), while J is the spin that characterizes the representation. The OPA of the ξ fields has been studied at first for standard representations with $2J$ a positive integer [2, 3], and we discuss this case in Sects. 3 and 5. On the other hand, the application of quantum groups to two dimensional gravity led us to go away [2–13] from this conventional situation. First one needs to deal with semi-infinite representations with continuous total spins. Second there are two dual quantum groups

$U_q(sl(2))$ and $U_{\widehat{q}}(sl(2))$ such that $q=\exp(ih)$, $\widehat{q}=\exp(i\widehat{h})$, and $h\widehat{h}=\pi^2$. For continuous spins, the complete quantum group structure noted $U_q(sl(2))\odot U_{\widehat{q}}(sl(2))$ is a non-trivial combination of the two Hopf algebras. Its overall Hopf algebra structure will be derived in Sect. 6 by making use of the present general scheme. The particular case of spin $-1/2$ will be considered in Sect. 4, where it is pointed out that the corresponding ξ fields are required for the description of the Cartan generator $\mathcal{O}[q^{-J_3}]$ (with right-action). Finally, let us note that our results will be weaker than a full realization of the general scheme of ref. [14] concerning a basic point. Since the algebra of our generators will differ from the standard $U_q(sl(2))$ one, we cannot yet address the question of the existence of an invariant vacuum $|0\rangle$ such that

$$\mathcal{O}(J^a)|0\rangle=\varepsilon(J^a)|0\rangle\,. \tag{1.16}$$

We will comment further on this point in Sect. 7.

2. Preamble: More General Definition of the Co-product Action for $U_q(sl(2))$

Let us now turn to two-dimensional gravity (Liouville theory). There the enveloping algebra is $U_q(sl(2))$. In this latter case, the generators are $J_\pm$ and $q^{\pm J_3}$, which satisfy

$$q^{J_3}J_\pm=q^{\pm1}J_\pm q^{J_3},\qquad [J_+,J_-]=\frac{q^{2J_3}-q^{-2J_3}}{q-q^{-1}},\qquad q^{J_3}q^{-J_3}=1\,, \tag{2.1}$$

and we have the coproduct

$$\Lambda(q^{\pm J_3})=q^{\pm J_3}\otimes q^{\pm J_3},\qquad \Lambda(J_\pm)=J_\pm\otimes q^{J_3}+q^{-J_3}\otimes J_\pm\,. \tag{2.2}$$

Thus one would write

$$\mathcal{O}(q^{\pm J_3})\Psi_M=\Psi_N[q^{\pm J_3}]_{NM}\mathcal{O}(q^{\pm J_3})\,,$$

$$\mathcal{O}(J_\pm)\Psi_M=\Psi_N[J_\pm]_{NM}\mathcal{O}(q^{J_3})+\Psi_N[q^{-J_3}]_{NM}\mathcal{O}(J_\pm)\,. \tag{2.3}$$

We now use upper case indices to agree with later notations. In the general scheme of Mack and Schomerus, it is assumed that the generators acting in the Hilbert space and the matrices of the transformation law of the fields obey the same algebra. Indeed, it is easy to verify that the coproduct action just written is compatible with the $U_q(sl(2))$ algebra for the generators

$$\mathcal{O}[q^{J_3}]\mathcal{O}[J_\pm]=q^{\pm1}\mathcal{O}[J_\pm]\mathcal{O}[q^{J_3}],\qquad [\mathcal{O}[J_+],\mathcal{O}[J_-]]=\frac{\mathcal{O}[q^{J_3}]^2-\mathcal{O}[q^{-J_3}]^2}{q-q^{-1}}\,. \tag{2.4}$$

However, this will not be true in our construction, which suggests more general alternatives. Thus we discuss on a general ground the possibility that the matrices satisfy the usual algebra, while the generators obey more general braiding relations. At the present stage of our understanding, we are not yet able to discuss the case of a general Hopf algebra. Although we believe that the property we are discussing is not specific to $U_q(sl(2))$, we restrict ourselves to this case from now on. Let us determine the most general algebra of the operators $\mathcal{O}[J_\pm]$ and $\mathcal{O}[q^{\pm J_3}]$ compatible with the coproduct action Eq. (2.3), from which we deduce immediately

$$(q\mathcal{O}[J_+]\mathcal{O}[q^{J_3}]-\mathcal{O}[q^{J_3}]\mathcal{O}[J_+])\Psi_M=\Psi_M(q\mathcal{O}[J_+]\mathcal{O}[q^{J_3}]-\mathcal{O}[q^{J_3}]\mathcal{O}[J_+])\,.$$

Therefore it follows that

$$q\mathcal{O}[J_+]\mathcal{O}[q^{J_3}] - \mathcal{O}[q^{J_3}]\mathcal{O}[J_+] = C_+ , \tag{2.5}$$

where C_+ is a central term which commutes with all the Ψ's. In the same way we derive the analogous relation

$$\mathcal{O}[q^{J_3}]\mathcal{O}[J_-] - q^{-1}\mathcal{O}[J_-]\mathcal{O}[q^{J_3}] = C_- . \tag{2.6}$$

Computing the action of $[\mathcal{O}[J_+], \mathcal{O}[J_-]]$ on Ψ_M we get

$$\begin{aligned}[\mathcal{O}[J_+], \mathcal{O}[J_-]]\Psi_M &= \Psi_N(q^{-2J_3})_{NM}[\mathcal{O}[J_+], \mathcal{O}[J_-]] + \Psi_N[J_+, J_-]_{NM}(\mathcal{O}[q^{J_3}])^2 \\ &\quad + \Psi_N((J_-q^{-J_3})_{NM}C_+ + (J_+q^{J_3})_{NM}C_-) .\end{aligned}$$

The last term can be "removed" by introducing the operator $\mathcal{O}[D]$ which acts as

$$\mathcal{O}[D]\Psi_M = \Psi_N(q^{-2J_3})_{NM}\mathcal{O}[D] + \Psi_N((J_-q^{-J_3})_{NM}C_+ + (J_+q^{J_3})_{NM}C_-) . \tag{2.7}$$

It is easy to see that it can be taken equal to:

$$\mathcal{O}[D] = (C_+\mathcal{O}[J_-] + C_-\mathcal{O}[J_+])\mathcal{O}[q^{-J_3}] . \tag{2.8}$$

We then get

$$\begin{aligned}([\mathcal{O}[J_+], \mathcal{O}[J_-]] - \mathcal{O}[D])\Psi_M &= \Psi_N(q^{-2J_3})_{NM}([\mathcal{O}[J_+], \mathcal{O}[J_-]] - \mathcal{O}[D]) \\ &\quad + \Psi_N[J_+, J_-]_{NM}(\mathcal{O}[q^{J_3}])^2 ,\end{aligned}$$

from which it follows that

$$\begin{aligned}[\mathcal{O}[J_+], \mathcal{O}[J_-]] &= \frac{\mathcal{O}[q^{J_3}]^2 - \mathcal{O}[q^{-J_3}]^2}{q - q^{-1}} + (C_3\mathcal{O}[q^{-J_3}] \\ &\quad + C_+\mathcal{O}[J_-] + C_-\mathcal{O}[J_+])\mathcal{O}[q^{-J_3}] ,\end{aligned} \tag{2.9}$$

where C_3 is also a central term which commutes with all the Ψ's. We note that $\mathcal{O}[D]$ is defined in fact up to a term $(\mathcal{O}[q^{-J_3}])^2$. This freedom is already taken into account in the term C_3.

At this stage we have derived the general operator algebra compatible with the action defined by Eq. (2.3). Clearly the commutation relations Eq. (2.4) are recovered for $C_\pm = C_3 = 0$, and we have found a three-parameter deformation of $U_q(sl(2))$. Let us discuss its properties by considering an arbitrary representation of this algebra by generators denoted by $\widetilde{J}_\pm, q^{\pm\widetilde{J}_3}$, satisfying

$$\begin{gathered}q\widetilde{J}_+q^{\widetilde{J}_3}\widetilde{J}_+ = C_+, \qquad q^{\widetilde{J}_3}\widetilde{J}_- - q^{-1}\widetilde{J}_-q^{\widetilde{J}_3} = C_- , \\ [\widetilde{J}_+, \widetilde{J}_-] = \frac{q^{2\widetilde{J}_3} - q^{-2\widetilde{J}_3}}{q - q^{-1}} + (C_3q^{-\widetilde{J}_3} + C_+\widetilde{J}_- + C_-\widetilde{J}_+)q^{-\widetilde{J}_3} , \\ [C_\pm, \widetilde{J}_a] = [C_3, \widetilde{J}_a] = 0 .\end{gathered} \tag{2.10}$$

At this point the question arises if the deformed algebra Eq. (2.10) is a Hopf algebra, in particular if we have a (co-associative) coproduct. This coproduct should be formulated in terms of the generators $\widetilde{J}_a$ only, in contrast to Eq. (2.3) where both the standard $U_q(sl(2))$ matrices $[J_a]_{NM}$ and the generators $\widetilde{J}_a$ appear. Evidently, this new coproduct is not of the form Eq. (2.2) as the latter does not conserve the deformed algebra.

The easiest way to answer this question is to show that, in general, any representation of this algebra may be re-expressed in terms of the original algebra given by Eq. (2.1) via a linear transformation of the generators. It is immediate to check that the following operators:

$$J_\pm = \rho\left(\widetilde{J}_\pm \pm \frac{C_\pm}{1-q^{\pm 1}} q^{-\widetilde{J}_3}\right), \qquad q^{\pm J_3} = \rho^{\pm}1 q^{\pm \widetilde{J}_3} \tag{2.11}$$

with

$$\rho^{-4} = (q-q^{-1})\left(C_+ C_- \frac{1+q}{1-q} - C_3\right) + 1$$

satisfy the original algebra[7], and the new generators still act on the Ψ_M fields according to Eq. (2.3) (with $\mathcal{O}(J_a)$ replaced by the J_a above). These relations can be inverted very easily

$$\widetilde{J}_\pm = \rho^{-1} J_\pm \mp \frac{C_\pm}{1-q^{\pm 1}} \rho q^{-J_3}, \qquad q^{\pm \widetilde{J}_3} = \rho^{\mp 1} q^{\pm J_3}\,. \tag{2.12}$$

The existence of such a linear transformation shows that at a formal level the present modification is a trivial extension of $U_q(sl(2))$. Two remarks are in order at this point. First it agrees with what is known for $h = 0$ from cohomology arguments[8]. Second this is really true only if we may diagonalize the central terms $C_\pm$ and work within an eigenspace where they may be replaced by numbers. The situation will be more involved in the coming field theoretic realization, where $C_\pm$ are operators in the Hilbert space of states. From the above mapping of the new algebra to the standard one it is very easy to derive the new coproduct

$$(q^{\widetilde{J}_3}) = \rho^{\pm 1} q^{\pm \widetilde{J}_3} \otimes q^{\pm \widetilde{J}_3}\,,$$

$$\widetilde{\Lambda}(\widetilde{J}_\pm) = \rho \widetilde{J}_\pm \otimes q^{\widetilde{J}_3} + \rho^{-1} q^{-\widetilde{J}_3} \otimes \widetilde{J}_\pm \pm \frac{C_\pm}{1-q^{\pm 1}} \rho q^{-\widetilde{J}_3} \otimes q^{\widetilde{J}_3}\,. \tag{2.13}$$

The action of $\mathcal{O}[J_\pm]$ and $\mathcal{O}[q^{\pm J_3}]$ can be defined with the new coproduct, the matrix elements that appear being the ones of the new generators of Eq. (2.12). Using Eq. (2.11), it can immediately be rewritten as

$$\widetilde{\Lambda}(q^{\pm \widetilde{J}_3}) = q^{\pm J_3} \otimes q^{\pm \widetilde{J}_3}, \qquad \widetilde{\Lambda}(\widetilde{J}_\pm) = J_\pm \otimes q^{\widetilde{J}_3} + q^{-J_3} \otimes \widetilde{J}_\pm\,. \tag{2.14}$$

Thus both coproducts acting in their own way on the Ψ's give in fact the same result. This is quite important since it shows that our definition of the action is umambiguous. This action is now compatible with the new group law in the sense defined by Eq. (2.10). The new coproduct is co-associative. Acting on the product of two Ψ's, it is easy to see that the action defined from Eq. (2.14) is preserved and that only matrix elements of the ordinary coproduct appear. This property is true not only for the generators but also for all the operators of the enveloping algebra.

[7] Provided that ρ is finite, if not the algebra can only be recast in a form with $C_\pm = 0$, $C_3 = 1/(q-q^{-1})$; we will not discuss this case here.

[8] We are indebted to L. Alvarez-Gaumé for mentioning this fact.

The profound reason is an invariance of the original coproduct structure constants Λ^a_{bc}. There exists a group of matrices X^a_b such that

$$\Lambda^a_{bc} X^d_a (X^{-1})^c_e = \Lambda^d_{be}$$

for all b. Then the generators defined by $\widetilde{J}^a = X^a_b J^a$ satisfy new commutation relations preserved by a new coproduct with structure constants given by

$$\widetilde{\Lambda}^a_{bc} = X^a_d \Lambda^d_{ef} (X^{-1})^e_b (X^{-1})^f_c \, .$$

The property of invariance of the Λ implies

$$\widetilde{\Lambda}^a_{bc} = \Lambda^a_{dc} (X^{-1})^d_b$$

so that

$$\widetilde{\Lambda}^a_{bc} \widetilde{J}^b \otimes \widetilde{J}^c = \Lambda^a_{bc} J^b \otimes \widetilde{J}^c \, .$$

In our case of $U_q(sl(2))$ it is easy to show that the general solution for the matrices X is precisely given by the three parameter relations of Eqs. (2.12), and that these transformations generate the group E_2.

The particular case where $C_3 = 1/(q - q^{-1})$. In the coming discussion, $\mathcal{O}[q^{-J_3}]$ will be on a completely different footing, and will not be introduced at all in the beginning. This will be possible since the generators will satisfy an algebra which is closely related with the one just written in the particular case $C_3 = 1/(q - q^{-1})$, where the coefficient of $\mathcal{O}[q^{-J_3}]^2$ on the right-hand side of Eq. (2.9) vanishes. In the present subsection, we show that indeed $\mathcal{O}[q^{-J_3}]$ may be completely eliminated – assuming that C_+C_- does not vanish – for this particular value[9] of C_3 from the algebra defined by Eqs. (2.5), (2.6), (2.9) if we use the operator $\mathcal{O}[D]$ introduced in Eq. (2.8):

$$\mathcal{O}[D] = C_+ \mathcal{O}[J_-] \mathcal{O}[q^{-J_3}] + C_- \mathcal{O}[J_+] \mathcal{O}[q^{-J_3}] \, .$$

What happens in practice is that $\mathcal{O}[q^{-J_3}]$ only appears in the particular combination $\mathcal{O}[D]$. The derivation goes as follows. First, using $\mathcal{O}[D]$ this algebra may be rewritten as

$$q \mathcal{O}[J_+] \mathcal{O}[q^{J_3}] - \mathcal{O}[q^{J_3}] \mathcal{O}[J_+] = C_+ \, , \tag{2.15}$$

$$\mathcal{O}[q^{J_3}] \mathcal{O}[J_-] - q^{-1} \mathcal{O}[J_-] \mathcal{O}[q^{J_3}] = C_- \, , \tag{2.16}$$

$$C_+ \mathcal{O}[J_-] + C_- \mathcal{O}[J_+] = \mathcal{O}[D] \mathcal{O}[q^{J_3}] \, , \tag{2.17}$$

$$\mathcal{O}[J_+] \mathcal{O}[J_-] - \mathcal{O}[J_-] \mathcal{O}[J_+] = \mathcal{O}[D] + \frac{(\mathcal{O}[q^{J_3}])^2}{q - q^{-1}} \, . \tag{2.18}$$

Second, making use of the definition Eq. (2.8), one verifies that the braiding relations of $\mathcal{O}[D]$ with the other generators are given by

$$\begin{aligned} \mathcal{O}[J_\pm] \mathcal{O}[D] - q^{\pm 1} \mathcal{O}[D] \mathcal{O}[J_\pm] &= \pm \frac{C_\pm}{q - q^{-1}} \mathcal{O}[q^{J_3}] \, , \\ \mathcal{O}[q^{J_3}] \mathcal{O}[D] - q^{\mp 1} \mathcal{O}[D] \mathcal{O}[q^{J_3}] &= \pm C_\mp (q - q^{-1}) \mathcal{O}[J_\pm] \, . \end{aligned} \tag{2.19}$$

[9] In fact we may always reduce the general situation to the present one by a suitable redefinition of the generators. For this value of C_3, $\rho = \infty$ is equivalent to $C_+C_- = 0$.

The key fact here is that $\mathcal{O}[q^{-J_3}]$ does not appear explicitly in the braiding relations just derived. Finally, substituting $\mathcal{O}[J_\pm]$ as given by Eqs. (2.19) into the previous commutation relations Eqs. (2.15)–(2.18), one derives further consistency relations

$$\begin{aligned}
&\mathcal{O}[q^{J_3}]\mathcal{O}[D]^2-(q+q^{-1})\mathcal{O}[D]\mathcal{O}[q^{J_3}]\mathcal{O}[D]+\mathcal{O}[D]^2\mathcal{O}[q^{J_3}]=C_+C_-\mathcal{O}[q^{J_3}]\,,\\
&\mathcal{O}[D]\mathcal{O}[q^{J_3}]^2-(q+q^{-1})\mathcal{O}[q^{J_3}]\mathcal{O}[D]\mathcal{O}[q^{J_3}]+\mathcal{O}[q^{J_3}]^2\mathcal{O}[D]=-C_+C_-(q-q^{-1})\,,\\
&\mathcal{O}[D]\mathcal{O}[q^{J_3}]^2\mathcal{O}[D]-\mathcal{O}[q^{J_3}]\mathcal{O}[D]^2\mathcal{O}[q^{J_3}]=-C_+C_-(q-q^{-1})\mathcal{O}[D]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-C_+C_-\mathcal{O}[q^{J_3}]^2\,.
\end{aligned}\tag{2.20}$$

It is easily seen that this last identity is not independent. It can be obtained by a combination of the first one multiplied by $\mathcal{O}[q^{J_3}]$ on the right and of the second multiplied by $\mathcal{O}[D]$ on the left. From this we can verify that the enveloping algebra of the generators $\mathcal{O}[J_\pm]$, $\mathcal{O}[D], \mathcal{O}[q^{J_3}]$ may be entirely derived without ever making use of Eq. (2.8), so that $\mathcal{O}[q^{-J_3}]$ has been completely eliminated as we wanted to show.

Recalling the action of $\mathcal{O}[D]$ on the Ψ_M fields,

$$\mathcal{O}[D]\Psi_M=\Psi_N(q^{-2J_3})_{NM}\mathcal{O}[D]+\Psi_N((J_-q^{-J_3})_{NM}C_++(J_+q^{J_3})_{NM}C_-)\,,$$

we see that we have a co-action of $\mathcal{O}[D]$ defined by

$$\widetilde{\Delta}(\mathcal{O}[D])=q^{-2J_3}\otimes\mathcal{O}[D]+(J_+q^{-J_3})\otimes C_-+(J_-q^{-J_3})\otimes C_+\,.$$

In deriving Eqs. (2.19), (2.20) from Eqs. (2.15)–(2.18) we have made use of the relation $\mathcal{O}[q^{J_3}]\mathcal{O}[q^{-J_3}]=1$. However, Eqs. (2.15)–(2.20) altogether define a consistent operator algebra which may be considered on its own, without introducing $\mathcal{O}[q^{-J_3}]$ at all, and it is this "weak" version of the original equations which will be realized by our generators – in a special way to be described in Sect. 3.3.2 below. In fact, our realization will necessitate a further generalization of the above considerations in that our generators will depend on position, like the fields on which they act. Although the derivation leading to Eqs. (2.15)–(2.20) is not strictly applicable in this situation, we will demonstrate explicitly that the latter, suitably interpreted, are realized by our generators.

For later reference, we note here already that a more general condition could be assumed in place of $\mathcal{O}[q^{J_3}]\mathcal{O}[q^{-J_3}]=1$, namely $\mathcal{O}[q^{J_3}]\mathcal{O}[q^{-J_3}]=C_0$, where C_0 is central. At the present formal level, this modification is trivial as it amounts only to a normalization change of $\mathcal{O}[q^{-J_3}]$, and thus the above arguments are unchanged (except that we should replace $\mathcal{O}[q^{-J_3}]\to\mathcal{O}[q^{-J_3}]C_0^{-1}$ in Eqs. (2.8), (2.9). However, it will acquire a non-trivial meaning for the field-theoretic realization discussed below (see Sect. 4). Note also that $\mathcal{O}[q^{J_3}]$ and $\mathcal{O}[q^{-J_3}]$ do not play the same role in the original algebra. In particular the expressions similar to Eqs. (2.5) and (2.6) with $\mathcal{O}[q^{J_3}]\to\mathcal{O}[q^{-J_3}]$, are not central, but proportional to $\mathcal{O}[q^{-J_3}]^2$.

Before beginning our construction let us clarify how the above formal structure will be realized by operators in the Hilbert space of states. These generators will in fact be operator-valued functions on the unit circle, much as for a Kac–Moody algebra. We will define the operators $\mathcal{O}[J_\pm]_\sigma\mathcal{O}[q^{J_3}]_\sigma\mathcal{O}[D]_{\sigma_1,\sigma_2}$ and so on, such that (what we will call) fixed-point realisation of the above structure hold. For instance, the fixed-point version of Eq. (2.18) is

$$\mathcal{O}[J_+]_{\sigma_1}\mathcal{O}[J_-]_{\sigma_2} - \mathcal{O}[J_-]_{\sigma_1}\mathcal{O}[J_+]_{\sigma_2} = \mathcal{O}[D]_{\sigma_1,\sigma_2} + \frac{\mathcal{O}[q^{J_3}]_{\sigma_1}\mathcal{O}[q^{J_3}]_{\sigma_2}}{q-q^{-1}} . \tag{2.21}$$

The rule should be clear on this example. One chooses a certain number of separate points (here σ_1, and σ_2), and replaces each term of Eq. (2.18) by the corresponding products with **the ordering of points being the same for every term**. As we will see, this fixed-point commutation relation is good enough to ensure the correct algebraic structure of the field transformation laws. Note that certain operators depend upon more than one point. This is true for $\mathcal{O}[D]$, and for the central terms. Clearly, these fixed point relations are very different from algebras of the Kac–Moody type (current algebras on the unit circle): in the latter case, one does exchange the points, and the central term is local, i.e. proportional to a Dirac distribution.

3. The Operator Realization of $U_q(sl(2))$

3.1. The Generators. Our basic tool will be the braiding relations of the family of chiral primaries $\xi_M^{(J)}$, with $2J$ a positive integer and $-J \leqq M \leqq J$. We work on the cylinder $0 \leqq \sigma \leqq 2\pi$, $-\infty \leqq \tau \leqq \infty$. Since the ξ fields are only functions of $\sigma - i\tau$ (or $\sigma - \tau$ in the Minkowski case), we may restrict ourselves to equal τ and take it to be zero. Thus we work on the unit circle $z = e^{i\sigma}$. It was shown in ref. [2] that, for $\sigma > \sigma'$,

$$\xi_M^{(J)}(\sigma)\xi_{M'}^{(J')}(\sigma') = \sum_{-J\leqq N\leqq j;\ -J'\leqq N'\leqq J'} (J,J')_{MM'}^{N'N}\xi_{N'}^{(J')}(\sigma')\xi_N^{(J)}(\sigma) , \tag{3.1}$$

where

$$(J,J')_{MM'}^{N'N} = (\langle J,M| \otimes \langle J',M'|)R(|J,N\rangle \times |J',N'\rangle) ,$$

$$R = e^{(-2ihJ_3\otimes J_3)}\left(1 + \sum_{n=1}^{\infty} \frac{(1-e^{2ih})^n e^{ihn(n-1)/2}}{\lfloor n\rfloor !} e^{-ihnJ_3}(J_+)^n \otimes e^{ihnJ_3}(J_-)^n\right) . \tag{3.2}$$

Since these formulae are basic for our discussion, it is worth commenting about their derivation. The original discussion of Gervais and Neveu (refs. [23, 2]) used the existence of two equivalent free fields, which was derived perturbatively, using a scheme that seems specific to the Liouville theory. However, more recently, the whole derivation was carried out again [4, 5], making a systematic use of the monodromy properties of the differential equations for Green functions, combined with the polynomial equations, and other standard axioms of the Moore Seiberg formalism. Thus, although it is not at the level of a mathematical theorem, the derivation of the ξ braiding relations is presently at a level of rigor which is usual in two dimensional conformal field theories. For instance its derivation is similar to the use of KZ equations in WZNW theories on the sphere. In addition, recently a Coulomb gas picture has been established [9, 13] which is similar in spirit to the one known from minimal models. For the following, we note that Eq. (3.1) is valid for $\sigma, \sigma' \in [0, 2\pi]$, although in refs. [2, 9, 13] it is stated to hold for $\sigma, \sigma' \in [0, \pi]$ only[10]. We have introduced states noted $|J,M\rangle$ which span the spin J representation of $U_q(sl(2))$, in order to write down the universal R matrix. We now show

[10] This is clear from the derivation in ref. [9] and also from the underlying transformation law for hypergeometric functions used in ref. [23].

that particular cases of the formulae just written are very similar to Eq. (2.3). There will be important differences which we will spell out in turn. In the following, the two Borel subalgebras $\mathscr{B}_+$ with elements J_+ and q_3^J, and $\mathscr{B}_-$ with elements J_- and q^{-J_3}, will be considered separately, as they will be realized in a different way.

3.1.1. The Borel Subalgebra $\mathscr{B}_+$. Let $\sigma_+ > \sigma$. A particular case of Eq. (3) is

$$\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+)\xi^{(J)}_M(\sigma) = q^M \xi^{(J)}_M(\sigma)\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+)\,, \tag{3.3}$$

$$\begin{aligned}\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_+)\xi^{(J)}_M(\sigma) &= q^{-M}\xi^{(J)}_M(\sigma)\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_+)\\ &+ \frac{(1-q^2)}{q^{\frac{1}{2}}}\langle J,M+1|J_+|J,M\rangle \xi^{(J)}_{M+1}(\sigma)\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+)\,.\end{aligned} \tag{3.4}$$

This exactly coincides with Eq. (2.3) if we identify, up to constants, $\xi^{(1/2)}_{1/2}(\sigma_+)$ with $\mathscr{O}(J_+)$, and $\xi^{(1/2)}_{-1/2}(\sigma_+)$ with $\mathscr{O}(q^{J_3})$. The crucial difference with the general transformation law is that the role of generators is played by fields that depend upon the worldsheet variable σ_+. This is possible since the braiding matrix of $\xi^{(1/2)}(\sigma_+)$ with a general field $\xi^{(J)}_M(\sigma)$ only depends upon the sign of $(\sigma_+ - \sigma)$. Thus we may realize the ($\mathscr{B}_+$ part of) the transformation Eq. (2.3) simply by the $\xi^{(1/2)}$ fields taken at an arbitrary point (within the periodicity interval $[0, 2\pi]$) such that this difference is positive. Accordingly, we will write, keeping in mind the σ_+ dependence,

$$\mathscr{O}[J_+]^{(R)}_{\sigma_+} \equiv \kappa^{(R_+)}_+ \xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_+), \qquad \mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+} \equiv \kappa^{(R_+)}_3 \xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+)\,, \tag{3.5}$$

where $\kappa^{(R_+)}_+$ and $\kappa^{(R_+)}_3$ are normalization constants to be specified below. For later convenience, we add a superscript R to indicate that the realization is by right-action – that is by acting to the right. We then obtain the action by coproduct of the form Eq. (2.3), that is

$$\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\xi^{(J)}_M(\sigma) = \xi^{(J)}_N(\sigma)[q^{J_3}]_{NM}\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\,, \tag{3.6}$$

$$\mathscr{O}[J_+]^{(R)}_{\sigma_+}\xi^{(J)}_M(\sigma) = \xi^{(J)}_N(\sigma)(q^{-J_3})_{NM}\mathscr{O}[J_+]^{(R)}_{\sigma_+} + \xi^{(J)}_N(\sigma)[J_+]_{NM}\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\,, \tag{3.7}$$

provided $\kappa^{(R_+)}_+$ and $\kappa^{(R_+)}_3$ satisfy

$$\frac{\kappa^{(R_+)}_+}{\kappa^{(R_+)}_3} = \frac{q^{\frac{1}{2}}}{1-q^2}\,. \tag{3.8}$$

Thus we have derived $\mathscr{B}_+$ transformations by right-action. The left-action discussed in the introduction will come out automatically if we braid starting from the product $\xi^{(J)}_M(\sigma)\xi^{(\frac{1}{2})}_{\pm\frac{1}{2}}(\sigma_+)$. Since we still have $\sigma_+ > \sigma$, we use the other braiding matrix. Recall that for $\sigma > \sigma'$ one has [2]

$$\xi^{(J')}_{M'}(\sigma')\xi^{(J)}_M(\sigma) = \sum_{-J\leqq N\leqq J;\ -J'\leqq N'\leqq J'} \overline{(J',J)}^{NN'}_{M'M}\xi^{(J)}_N(\sigma)\xi^{(J')}_{N'}(\sigma')\,, \tag{3.9}$$

where

$$\overline{(J',J)}_{M'M}^{NN'} = ((J,J')_{MM'}^{N'N})^* , \tag{3.10}$$

and $*$ means complex conjugate of the matrix elements. From this it is easy to verify that the left-action is given by Eq. (1.12) with $\widetilde{J}_+ = -qJ_+$, $\widetilde{q^{J_3}} = q^{-J_3}$, which are obtained from $\mathcal{B}_+$ by the antipode map. This is expected from the general argument given in the introduction. It will be convenient to denote the corresponding operators by $\mathcal{O}[J_+]^{(L)}_{\sigma_+}$, and $\mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_+}$. For a better readability of the formulae below, we always normalize the operators so that the symbols written in the square brackets are exactly equal to the matrix that appears in the coproduct action. Thus we absorb the proportionality coefficient between $\widetilde{J}_+$ and J_+ by changing the definition of the κ coefficients. Altogether, we finally have

$$\mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_+} = \kappa_3^{(L_+)}\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+), \qquad \mathcal{O}[J_+]^{(L)}_{\sigma_+} = \kappa_+^{(L_+)}\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_+), \qquad \frac{\kappa_+^{(L_+)}}{\kappa_3^{(L_+)}} = \frac{q^{-\frac{1}{2}}}{1-q^{-2}} , \tag{3.11}$$

$$\xi^{(J)}_M(\sigma)\mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_+} = \mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_+}\xi^{(J)}_N(\sigma)[q^{-J_3}]_{NM} ,$$

$$\xi^{(J)}_M(\sigma)\mathcal{O}[J_+]^{(L)}_{\sigma_+} = \mathcal{O}[J_+]^{(L)}_{\sigma_+}\xi^{(J)}_N(\sigma)[q^{J_3}]_{NM} + \mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_+}\xi^{(J)}_N(\sigma)[J_+]_{NM} . \tag{3.12}$$

3.1.2. The Borel Subalgebra $\mathcal{B}_-$. $\mathcal{B}_-$ will be realized by letting the $\xi^{(1/2)}$ fields act again, provided we reverse the role of left and right and make use of the fields $\xi^{(\frac{1}{2})}_{\pm\frac{1}{2}}(\sigma_-)$ with $\sigma_- < \sigma$. Since the discussion is very similar to the previous one, we will be brief. We let

$$\mathcal{O}[J_-]^{(L)}_{\sigma_-} \equiv \kappa_-^{(L_-)}\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_-), \qquad \mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_-} \equiv \kappa_3^{(L_-)}\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_-), \qquad \frac{\kappa_-^{(L_-)}}{\kappa_3^{(L_-)}} = \frac{q^{\frac{1}{2}}}{1-q^{-2}} , \tag{3.13}$$

$$\mathcal{O}[J_-]^{(R)}_{\sigma_-} \equiv \kappa_-^{(R_-)}\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_-), \qquad \mathcal{O}[q^{J_3}]^{(R)}_{\sigma_-} \equiv \kappa_3^{(R_-)}\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_-), \qquad \frac{\kappa_-^{(R_-)}}{\kappa_3^{(R_-)}} = \frac{q^{-\frac{1}{2}}}{1-q^{-2}} . \tag{3.14}$$

The action by coproduct is

$$\xi^{(J)}_M(\sigma)\mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_-} = \mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_-}\xi^{(J)}_N(\sigma)[q^{-J_3}]_{NM} , \tag{3.15}$$

$$\xi^{(J)}_M(\sigma)\mathcal{O}[J_-]^{(L)}_{\sigma_-} = \mathcal{O}[-J_-]^{(L)}_{\sigma_-}\xi^{(J)}_N(\sigma)(q^{J_3})_{NM} + \mathcal{O}[q^{-J_3}]^{(L)}_{\sigma_-}\xi^{(J)}_N(\sigma)[J_-]_{NM} , \tag{3.16}$$

$$\mathcal{O}[q^{J_3}]^{(R)}_{\sigma_-}\xi^{(J)}_M(\sigma) = \xi^{(J)}_N(\sigma)[q^{J_3}]_{NM}\mathcal{O}[q^{J_3}]^{(R)}_{\sigma_-} , \tag{3.17}$$

$$\mathcal{O}[J_-]^{(R)}_{\sigma_-}\xi^{(J)}_M(\sigma) = \xi^{(J)}_N(\sigma)(q^{-J_3})_{NM}\mathcal{O}[J_-]^{(R)}_{\sigma_-} + \xi^{(J)}_N(\sigma)[J_-]_{NM}\mathcal{O}[q^{J_3}]^{(R)}_{\sigma_-} . \tag{3.18}$$

We see that $\mathcal{B}_-$ is generated by left-action, while its antipode is generated by right-action. This different treatment of the two Borel subalgebras comes from the fact that for $\mathcal{B}_+$ (resp. $\mathcal{B}_-$), q^{-J_3} (resp. q^{J_3}) does not belong to the algebra, so that only right- (resp. left-) action – where $\mathcal{O}[q^{-J_3}]$ (resp. $\mathcal{O}[q^{J_3}]$) does not appear – may be written.

3.2. Symmetries of the Operator-Product Algebra. We have begun to see, and it will be more and more evident in the following, that the operators $\mathcal{O}[J_+]^{(R)}_{\sigma+}$, $\mathcal{O}[q^{J_3}]^{(R)}_{\sigma+}$, $\mathcal{O}[J_-]^{(L)}_{\sigma-}$, $\mathcal{O}[q^{-J_3}]^{(L)}_{\sigma-}$, and so on, play the same role as the generators introduced in ref. [14] and recalled in the introduction. From now on we will call them the generators. The basic point is that the product of ξ fields will satisfy relations of the type of Eq. (1.8), which must be compatible with the fusion and braiding relations. Consider a product of two general fields $\xi^{(J_1)}_{M_1}(\sigma_1)\xi^{(J_2)}_{M_2}(\sigma_2)$. If we choose $\sigma_+ > \sigma_1$, $\sigma_2 > \sigma_-$, the action of the generators on each field will be given by the previous analysis, and so the σ_+ dependence will be irrelevant. Accordingly, we see that Eq. (1.8) will apply for $\mathscr{B}_+$, and we get for $J^a \in \mathscr{B}_+$,

$$\begin{aligned}&\mathcal{O}(J^a)_{\sigma+}\xi^{(J_1)}_{M_1}(\sigma_1)\xi^{(J_2)}_{M_2}(\sigma_2)\\&\quad=\xi^{(J_1)}_{N_1}(\sigma_1)\xi^{(J_2)}_{N_2}(\sigma_2)\Lambda^a_{bc}\{\Lambda^b_{de}[J^d]_{N_1M_1}[J^e]_{N_2M_2}\}\mathcal{O}(J^c)_{\sigma+}\,.\end{aligned}\tag{3.19}$$

Similarly, for $J^a \in \mathscr{B}_-$ Eq. (1.15) will apply and we get

$$\begin{aligned}&\xi^{(J_1)}_{M_1}(\sigma_1)\xi^{(J_2)}_{M_2}(\sigma_2)\mathcal{O}(J^a)_{\sigma-}\\&\quad=\Lambda^a_{bd}\mathcal{O}(J^b)\xi^{(J_1)}_{N_1}(\sigma_1)\xi^{(J_2)}_{N_2}(\sigma_2)\Lambda^d_{ec}[\widetilde{J}^c]_{N_1M_1}[\widetilde{J}^e]_{N_2M_2}\,.\end{aligned}\tag{3.20}$$

Let us next explicitly verify that these transformation laws are consistent with the OPA of the ξ fields. First, in ref. [4] the complete fusion of the ξ fields was shown to be given (in the coordinates of the sphere) by

$$\begin{aligned}\xi^{(J_1)}_{M_1}(\sigma_1)\xi^{(J_2)}_{M_2}(\sigma_2)&=\sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g^{J_{12}}_{J_1J_2}(J_1,M_1;J_2,M_2|J_{12})\\&\quad\times\sum_{\{\nu\}}\xi^{(J_{12},\{\nu\})}_{M_1+M_2}(\sigma_2)\langle\varpi_{J_{12}},\{\nu\}|V^{(J_1)}_{J_2-J_{12}}(e^{i\sigma_1}-e^{i\sigma_2})|\varpi_{J_2}\rangle\,,\end{aligned}\tag{3.21}$$

where $(J_1,M_1;J_2,M_2|J_{12})$ are the q-Clebsch–Gordan coefficients, and $g^{J_{12}}_{J_1J_2}$ are the so-called coupling constants, which depend on the spins only. The primary fields $V^{(J)}_m(z)$ whose matrix elements appear on the right-hand side are the so-called Bloch wave operators, with diagonal monodromy, which are linearly related to the ξ fields. We will come back to them in Sect. 5. Let us next apply this relation to the two sides of Eqs. (3.19) and (3.20). This yields the same consistency condition for both B_+ and B_-:

$$\begin{aligned}&\sum_{N_1+N_2=N_{12}}(J_1,N_1;J_2,N_2|J_{12})\Lambda^b_{de}[J^d]_{N_1M_1}[J^e]_{N_2M_2}\\&\quad=(J_1,M_1;J_2,M_2|J_{12})[J^b]_{N_{12}M_{12}}\,,\end{aligned}\tag{3.22}$$

which is just the standard form of the recurrence relation for the $3j$ symbols. Equation 3.22 expresses the fact that the $3j$ symbols realize the decomposition into irreducible representations of q tensorial products. Similarly, we apply Eq. (3.1) to both sides of Eqs. (3.19) and (3.20), assuming for definiteness that $\sigma_1 > \sigma_2$. One obtains the consistency condition

$$(J_1,J_2)^{P_2P_1}_{N_1N_2}\Lambda^b_{de}[J^d]_{N_1M_1}[J^e]_{N_2M_2}=\Lambda^b_{de}[J^d]_{P_2N_2}[J^e]_{P_1N_1}(J_1,J_2)^{N_2N_1}_{M_1M_2}\,.\tag{3.23}$$

According to Eq. (3.2), this is equivalent to the condition that the universal R matrix interchanges the two coproducts. As is well known, this is true by definition. A similar conclusion is reached if we choose $\sigma_1 < \sigma_2$ instead. It is clear from their derivation that Eqs. (3.22) and (3.23) are consequences of the commutativity of fusion and braiding, and of the Yang–Baxter equation, respectively, and thus of the polynomial equations. Equations (3.22) and (3.23) tell us that given the existence of a relation of the form of Eq. (1.7) in the theory – realized here by Eqs. (3.3), (3.4) – the operator algebra of the ξ fields has to be covariant under the action of the quantum group given by

$$\xi_M^{(J)}(\sigma) \to \xi_N^{(J)}(\sigma)[J^a]_{NM} . \tag{3.24}$$

For a product of fields, the statement of covariance becomes

$$\xi_{M_1}^{(J_1)}(\sigma_1)\xi_{M_2}^{(J_2)}(\sigma_2) \to \xi_{N_1}^{(J_1)}(\sigma_1)\xi_{N_2}^{(J_2)}(\sigma_2)\Lambda_{de}^b[J^d]_{N_1M_1}[J^e]_{N_2M_2} \tag{3.25}$$

according to Eq. (3.19). The quantum group action so defined coincides with the one introduced in ref. [2] without derivation.

In the present case the generators are themselves given by the simplest ξ fields of spin $\frac{1}{2}$, and we thus find ourselves in a bootstrap situation where we could try to derive the braiding and fusion of ξ fields with arbitrary spins from those of $\xi_{\pm 1/2}^{(1/2)}$ and $\xi_M^{(J)}$. In fact, this works for the case where all spins are half-integer positive, thus multiples of the spins of the generator, as was shown in refs. [2, 4, 5] using also the associativity of the operator algebra. We remark that on the other hand, the exchange algebra in the more general case of arbitrary continuous spins was derived by an entirely different, direct constructive method in refs. [9, 12] (within the Bloch wave picture). Notice also that the considerations above apply beyond the level of primaries, as Eq. (3.21) involves the contributions of all the descendants as well. Their behaviour is governed by the general Moore–Seiberg formalism [16]. The last term of Eq. (3.21) is a number, and thus not acted upon when we derive the braiding with the operators $\mathcal{O}(J^a)_{\sigma_+}$. This is consistent with the quantum group structure since it does not depend on the magnetic quantum numbers M_i. Thus the quantum group structure of all the descendants of the ξ fields is the same. As a matter of fact, we may use the orthogonality of the $3j$ symbols to transform Eq. (3.21) into

$$\sum_{M_1+M_2=M_{12}} (J_1, M_1; J_2, M_2|J_{12})\xi_{M_1}^{(J_1)}(\sigma_1)\xi_{M_2}^{(J_2)}(\sigma_2)$$
$$= g_{J_1J_2}^{J_{12}} \sum_{\{v\}} \xi_{M_1+M_2}^{(J_{12},\{v\})}(\sigma_2)\langle\varpi_{J_{12}}, \{v\}|V_{J_2-J_{12}}^{(J_1)}(e^{i\sigma_1} - e^{i\sigma_2})|\varpi_{J_2}\rangle . \tag{3.26}$$

Let us denote the left-hand side by $\xi_{M_{12}}^{[J_1,J_2](J_{12})}(\sigma_1, \sigma_2)$. It follows from Eqs. (3.19), (3.20) and (3.21) that under the action of the generators $\mathcal{O}[J_+]_{\sigma_+}^{(R)}$, $\mathcal{O}[q^{J_3}]_{\sigma_+}^{(R)}$, $\mathcal{O}[J_-]_{\sigma_-}^{(R)}$, $\mathcal{O}[q^{-J_3}]_{\sigma_-}^{(R)}$, with $\sigma_+ > \sigma_1, \sigma_2 > \sigma_-$, their transformation laws are similar to Eqs. (3.7), and (3.18) with spin J_{12}. Following the same line as above, this finally shows that the braiding matrix of this field with any covariant field of spin, say J' must obey a relation of the form Eq. (3.23), and thus be the corresponding universal R matrix. In particular, we see that, since this R matrix is equal to 1 if

$J_{12}=0$, $\xi_0^{[J,J](0)}(\sigma_1,\sigma_2)$ commutes with any $\xi_M^{(J)}(\sigma)$ field. We will come back to this important fact below.

3.3. The Algebra of the Generators. There are two levels which we discuss in turn.

3.3.1. The Algebra within $\mathscr{B}_+$. Consider first $\mathscr{B}_+$. The novel feature of the present generators is that they depend upon σ. When we discuss their algebra, we could use the fusion relations to consider their products at the same point. This is not necessary since the above quantum group action depends only on the ordering between the σ of the generator and the σ of the covariant field, and it will pay not to do so. Thus, when we discuss quadratic relations within $\mathscr{B}_+$, we introduce two points σ_+ and σ'_+, both larger than σ, and for $J^a, J^b\in\mathscr{B}_+$, we have a priori four products $\mathscr{O}(J^a)_{\sigma_+}\mathscr{O}(J^b)_{\sigma'_+}$, $\mathscr{O}(J^a)_{\sigma'_+}\mathscr{O}(J^b)_{\sigma_+}$, $\mathscr{O}(J^b)_{\sigma_+}\mathscr{O}(J^a)_{\sigma'_+}$, $\mathscr{O}(J^b)_{\sigma'_+}\mathscr{O}(J^a)_{\sigma_+}$. When we let them act, for instance, to the right, it is clear that the rightmost will act first irrespective of which is at σ and which is at σ'. On the other hand, for each choice of ordering between σ and σ', it follows from the braiding relations Eq. (3.1) or (3.9) that we have equations of the type

$$\mathscr{O}(J^a)_{\sigma_+}\mathscr{O}(J^b)_{\sigma'_+}=\rho^{ab}_{cd}\mathscr{O}(J^c)_{\sigma'_+}\mathscr{O}(J^d)_{\sigma_+}\,, \tag{3.27}$$

where ρ is a numerical matrix. Thus we need only discuss relations between the products $\mathscr{O}(J^a)_{\sigma_+}\mathscr{O}(J^b)_{\sigma'_+}$ and $\mathscr{O}(J^b)_{\sigma_+}\mathscr{O}(J^a)_{\sigma'_+}$, with $a\neq b$. We will refer to this particular type of braiding relations as fixed-point (FP) commutation relations. For the specific case we are discussing, this means that we have to compare the action of $\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\mathscr{O}[J_+]^{(R)}_{\sigma'_+}$ and $\mathscr{O}[J_+]^{(R)}_{\sigma_+}\mathscr{O}[q^{J_3}]^{(R)}_{\sigma'_+}$. Of course this amounts to looking for the operator equivalent of the matrix commutation relation

$$[q^{J_3}]_{MP}[J_+]_{PN}=q[J_+]_{MP}[q^{J_3}]_{PN} \tag{3.28}$$

that holds in any spin J representation. Making use of the explicit form of the q Clebsch–Gordan coefficients, one sees that

$$\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\mathscr{O}[J_+]^{(R)}_{\sigma'_+}-q\mathscr{O}[J_+]^{(R)}_{\sigma_+}\mathscr{O}[q^{J_3}]^{(R)}_{\sigma'_+}=-q^{\frac{1}{2}}\kappa_+^{(R+)}\kappa_3^{(R+)}\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+,\sigma'_+)\,, \tag{3.29}$$

where $\xi_0^{[1/2,1/2](0)}(\sigma_+,\sigma'_+)$ is the left-hand side of Eq. (3.26) with $J_1=J_2=\frac{1}{2}$, $J_{12}=M_{12}=0$. Thus the right-hand side does not vanish. However, it follows from the above discussion that

$$\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+,\sigma'_+)\xi_M^{(J)}(\sigma)=\xi_M^{(J)}(\sigma)\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+,\sigma'_+)\,. \tag{3.30}$$

Thus our generators satisfy the $\mathscr{B}_+$ FP commutation relations up to a central term, and we have found the equivalent Eq. (2.5) of Sect. 2, where C_+ is replaced by $\xi_0^{[1/2,1/2](0)}(\sigma_+,\sigma'_+)$.

Concerning $\mathscr{B}_-$ the discussion of the left-action is essentially the same as for the right-action realization of $\mathscr{B}_+$. One finds

$$\mathscr{O}[J_-]^{(L)}_{\sigma_-}\mathscr{O}[q^{-J_3}]^{(L)}_{\sigma'_-}-q\mathscr{O}[q^{-J_3}]^{(L)}_{\sigma_-}\mathscr{O}[J_-]^{(L)}_{\sigma'_-}=-\kappa_-^{(L-)}\kappa_3^{(L-)}q^{\frac{1}{2}}\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+,\sigma'_+)\,. \tag{3.31}$$

Note that, since we now act to the left, it is the leftmost operator that acts first. Thus on the left-hand side of this equation the ordering is the reverse of the one of the matrix relation. On the other hand, Sect. 2 only dealt with right-action. Making use of Eqs. (3.13), it is easy to write the following relation equivalent to Eq. (3.31).

$$\mathcal{O}[q^{J_3}]^{(R)}_{\sigma_-}\mathcal{O}[J_-]^{(R)}_{\sigma'_-} - q^{-1}\mathcal{O}[J_-]^{(R)}_{\sigma_-}\mathcal{O}[q^{J_3}]^{(R)}_{\sigma'_-} = \kappa_-^{(R-)}\kappa_3^{(R-)}q^{-\frac{1}{2}}\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_-,\sigma'_-). \quad (3.32)$$

This is the operator (FP) realization of Eq. (2.6) of Sect. 2.

3.3.2. The Complete Algebra. In order to really compare two successive actions we have to act from the same side for both, so that we will combine one Borel algebra with the antipode of the other. Let us act to the right following Sect. 2. A priori, we have a problem to extend the notion of FP commutation relations: so far the Borel subalgebras $\mathscr{B}_+$ and $\mathscr{B}_-$ – as well as their antipodes – are defined by using points larger and smaller than σ, respectively.

Let us open a parenthesis to answer a question which may have come to the mind of the reader. The fact that we only have two generators $\mathcal{O}[q^{J_3}]^{(R)}_{\sigma_+}$ and $\mathcal{O}[J_+]^{(R)}_{\sigma_+}$ is of course due to our use of the $\xi^{(1/2)}$ field which have only two components. A priori, we could start from $\xi^{(J)}_M(\sigma_+)$, with $-J \leqq M \leqq J$. This does not help, however, since their braiding only defines the coproduct action of the enveloping algebra of $\mathscr{B}_+$, in agreement with the fact that the $\xi^{(J)}$ fields may be obtained by fusion of the $\xi^{(1/2)}$ fields. At the level of $J = 1$, for instance, one has the correspondence $\xi^{(1)}_{-1} \sim \mathcal{O}[q^{2J_3}]$, $\xi^{(1)}_0 \sim \mathcal{O}[J_+q^{J_3}]$, $\xi^{(1)}_1 \sim \mathcal{O}[J_+q^{J_3}]$. Thus the other Borel subalgebra never appears.

Making use of the monodromy. At this point, we will make use of the monodromy properties of the $\xi^{(1/2)}$ fields. As already recalled, these fields are linearly related to Bloch wave operators with diagonal monodromy – the explicit formulae will be recalled in Sect. 5. From this, it is straightforward to deduce that

$$\begin{aligned} \xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma + 2\pi) &= \xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma)\,, \\ \xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma + 2\pi) &= -q\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma) + 2q^{\frac{1}{2}}\cos(h\varpi)\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma)\,, \end{aligned} \quad (3.33)$$

where ϖ is the (rescaled) zero-mode momentum of the Bäcklund free field. Of course, we also have the inverse relation

$$\begin{aligned} \xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma - 2\pi) &= 2q^{-\frac{1}{2}}\cos(h\varpi)\xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma) - q^{-1}\xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma)\,, \\ \xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma - 2\pi) &= \xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma)\,. \end{aligned} \quad (3.34)$$

For the following, it is important to stress that ϖ is an operator, with non-trivial commutation relations with the $\xi^{(1/2)}$ fields. It seems natural to conjecture that if the fields $\xi^{(1/2)}_\alpha(\sigma)$ satisfy braiding relations with $\xi^{(J)}_M$ given by the R matrix $(\frac{1}{2},J)$ (resp. $\overline{(\frac{1}{2},J)}$), then the translated fields $\xi^{(1/2)}_\alpha(\sigma - 2\pi)$ (resp. $\xi^{(1/2)}_\alpha(\sigma + 2\pi)$) satisfy braiding relations given by the R matrix $\overline{(\frac{1}{2},J)}$ (resp. $(\frac{1}{2},J)$)[11]. To prove this we need to know how to commute $\cos(h\varpi)$ with $\xi_M^{(J)}$. The commutation of the fieldss$\xi_\alpha^{(1/2)}(\sigma)$ is easily derived from the condition that the translated fields $\xi_\alpha^{(1/2)}(\sigma \pm 2\pi)$ have the same braiding as the fields $\xi_\alpha^{(1/2)}(\sigma)$ and the consistency of the commutation of $\cos(h\varpi)$ with the translated fields. One obtains[12]

$$2\cos(h\varpi)\xi_{\pm\frac{1}{2}}^{(\frac{1}{2})} = 2q^{\mp 1}\xi_{\pm\frac{1}{2}}^{(\frac{1}{2})}\cos(h\varpi) \pm q^{\mp\frac{1}{2}}(q-q^{-1})\xi_{\mp\frac{1}{2}}^{(\frac{1}{2})} .$$

The consistency of the commutation of $\cos(h\varpi)$ with the fusion of two fields ξ leads to

$$2\cos(h\varpi)\xi_M^{(J)} = 2q^{-2M}\xi_M^{(J)}\cos(h\varpi) + \sum_N \xi_N^{(J)} E_{NM}^{(J)} ,$$

where the matrices $E_{NM}^{(J)}$ must satisfy

$$(J_1,M_1;J_2,M_2|J_{12})E_{N_{12}M_{12}}^{(J_{12})} = \sum_{N_1+N_2=N_{12}} (J_1,M_1;J_2,N_2|J_{12})E_{N_2M_2}^{(J_2)}q^{-2M_1}\delta_{N_1M_1}$$
$$+ (J_1,N_1;J_2,M_2|J_{12})E_{N_1M_1}^{(J_1)}\delta_{N_2M_2} .$$

This defines a recurrence relation which determines all the matrix elements $E_{NM}^{(J)}$ from $E_{\pm\frac{1}{2}\mp\frac{1}{2}}^{(\frac{1}{2})}$. Comparing with Eq. (3.22), we see that actually the matrices $(J_\pm q^{-J_3})_{MN}$ satisfy the same recursion relation. Therefore

$$E_{NM}^{(J)} = a(J_+q^{-J_3})_{NM} + b(J_-q^{-J_3})_{NM} .$$

The coefficients a and b are determined from the commutation with $\xi_{\pm 1/2}^{(1/2)}$ and we get finally

$$2\cos(h\varpi)\xi_M^{(J)} = 2\xi_N^{(J)}(q^{-2J_3})_{NM}\cos(h\varpi)$$
$$+ (q-q^{-1})\sum_N \xi_N^{(J)}((J_- - J_+)q^{-J_3})_{NM} . \qquad (3.35)$$

Equation (3.35) defines a coproduct action of $2\cos(h\varpi)$ on $\xi_M^{(J)}$ with

$$\Delta(2\cos(h\varpi)) = q^{-2J_3} \otimes 2\cos(h\varpi)$$
$$+ (q-q^{-1})(J_-q^{-J_3} - J_+q^{-J_3}) \otimes \mathrm{Id} . \qquad (3.36)$$

For the action on two ξ it is sufficient to replace the operators q^{-2J_3} and $(J_-q^{-J_3} - J_+q^{-J_3})$ by their ordinary coproduct. Now it is straightforward to prove our conjecture and this allows us to construct the missing generators $\mathcal{O}[J_-]_{\sigma_+}^{(R)}$ and $\mathcal{O}[J_+]_{\sigma_-}^{(R)}$.

[11] Actually this is an immediate consequence of the translation invariance of the braiding matrix, whose position dependence is governed by the step function $\varepsilon(\sigma-\sigma')$. Using the full region of validity $\sigma,\sigma' \in [0,2\pi]$ of Eq. (3.1) according to the remark made there, the assertion then trivially follows. However we provide below an independent, explicit proof using only the validity of Eq. (3.1) for $\sigma,\sigma' \in [0,\pi]$.

[12] This formula could also be derived directly from the definition of the $\xi^{\frac{1}{2}}$ fields in terms of the $\psi^{\frac{1}{2}}$ fields. It is however not necessary to make reference to the $\psi^{\frac{1}{2}}$ fields; at this stage we could even forget that ϖ is the (rescaled) zero-mode momentum of the Backlund free field.

They are given by

$$\mathscr{O}[J_-]^{(R)}_{\sigma_+} = \kappa_-^{(R+)} \xi^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+ - 2\pi)\,, \tag{3.37}$$

$$\mathscr{O}[J_+]^{(R)}_{\sigma_-} = \kappa_+^{(R+)} \xi^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_- + 2\pi)\,, \tag{3.38}$$

with

$$\frac{\kappa_-^{(R+)}}{\kappa_3^{(R+)}} = \frac{\kappa_-^{(R-)}}{\kappa_3^{(R-)}} = \frac{q^{-\frac{1}{2}}}{1-q^{-2}}\,,$$

$$\frac{\kappa_+^{(R+)}}{\kappa_3^{(R+)}} = \frac{\kappa_+^{(R-)}}{\kappa_3^{(R-)}} = \frac{q^{\frac{1}{2}}}{1-q^{2}}\,. \tag{3.39}$$

The explicit expressions for the monodromy of $\xi^{(1/2)}_{\pm 1/2}(\sigma)$ lead to the following relation between $\mathscr{O}[J_\pm], \mathscr{O}[q^{J_3}]$ and $\cos(h\varpi)$:

$$2\cos(h\varpi)\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_\pm} = (q-q^{-1})(\mathscr{O}[J_-]^{(R)}_{\sigma_\pm} - \mathscr{O}[J_+]^{(R)}_{\sigma_\pm})\,. \tag{3.40}$$

It is easy to check that the action of $\cos(h\varpi)$ as defined by Eq. (3.40) is completely equivalent to the one given in Eq. (3.35), as it should be.

Finally, we verify that a FP algebra comes out, which is a realization of the algebra in Eqs. (2.15)–(2.19) of Sect. 2. First, comparing Eq. (2.5) with Eq. (3.29) at σ_+, and Eq. (2.6) with Eq. (3.32) at $\sigma_+ - 2\pi$, we see that we may identify

$$C_+ = q^{\frac{1}{2}} \kappa_+^{(R+)} \kappa_3^{(R+)} \xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+, \sigma'_+)\,,$$

$$C_- = q^{-\frac{1}{2}} \kappa_-^{(R-)} \kappa_3^{(R-)} \xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+ - 2\pi, \sigma'_+ - 2\pi)\,. \tag{3.41}$$

For C_-, one uses the monodromy properties Eqs. (3.34) to verify that

$$\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+ - 2\pi, \sigma'_+ - 2\pi) = \xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+, \sigma'_+)\,.$$

We therefore have (cf. Eq. (3.39))

$$C_- = -\left(\frac{\kappa_3^{(R-)}}{\kappa_3^{(R+)}}\right)^2 C_+\,. \tag{3.42}$$

Now we can establish the FP equivalent of Eq. (2.18), which takes the form

$$\mathscr{O}[J_+]^{(R)}_{\sigma_+}\mathscr{O}[J_-]^{(R)}_{\sigma'_+} - \mathscr{O}[J_-]^{(R)}_{\sigma_+}\mathscr{O}[J_+]^{(R)}_{\sigma'_+} = \mathscr{O}[D]^{(R)}_{\sigma_+,\sigma'_+} + \frac{\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\mathscr{O}[q^{J_3}]^{(R)}_{\sigma'_+}}{q-q^{-1}}\,. \tag{3.43}$$

One finds that $\mathscr{O}[D]$ is given by the remarkable expression

$$\mathscr{O}[D]^{(R)}_{\sigma_+,\sigma'_+} = \frac{1}{q-q^{-1}}\, 2\cos(h\varpi) C_+(\sigma_+, \sigma'_+)\,, \tag{3.44}$$

which shows how the generator introduced in Sect. 2 is realized. Next multiply both sides of Eq. (3.40) by $C_+(\sigma'_+, \sigma''_+)$. This gives the FP version of Eq. (2.17):

$$C_+(\sigma'_+, \sigma''_+)\mathscr{O}[J_-]^{(R)}_{\sigma_+} + C_-(\sigma'_+, \sigma'_+)\mathscr{O}[J_+]^{(R)}_{\sigma_+} = \mathscr{O}[D]^{(R)}_{\sigma'_+,\sigma''_+}\mathscr{O}[q^{J_3}]^{(R)}_{\sigma_+}\,, \tag{3.45}$$

from which one finally gets

$$C_+ = -C_- \,. \tag{3.46}$$

This leads to the condition $(\kappa_3^{(R+)})^2 = (\kappa_3^{(R-)})^2$. We choose[13]

$$\kappa_3^{(R+)} = \kappa_3^{(R-)} \,. \tag{3.47}$$

The right-action of $\mathcal{O}[D]^{(R)}_{\sigma_+,\sigma'_+}$ is found to read

$$\begin{aligned}\mathcal{O}[D]^{(R)}_{\sigma_+,\sigma'_+}\xi_M^{(J)}(\sigma) =& \xi_M^{(J)}(\sigma)q^{-2M}\mathcal{O}[D]^{(R)}_{\sigma_+,\sigma'_+}\\ &+ \xi_N^{(J)}(\sigma)[J_- - J_+]_{NM}q^{-M}C_+(\sigma_+,\sigma'_+)\,,\end{aligned} \tag{3.48}$$

as may be derived either from Eq. (3.43) or from the definition Eq. (3.44). It takes the form of a coproduct action, if we define the coproduct of D by

$$\widetilde{\Delta}(D) = q^{-2J_3}\otimes D + (J_- - J_+)q^{-J_3}\otimes C_+ \,. \tag{3.49}$$

In conclusion we have derived a FP realization of the algebra Eqs. (2.15)–(2.19) of Sect. 2, where $\mathcal{O}[q^{J_3}]$ and $\mathcal{O}[J_+]$ depend upon one point, while $C_\pm$ and $\mathcal{O}[D]$ depends upon two points. Clearly this number of points may be identified with a sort of additive grading of the algebra introduced in Sect. 2, such that Eqs. (2.15), (2.16), (2.18) have grading 2, Eqs. (2.17), (2.19) grading 3, and Eqs. (2.20) gradings 5 and 4 respectively. In the present realization, the equations of grading larger than 2 are not directly FP realized, but the following simplified versions are: In fact we are in the special case where $C_- = -C_+$ are equal and where $\mathcal{O}[D]^{(R)}_{\sigma_+,\sigma'_+}$ is equal to either of them up to an (ϖ dependent) operator that does not depend upon the points. Thus we may divide both sides of Eq. (2.20) by $C_\pm$, which reduces the grading (number of points) to 1. Similarly, equations of higher grading are to be divided by appropriate powers of $C_\pm$, so that in the end the grading is always $\leqq 2$. These simplified relations hold for the present construction.

Let us summarize our present results. We have obtained two types of representations for the operators $J_\pm, q^{J_3}$ in terms of the fields $\xi_\alpha^{(1/2)}(\sigma_+)$ and $\xi_\alpha^{(1/2)}(\sigma_-)$:

$$\begin{aligned}\frac{\mathcal{O}[q^{J_3}]^{(R)}_{\sigma_+}}{\kappa_3^{(R+)}} &= \xi_{-\frac12}^{(\frac12)}(\sigma_+) = \xi_{\frac12}^{(\frac12)}(\sigma_+ - 2\pi)\,,\\ \frac{\mathcal{O}[q^{J_+}]^{(R)}_{\sigma_+}}{\kappa_+^{(R+)}} &= \xi_{\frac12}^{(\frac12)}(\sigma_+) = q^{\frac12}(q^\varpi + q^{-\varpi})\xi_{\frac12}^{(\frac12)}(\sigma_+ - 2\pi) - q\xi_{-\frac12}^{(\frac12)}(\sigma_+ - 2\pi)\,,\\ \frac{\mathcal{O}[q^{J_-}]^{(R)}_{\sigma_+}}{\kappa_-^{(R+)}} &= q^{-\frac12}(q^\varpi + q^{-\varpi})\xi_{\frac12}^{(\frac12)}(\sigma_+) - q^{-1}\xi_{\frac12}^{(\frac12)}(\sigma_+) = \xi_{-\frac12}^{(\frac12)}(\sigma_+ - 2\pi)\,.\end{aligned} \tag{3.50}$$

The representation for σ_- is obtained by the exchange of σ_+ and $\sigma_- + 2\pi\,(\sigma_+ - 2\pi$ and $\sigma_-)$. We have seen also that $(q^\varpi + q^{-\varpi})$ becomes part of the enveloping algebra through the relation

$$(q^\varpi + q^{-\varpi})\mathcal{O}[q^{J_3}] = (q - q^{-1})(\mathcal{O}[J_-] - \mathcal{O}[J_+])\,. \tag{3.51}$$

[13] This is of course consistent with the fact that q^{-J_3} is the antipode of q^{J_3}.

We remark that one could derive FP commutation relations not only for operators at points $(\sigma_+\sigma'_+)$ or (σ_-,σ'_-) but also at points (σ_+,σ'_-) or (σ_-,σ'_+). This would introduce new central terms, which commute with all the $\xi_M^{(J)}(\sigma)$ fields, namely

$$\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_- + 2\pi, \sigma'_+) = \xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_-, \sigma'_+ - 2\pi)\,,$$

$$\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+, \sigma'_- + 2\pi) = \xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+, -2\pi, \sigma'_-)\,. \tag{3.52}$$

Another viewpoint. Could we work in a way that would be more symmetric between $\mathscr{B}_+$ and $\mathscr{B}_-$, without having to use the monodromy to "transport" σ_- to σ_+ or vice versa? This is indeed possible by starting from an expression of the form $\mathcal{O}(J^a)_{\sigma_+}\xi_M^{(J)}(\sigma)\mathcal{O}(J^b)_{\sigma_-}$, with $J^a \in B_+$, and $J^b \in B_-$. Then, the relationship between the two orderings of the action of J^a and J^b is a consequence of the Yang–Baxter equation associated with the braiding operation $\xi_\alpha^{(1/2)}(\sigma_+)\xi_M^{(J)}(\sigma)\xi_\beta^{(1/2)}(\sigma_-) \to \xi_\gamma^{(1/2)}(\sigma_-)\xi_N^{(J)}(\sigma)\xi_\delta^{(1/2)}(\sigma_+)$. In particular, choosing $\alpha = \gamma = \frac{1}{2}$, $\beta = \delta = -\frac{1}{2}$ gives back the equation $\langle J,M|[J_+,J_-]|J,M\rangle = \lfloor 2M \rfloor$ satisfied by the matrix representations.

4. The Extended Framework and the Construction of $\mathcal{O}[q^{-J_3}]$

In the previous sections, we have exclusively considered $\xi_M^{(J)}$ fields with half-integer positive spins. Although we obtained the combination $\mathcal{O}[D] = (C_+\mathcal{O}[J_-] + C_-\mathcal{O}[J_+])\mathcal{O}[q^{-J_3}]$ (with $C_+ = -C_-$), it was not possible within this framework to construct the operator $\mathcal{O}[q^{-J_3}]$ itself for right-action (or $\mathcal{O}[q^{J_3}]$ for left-action). The reason for this is in fact easy to understand already by a classical scale argument: Consider Eqs. (2.8) and (2.18). Since $\mathcal{O}[J_-]$ and $\mathcal{O}[J_+]$ are realized by $\xi^{(1/2)}$ fields that have classical scale dimension $-1/2$, it follows from Eq. (2.18) that $\mathcal{O}[D]$ has scale dimension -1, and this of course true for our realization Eq. (3.44). On the other hand, Eq. (2.8) then tells us immediately that the scale dimension of $\mathcal{O}[q^{-J_3}]$ must be $+1/2$, as C_+ is realized by operators with total dimension -1[14]. But this is just the classical dimension of the ξ fields with spin $-1/2$. Thus we are lead to consider the extended framework described in refs. [9, 13], where in particular negative half-interger spins can be considered. However, for our purposes here it will be sufficient to know that their braiding with any $\xi_{M'}^{(J')}$ is still given by Eq. (3.1) now specified to representations of spin $-\frac{1}{2}$ and J', and that the leading-order fusion of $\xi_{\pm 1/2}^{(-1/2)}$ with $\xi_M^{(J)}$ is also described correctly by the formula for positive half-integer spins [2], so that ($z := e^{i\sigma}$)

$$\xi_{\pm\frac{1}{2}}^{(-\frac{1}{2})}(\sigma)\xi_{M'}^{(J')}(\sigma') \sim \left(1 - \frac{z'}{z}\right)^{J'h/\pi} q^{-\frac{1}{2}M' \mp \frac{1}{2}J'} \sqrt{\frac{\binom{2J'}{J'+M'}}{\binom{2J'-1}{J'-\frac{1}{2}+M'\pm\frac{1}{2}}}}\,\xi_{\pm\frac{1}{2}+M'}^{(J'-\frac{1}{2})}(\sigma')\,. \tag{4.1}$$

[14] Since the coefficient of $\mathcal{O}[q^{-J_3}]^2$ is zero in our realization, this causes no conflict with Eq. (2.9).

The basic observation is now that $\xi^{(-1/2)}_{1/2}$, which is the formal inverse of $\xi^{(1/2)}_{-1/2}$ according to Eq. (4.1), has braiding relations with the $\xi^{(J)}_M$ which are exactly those appropriate for $\mathcal{O}[q^{-J_3}]^{(R)}$, i.e.

$$\xi^{(-\frac{1}{2})}_{\frac{1}{2}}(\sigma_+)\xi^{(J)}_M(\sigma) = q^{-M}\xi^{(J)}_M(\sigma)\xi^{(-\frac{1}{2})}_{\frac{1}{2}}(\sigma_+)\,. \tag{4.2}$$

Similarly, from the R matrix for the other case $\sigma_- < \sigma$ one obtains

$$\xi^{(-\frac{1}{2})}_{-\frac{1}{2}}(\sigma_-)\xi^{(J)}_M(\sigma) = q^{-M}\xi^{(J)}_M(\sigma)\xi^{(-\frac{1}{2})}_{-\frac{1}{2}}(\sigma_-)\,. \tag{4.3}$$

Thus we should identify

$$\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma_+} = \kappa^{(R+)}_{-3}\xi^{(-\frac{1}{2})}_{\frac{1}{2}}(\sigma_+)\,, \tag{4.4}$$

in analogy with Eq. (3.5), and

$$\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma_-} = \kappa^{(R-)}_{-3}\xi^{(-\frac{1}{2})}_{-\frac{1}{2}}(\sigma_-)\,. \tag{4.5}$$

Analogous formulae of course describe the realization of $\mathcal{O}[q^{J_3}]$ for left-action. From Eq. (4.1) and the first of Eqs. (3.33) it follows that

$$\xi^{(-\frac{1}{2})}_{\frac{1}{2}}(\sigma_+) = \xi^{(-\frac{1}{2})}_{\frac{1}{2}}(\sigma_+ - 2\pi)\,. \tag{4.6}$$

Thus we are lead to identify

$$\kappa^{(R+)}_{-3} = \kappa^{(R-)}_{-3}\,, \tag{4.7}$$

similarly to Eq. (3.47) of Sect. 3.3.2, and analogously for the left-action case. Notice however that the coefficients $\kappa^{(R+)}_{-3} = \kappa^{(R-)}_{-3}$ (resp. $\kappa^{(L+)}_{3} = \kappa^{(L-)}_{3}$) are not fixed in terms of $\kappa^{(R+)}_{+}, \kappa^{(R-)}_{-}$, as was the case for their counterparts $\kappa^{(R+)}_{3} = \kappa^{(R-)}_{3}$ (cf. Eq. (3.8), (3.14)). This is of course a consequence of the fact that $\mathcal{O}[q^{-J_3}]^{(R)}$ does not appear in the coproduct action of the other generators. Since the coproduct Eq. (2.3) is asymmetric in $\mathcal{O}[q^{J_3}]$ and $\mathcal{O}[q^{-J_3}]$, the analogue of Eq. (2.5) is not valid. Rather, multiplying it by $\mathcal{O}[q^{-J_3}]$ from both sides, one would conclude that

$$q\mathcal{O}[q^{-J_3}]\mathcal{O}[J_+] - \mathcal{O}[J_+]\mathcal{O}[q^{-J_3}] = C_+(\mathcal{O}[q^{-J_3}])^2\,. \tag{4.8}$$

However, in our FP realization this cannot be true since the grading of the right-hand side is 4, while that of the left-hand side is 2. The proper FP analogue of Eq. (4.8) is obtained if – as announced already at the end of Sect. 2 – we now introduce a new central charge C_0, defined by

$$C_0(\sigma_+, \sigma'_+) = \mathcal{O}[q^{J_3}]^{(R)}_{\sigma_+}\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'_+}\,. \tag{4.9}$$

It is obvious that C_0 commutes with all the $\xi^{(J)}_M$, and one can make an expansion of this new central around $\sigma_+ = \sigma'_+$, as we will do for C_+, to verify that it is given by a sum of local quantities with respect to the ξ fields. If we now multiply Eq. (3.29) by $\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma''_+}$ and $\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_+}$ from the left and from the right respectively, we obtain

$$C_0(\sigma''_+,\sigma_+)\mathcal{O}[J_+]^{(R)}_{\sigma'_+}\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_+} - q\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma''_+}\mathcal{O}[J_+]^{(R)}_{\sigma_+}C_0(\sigma'_+,\sigma'''_+)$$
$$= -\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma''_+}C_+(\sigma_+,\sigma'_+)\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_+} . \tag{4.10}$$

Notice that the points $\sigma''_+ > \sigma_+ > \sigma'_+ > \sigma'''_+$ appears in the same sequence in both terms on the left-hand side of the above equation, in accord with the fixed-point prescription for the case of grading 4. Similarly, we have in place of Eq. (3.32),

$$C_0(\sigma''_-,\sigma_-)\mathcal{O}[J_-]^{(R)}_{\sigma'_-}\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_-} - q\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma''_-}\mathcal{O}[J_-]^{(R)}_{\sigma_-}C_0(\sigma'_-,\sigma'''_-)$$
$$= -\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma''_-}C_+(\sigma_-,\sigma'_-)\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_-} , \tag{4.11}$$

with $\sigma''_- < \sigma_- < \sigma'_- < \sigma'''_-$. With $\mathcal{O}[q^{-J_3}]$ at our disposal, we can now give a concrete sense also to Eq. (2.8) within our realization. Indeed, multiplying the FP equivalent of Eq. (2.17), that is Eq. (3.45), by $\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_+}$ from the right, one obtains

$$\mathcal{O}[D]^{(R)}_{\sigma'_+,\sigma''_+}C_0(\sigma_+,\sigma'''_+) = C_+(\sigma'_+,\sigma''_+)\mathcal{O}[J_-]^{(R)}_{\sigma_+}\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_+}$$
$$- C_+(\sigma'_+,\sigma''_+)\mathcal{O}[J_+]^{(R)}_{\sigma_+}\mathcal{O}[q^{-J_3}]^{(R)}_{\sigma'''_+} . \tag{4.12}$$

Thus we now have a FP realization of the full algebra, with both $\mathcal{O}[q^{J_3}]$ and $\mathcal{O}[q^{-J_3}]$ available for right- as well as for left-action, and this completes our considerations on the operator realization of the quantum group action.

5. Study of the Central Term

The existence of a non-trivial operator that commutes with all the $\xi^{(J)}_M(\sigma)$ (with a suitable range of σ) may seem surprising. Let us therefore investigate the structure of the central term in Eqs. (3.29), (3.32) in some more detail. For this purpose, it is convenient to re-express it in terms of Bloch wave operators. The simplest formulae arise by using the ψ fields introduced in ref. [2]. These are related to the ξ fields as follows:

$$\xi^{(J)}_M(\sigma) := \sum_{-J\leq m\leq J} |J,\varpi)^m_M \psi^{(J)}_m(\sigma) , \tag{5.1}$$

$$|J,\varpi)^m_M := \sqrt{\binom{2J}{J+M}}\, e^{ihm/2}\sum_t e^{iht(\varpi+m)}\binom{J-M}{(J-M+m-t)/2}\binom{J+M}{(J+M+m+t)/2} , \tag{5.2}$$

$$\binom{P}{Q} := \frac{\lfloor P\rfloor!}{\lfloor Q\rfloor!\lfloor P-Q\rfloor!}, \qquad \lfloor n\rfloor! := \prod_{r=1}^{n}\lfloor r\rfloor ,$$

where the variable t takes all values such that the entries of the binomial coefficients are non-negative integers. (We consider only the case of half-integer positive spin here.) The symbol ϖ denotes the rescaled Liouville zero-mode momentum. Using this relation for $J = \frac{1}{2}$, one finds

$$\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+, \sigma'_+) = 2i\{\sin[h(\varpi+1)]\psi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+)\psi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma'_+) - \sin[h(\varpi-1)]\psi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+)\psi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma'_+)\}\,. \tag{5.3}$$

In general the index m of the ψ fields characterizes the shift of ϖ:

$$\psi_m^{(J)}(\sigma)f(\varpi) = f(\varpi + 2m)\psi_m^{(J)}(\sigma)\,. \tag{5.4}$$

This shows that the central term commutes with ϖ. It then follows from the (inverse of) Eq. (5.1) that it also commutes with any $\psi_M^{(J)}(\sigma)$ field, if $\sigma_+, \sigma'_+ > \sigma$.

How is this possible? First consider the classical ($h = 0$ case). In this limit $h\varpi$ is kept fixed so that

$$\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(\sigma_+, \sigma'_+) \propto \psi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+)\psi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma'_+) - \psi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+)\psi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma'_+)\,.$$

Classically, $\psi_{\pm 1/2}^{(1/2)}$ are solutions of the Schrödinger equation $(-\partial_\sigma^2 + T)\psi_{\pm 1/2}^{(1/2)}(\sigma) = 0$. Making use of this fact, it is easy to Taylor-expand. The first terms read

$$\begin{aligned}
&\psi_{\frac{1}{2}}^{(\frac{1}{2})}(x+\varepsilon)\psi_{-\frac{1}{2}}^{(\frac{1}{2})}(x) - \psi_{-\frac{1}{2}}^{(\frac{1}{2})}(x+\varepsilon)\psi_{\frac{1}{2}}^{(\frac{1}{2})}(x)\\
&\sim \left(\psi_{\frac{1}{2}}^{(\frac{1}{2})'}(x)\psi_{-\frac{1}{2}}^{(\frac{1}{2})}(x) - \psi_{-\frac{1}{2}}^{(\frac{1}{2})'}(x)\psi_{\frac{1}{2}}^{(\frac{1}{2})}(x)\right)\\
&\times \left\{ \varepsilon + \frac{\varepsilon^3}{3!}T(x) + \frac{\varepsilon^4}{4!}2T'(x) + \frac{\varepsilon^5}{5!}(T''(x) + 3T^2(x)) \right.\\
&\left. + \frac{\varepsilon^6}{6!}(4T'''(x) + 6T(x)T'(x)) + \cdots \right\}.
\end{aligned} \tag{5.5}$$

The first factor is the Wronskian, which is a constant, say 1. We see that the classical central term has an expansion in $\sigma_+ - \sigma'_+$, where the coefficients are polynomials in $T(\sigma'_+)$ and its derivatives. Since $\psi_m^{(J)}(\sigma)$ is a primary with weight Δ_J, its Poisson bracket with $T(\sigma'_+)$ reads

$$\{T(\sigma'_+), \psi_m^{(J)}(\sigma)\}_{\text{P.B.}} = -4\pi\gamma\left(\delta(\sigma - \sigma'_+)\frac{\partial}{\partial\sigma} + \Delta_J\delta'(\sigma - \sigma'_+)\right)\psi_m^{(J)}(\sigma)\,.$$

It thus follows that the Poisson bracket of each term of the expansion Eq. (5.5) is a sum of derivatives of delta functions, which indeed vanishes for $\sigma_+, \sigma'_+ > \sigma$, as we wanted to verify.

Let us return to the quantum level. We expect that a similar mechanism will be at work. Indeed, the operators $\psi^{(1/2)}$ satisfy a quantum version of the Schrödinger equation. The expansion Eq. (3.21) of $\xi_0^{[1/2,1/2](0)}$ involves the descendants of unity which begin with T. One may expect that a general term will be given by an ordered polynomial in the (derivatives of) T. The simplest way to give it a meaning is to order with respect to the Fourier modes of this operator, in which case the expectation value between highest-weight states of each term in the expansion will be given by the expectation value of a polynomial of L_0. Let us verify this explicitly. Using the differential equation satisfied by $\psi^{(1/2)}$ fields one may express the expectation value of the central term as a hypergeometric function. The overall normalization of the ψ field is derived in ref. [2]. Using formulae given in refs. [2,4], one sees that

$$\langle\varpi|\psi^{(\frac{1}{2})}_{\pm\frac{1}{2}}(z)\psi^{(\frac{1}{2})}_{\mp\frac{1}{2}}(z')|\varpi\rangle = d_\pm(\varpi) z'^{-\Delta_{\frac{1}{2}}-\Delta(\varpi)} z^{-\Delta_{\frac{1}{2}}+\Delta(\varpi)}$$
$$\times\left(\frac{z'}{z}\right)^{\Delta(\varpi\pm1)}\left(1-\frac{z'}{z}\right)^{-h/2\pi} F\left(-\frac{h}{\pi},-\frac{h}{\pi}(1\pm\varpi);\ 1\mp\frac{h\varpi}{\pi};\ \frac{z'}{z}\right), \quad (5.6)$$

where $F(a,b;c;z)$ is the standard hypergeometric function, and

$$d_+ = \Gamma\left(\varpi\frac{h}{\pi}\right)\Gamma\left(-(\varpi+1)\frac{h}{\pi}\right), \qquad d_- = \Gamma\left(-\varpi\frac{h}{\pi}\right)\Gamma\left((\varpi-1)\frac{h}{\pi}\right). \quad (5.7)$$

The notation $\Delta(\varpi)$ is such that

$$L_0|\varpi\rangle = \Delta(\varpi)|\varpi\rangle = \frac{h}{4\pi}(\varpi_0^2-\varpi^2)|\varpi\rangle \quad (5.8)$$

with $\varpi_0 = 1+\pi/h$. The letter Γ represents the usual (not q-deformed) Gamma function. The basic relation underlying the fusion for the present case is the well-known relation between hypergeometric functions:

$$F(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-x)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}$$
$$\times F(c-a,c-b;c-a-b+1;1-x)\,. \quad (5.9)$$

The particular combination of $\psi^{(1/2)}$ fields appearing in the central term is such that the second term vanishes, and we are left with

$$\langle\varpi|\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(z,z')|\varpi\rangle = \left(\frac{z}{z'}\right)^{h(\varpi-\varpi_0)/2\pi}(z-z')^{1+3h/2\pi}$$
$$\times\frac{\Gamma(-1-\frac{2h}{\pi})}{\Gamma(-\frac{h}{\pi})}\frac{\lfloor\varpi+1\rfloor+\lfloor\varpi-1\rfloor}{\lfloor\varpi\rfloor}F\left(u_0-u,u_0;2u_0;1-\frac{z'}{z}\right), \quad (5.10)$$

where we have let $u = h\varpi/\pi$, $u_0 = h\varpi_0/\pi$, and $\varpi_0 = 1+\pi/h$. Making use of one of the standard quadratic transformations of Goursat's table, and writing explicitly the power series expansion of the resulting hypergeometric function, we find (see the Appendix):

$$\langle\varpi|\xi_0^{[\frac{1}{2},\frac{1}{2}](0)}(z,z')|\varpi\rangle = (z-z')^{1+3h/2\pi}\frac{\Gamma\left(-1-\frac{2h}{\pi}\right)}{\Gamma\left(-\frac{h}{\pi}\right)}\frac{\lfloor\varpi+1\rfloor+\lfloor\varpi-1\rfloor}{\lfloor\varpi\rfloor}$$

$$\times\sum_{\nu=0}^{\infty}\left(\frac{(z-z')^2}{4zz'}\right)^{\nu}\left(\frac{h}{\pi}\right)^{\nu}\prod_{l=1}^{\nu}\left\langle\varpi\right| - \frac{\Delta(\varpi_{2l})-L_0}{(u_0+\frac{1}{2})_\nu \nu!}\left|\varpi\right\rangle , \tag{5.11}$$

where $\varpi_{2l}\equiv\varpi_0+2l$ is the momentum of the highest-weight state whose weight is equal to the conformal weight of the fields $\psi_m^{(l)}$, and $(a)_\nu=\prod_{r=0}^{\nu-1}(a+r)$. The right-hand side is entirely expressed in terms of matrix elements of powers of L_0, as we had anticipated. Two mathematical properties of the above hypergeometric function are remarkable, although their meaning is not clear at this time. First, the form of the arguments is that of a so-called Gauss series $F(a,b;a+b+\frac{1}{2};t)$. Hence[15] its square is a hypergeometric function of the type ${}_3F_2$, and thus the square of the expectation value Eq. (5.10) again satisfies a linear differential equation, now of third order. Second, Eq. (5.11) becomes particularly simple at the point 1, corresponding to $\sigma_+-\sigma'_+=\pi$, where it reduces to a product of Γ functions. See again the Appendix for details.

6. The General Quantum Group Structure

In this section we turn at once to the most general structure. On the one hand, we include the dual Hopf algebra $U_{\widehat{q}}(sl(2))$, with $\widehat{q}=\exp(i\widehat{h})$ and $h\widehat{h}=\pi^2$, on the other hand we deal with the semi-infinite representations with continuous spins introduced in refs. [9, 10, 13]. In this case the Hopf algebra structure noted $U_q(sl(2))\odot U_{\widehat{q}}(sl(2))$ is novel, since it cannot be reduced to a simple graded tensor product of $U_q(sl(2))$ and $U_{\widehat{q}}(sl(2))$. Indeed, although ξ depends upon four quantum numbers for the half-integer case – that is two total spins $J,\widehat{J}$, and two magnetic numbers $M,\widehat{M}$ – there are only three independent quantum numbers for continuous spins: the so-called effective total spin J^e and two screening numbers $N,\widehat{N}$, which are positive integers. Within the Bloch wave basis of operators with diagonal monodromy, the corresponding quantum group symbols and their relations to the operator algebra have been worked out in refs. [10, 13]. We will not attempt here to present the corresponding derivations within the covariant operator basis (this will be done in ref. [19]), but rather concentrate on the Hopf algebra structure of the extended quantum group, and simply quote formulae from ref. [19] where necessary.

The most convenient parametrization of the general ξ fields is obtained by writing them as $\xi^{(J^e)}_{M^\circ\widehat{M}^\circ}$, with $M^\circ=N-J^e$ and $\widehat{M}^\circ=\widehat{N}-\widehat{J}^e$, with $\widehat{J}^e=J^eh/\pi$. In effect one has two semi-infinite lowest-weight representations since J^e+M°, and $\widehat{J}^e+\widehat{M}^\circ$ are arbitrary positive integers. The general braiding matrix takes the form [19]

$$((J^e,J^{e\prime}))^{M_2^\circ\widehat{M}_2^\circ;M_1^\circ\widehat{M}_1^\circ}_{M^\circ\widehat{M}^\circ;M^{\circ\prime}\widehat{M}^{\circ\prime}} = q^{J^eJ^{e\prime}}\widehat{q}^{\widehat{J}^e\widehat{J}^{e\prime}}(J^e,J^{e\prime})^{M_2^\circ M_1^\circ}_{M^\circ M^{\circ\prime}}(\widehat{J}^e,\widehat{J}^{e\prime})^{\widehat{M}_2^\circ\widehat{M}_1^\circ}_{\widehat{M}^\circ\widehat{M}^{\circ\prime}} , \tag{6.1}$$

[15] This was discovered by Clausen in 1828 (!).

where the two factors are suitable extensions of the universal R matrix of $U_q(sl(2))$ and $U_{\widehat{q}}(sl(2))$ respectively. This general formula gives, in particular, the braiding of the spin $\frac{1}{2}$ fields with a general ξ field, and thus determines the coproduct action in our scheme. First consider one of the two Borel subalgebras. Keeping the same definition of the operators $\mathscr{O}(J^a)$ for $J^a \in \mathscr{B}_+$, one gets

$$\mathscr{O}[q^{J_3}]^{(R)}_{\sigma+} \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma) = \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma) e^{ihM^\circ}(-1)^{\widehat{N}}$$

$$\mathscr{O}[q^{J_3}]^{(R)}_{\sigma+} \mathscr{O}[J_+]^{(R)}_{\sigma+} \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma) = \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma') e^{-ihM^\circ}(-1)^{\widehat{N}} \mathscr{O}[J_+]^{(R)}_{\sigma+}$$

$$+ \xi^{(J^e)}_{M^\circ+1, \widehat{M}^\circ}(\sigma) \sqrt{\lfloor J^e - M^\circ \rfloor \lfloor J^e + M^\circ + 1 \rfloor} \times (-1)^{\widehat{N}} \mathscr{O}[q^{J_3}]^{(R)}_{\sigma+} . \tag{6.2}$$

This takes the general form Eq. (1.7), if we introduce the matrix representation

$$[\underline{J}_+]_{P^\circ \widehat{P}^\circ; M^\circ \widehat{M}^\circ} = \delta_{P^\circ M^\circ+1} \sqrt{\lfloor J^e - M^\circ \rfloor \lfloor J^e + M^\circ + 1 \rfloor} \delta_{\widehat{P}^\circ \widehat{M}^\circ}(-1)^{\widehat{N}} ,$$

$$[\underline{q}^{J_3}]_{P^\circ \widehat{P}^\circ; M^\circ \widehat{M}^\circ} = \delta_{P^\circ; M^\circ} \delta_{\widehat{P}^\circ; \widehat{M}^\circ} e^{ihM^\circ}(-1)^{\widehat{N}} \equiv [q^{J_3}]_{P^\circ M^\circ}(-1)^{\widehat{N}} , \tag{6.3}$$

and keep the same coproduct as before. We use underlined letters for the full generators. This may be rewritten as

$$[\underline{J}_+]_{P^\circ \widehat{P}^\circ; M^\circ \widehat{M}^\circ} = [J_+]_{P^\circ M^\circ} [(-1)^{\widehat{N}}]_{\widehat{P}^\circ \widehat{M}^\circ} ,$$

$$[\underline{q}^{J_3}]_{P^\circ \widehat{P}^\circ; M^\circ \widehat{M}^\circ} = [q^{J_3}]_{P^\circ M^\circ} [(-1)^{\widehat{N}}]_{\widehat{P}^\circ \widehat{M}^\circ} , \tag{6.4}$$

where there appear the standard matrices $[J^a]_{P^\circ M^\circ}$ of the representation of $U_q(sl(2))$ with spin J^e, and the diagonal matrix $(-1)^{\widehat{N}}$.

Now in addition we have similar definitions where the roles of hatted and unhatted quantum numbers are exchanged. Letting

$$\mathscr{O}[\widehat{J}_+]^{(R)}_{\sigma+} \equiv \widehat{\kappa}_+ \widehat{\xi}^{(\frac{1}{2})}_{\frac{1}{2}}(\sigma_+), \qquad \mathscr{O}[\widehat{q}^{\widehat{J}_3}]^{(R)}_{\sigma+} \equiv \widehat{\kappa}_3 \widehat{\xi}^{(\frac{1}{2})}_{-\frac{1}{2}}(\sigma_+) , \tag{6.5}$$

$$\frac{\widehat{\kappa}_+}{\widehat{\kappa}_3} = \frac{\widehat{q}^{\frac{1}{2}}}{1 - \widehat{q}^2} , \tag{6.6}$$

we get another coproduct action

$$\mathscr{O}[\widehat{q}^{\widehat{J}_3}]^{(R)}_{\sigma+} \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma) = \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma) e^{i\widehat{h}\widehat{M}^\circ}(-1)^{N} \mathscr{O}[\widehat{q}^{\widehat{J}_3}]^{(R)}_{\sigma+},$$

$$\mathscr{O}[\widehat{J}_+]^{(R)}_{\sigma+} \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma) = \xi^{(J^e)}_{M^\circ \widehat{M}^\circ}(\sigma') e^{-i\widehat{h}\widehat{M}^\circ}(-1)^{N} \mathscr{O}[\widehat{J}_+]^{(R)}_{\sigma+}$$

$$+ \xi^{(J^e)}_{M^\circ, \widehat{M}^\circ+1}(\sigma) \sqrt{\hat{\lfloor} \widehat{J}^e - \widehat{M}^\circ \hat{\rfloor} \hat{\lfloor} \widehat{J}^e + \widehat{M}^\circ + 1 \hat{\rfloor}} (-1)^{N} \mathscr{O}[\widehat{q}^{\widehat{J}_3}]^{(R)}_{\sigma+} , \tag{6.7}$$

where we have introduced the q-deformed numbers with parameter $\widehat{h}$,

$$\hat{\lfloor} x \hat{\rfloor} = \sin(\widehat{h}x)/\sin\widehat{h} .$$

This takes the general form of Eq. (1.7), if we introduce the matrix representation

$$[\widehat{\underline{J}_+}]_{P^\circ\widehat{P}^\circ;M^\circ\widehat{M}^\circ} = \delta_{P^\circ;M^\circ}(-1)^N\sqrt{\lfloor \widehat{J}^e - \widehat{M}^\circ \rfloor \lfloor \widehat{J}^e + \widehat{M}^\circ + 1 \rfloor}\,,$$

$$[\widehat{\underline{q}^{J_3}}]_{P^\circ\widehat{P}^\circ;M^\circ\widehat{M}^\circ} = \delta_{P^\circ;M^\circ}\delta_{\widehat{P}^\circ;\widehat{M}^\circ}(-1)^N e^{ih\widehat{M}^\circ}\,. \tag{6.8}$$

In terms of the hatted generators, the coproduct takes the same expression as for the unhatted generators. This may be rewritten as

$$[\widehat{\underline{J}_+}]_{P^\circ\widehat{P}^\circ;M^\circ\widehat{M}^\circ} = [(-1)^N]_{P^\circ;M^\circ}[\widehat{J}_+]_{\widehat{P}^\circ\widehat{M}^\circ}\,,$$

$$[\widehat{\underline{q}^{J_3}}]_{P^\circ\widehat{P}^\circ;M^\circ\widehat{M}^\circ} = [(-1)^N]_{P^\circ;M^\circ}[\widehat{q}^{\widehat{J}_3}]_{\widehat{P}^\circ\widehat{M}^\circ}\,, \tag{6.9}$$

where there appear the standard matrices $[\widehat{J}^a]_{\widehat{P}^\circ\widehat{M}^\circ}$ of the representation of $U_{\widehat{q}}(sl(2))$ with spin $\widehat{J}^e$. What is the extended B_+ algebra? Clearly each pair J_+, q^{J_3}, and $\widehat{J}_+$, $\widehat{q}^{\widehat{J}_3}$ of matrices satisfies the same algebra as before. For the mixed relations one trivially obtains

$$\underline{J}_+\widehat{\underline{J}}_+ = \widehat{\underline{J}}_+\underline{J}_+, \qquad \underline{q}^{J_3}\widehat{\underline{q}}^{\widehat{J}_3} = \widehat{\underline{q}}^{\widehat{J}_3}\underline{q}^{J_3}\,,$$

$$\underline{J}_+\widehat{\underline{q}}^{\widehat{J}_3} = -\widehat{\underline{q}}^{\widehat{J}_3}\underline{J}_+, \qquad \widehat{\underline{J}}_+\underline{q}^{J_3} = -\underline{q}^{J_3}\widehat{\underline{J}}_+\,. \tag{6.10}$$

The case of $\mathscr{B}_-$ is treated in exactly the same way, and we will be very brief. One finds formulae similar to the above with $J_+ \to J_-$ and $q^{J_3} \to q^{-J_3}$. It is easy to see that we still have the matrix commutation relations

$$[\underline{J}_+,\underline{J}_-] = \frac{(\underline{q}^{J_3})^2 - (\underline{q}^{-J_3})^2}{q - q^{-1}}, \qquad [\widehat{\underline{J}}_+,\widehat{\underline{J}}_-] = \frac{(\widehat{\underline{q}}^{\widehat{J}_3})^2 - (\widehat{\underline{q}}^{-\widehat{J}_3})^2}{\widehat{q} - \widehat{q}^{-1}}\,. \tag{6.11}$$

Again the above transformations generate Hopf quantum symmetries of the operator algebra. In ref. [19], the generalized $3j$ symbols are shown to be given by

$$\begin{aligned}&(\underline{J}_1,\underline{M}_1^\circ,\underline{J}_2,\underline{M}_2^\circ|\underline{J}_{12})\\ &= \left(J_1^e,M_1^\circ,J_2^e,M_2^\circ|J_{12}^e + \widehat{p}\frac{\pi}{h}\right)\left(\widehat{J}^{e_1},\widehat{M}_1^\circ,\widehat{J}^{e_2},\widehat{M}_2^\circ|\widehat{J}^{e_{12}} + p\frac{\widehat{h}}{\pi}\right).\end{aligned} \tag{6.12}$$

On the right-hand side the factors are the standard $3j$ symbols of $U_q(sl(2))$ and $U_{\widehat{q}}(sl(2))$ respectively, suitably generalized. We use underlined letters to denote pairs of quantum numbers: $\underline{M}_1^\circ$ stands for the pair $M_1^\circ\widehat{M}_1^\circ$, and so on. For total spins, we use the same convention, i.e. $\underline{J}$ stands for J^e and $\widehat{J}^e$. This latter convention is convenient even though J^e and $\widehat{J}^e$ are not independent ($\widehat{J}^e = J^e\frac{h}{\pi}$), since each of them plays the role of a total spin, as is clear from the right-hand side. The above Clebsch–Gordan coefficient is non-zero iff

$$J_1^e + J_2^e - J_{12}^e = p + \widehat{p}\frac{\pi}{h}\,, \tag{6.13}$$

with p and $\widehat{p}$ arbitrary positive integers. Using the same argument as in Sect. 2.2, we deduce that the above generalized $3j$'s should satisfy the recurrence relation

$$
\begin{aligned}
&(\underline{J}_1, \underline{M}_1^\circ, \underline{J}_2, \underline{M}_2^\circ | \underline{J}_{12})[J^a]_{\underline{P}_{12}^\circ, \underline{M}_{12}^\circ} \\
&= \sum_{\underline{P}_1^\circ + \underline{P}_2^\circ = \underline{P}_{12}^\circ} (\underline{J}_1, \underline{P}_1^\circ, \underline{J}_2, \underline{P}_2^\circ | \underline{J}_{12})[\Lambda(J^a)]_{\underline{P}_1^\circ \underline{P}_2^\circ, \underline{M}_1^\circ \underline{M}_2^\circ}\,, \qquad (6.14)
\end{aligned}
$$

where $\underline{M}_{12}^\circ = \underline{M}_1^\circ + \underline{M}_2^\circ$. Using the recurrence relation for the ordinary $3j$ symbols together with the fact that p and $\widehat{p}$ are integer, one may verify this relation. Similarly, one may verify that the general braiding matrix is the universal R matrix since it satisfies the appropriate generalization of Eq. (3.23). Thus we have derived the full Hopf algebra structure of $U_q(sl(2)) \odot U_{\widehat{q}}(sl(2))$ in the most general case. This could not be achieved before, despite several attempts to guess the answer [2, 20, 5]. We remark that the above structure can be derived independently on the basis of the extended quantum group symbols alone [19]. This generalized quantum group structure is particularly important in the strong coupling regime of Liouville theory [10, 11]. In particular, there is an interesting special case where $h + \widehat{h} = 0$ ($C = 13$ for Liouville theory). Then $\widehat{q} = 1/q$, and there is a relationship between the $\mathscr{B}_-$, and the hatted Borel $\widehat{\mathscr{B}}_+$. For instance, the corresponding elements associated with $q^{J_3} \in \mathscr{B}_+$ are q^{-J_3}, and $\widehat{q}^{J_3}$, respectively, and there are further remarkable properties. In general, however, q^{-1} and $\widehat{q}$ are unrelated (we have $qq^{-1} = 1$, and $\ln(q)\ln(\widehat{q}) = -\pi^2$), so that these two subalgebras have no simple connection.

7. Outlook

Our original observation was that the braiding properties of the $\xi_M^{(J)}$ fields allow us to use the $\xi^{(\frac{1}{2})}$ fields as generators of the quantum group symmetry. This idea led us to consider coproduct realizations with novel features. In particular our generators are position-dependent, and the algebra of the field transformation laws (here $U_q(sl(2))$) was seen to follow from FP commutation relations where only the quantum numbers of the generators are exchanged, and not the operators themselves. This FP algebra differs from $U_q(sl(2))$, but we showed in general that the algebra of the field transformation laws and that of the field generators need not be identical: the latter may be a suitable extension of the former by central operators that commute with all the fields. Our FP algebra was found to be precisely a realization of such a central extension of $U_q(sl(2))$. In establishing this, we had to overcome the fact that, in the present approach, $U_q(sl(2))$ is split into its two natural Borel subalgebras, which are most directly realized by right- and left-actions, respectively. This was possible using the monodromy properties of the ξ fields, and thus the Liouville zero mode momentum ϖ took part in the algebra, which now includes a new generator which we called D. This last point is rather interesting, since so far ϖ did not play any role in the quantum group structure although its spectrum of eigenvalues determines the spectrum of Verma modules. Of course the ultimate aim of the operator realization is to understand how the Hilbert space is organized by the quantum group symmetry, and thus how the generators act on the Verma modules. The fact that ϖ appears in some of our generators may contain a clue to this problem. There still remain many related questions, especially the existence of a vacuum state $|0\rangle$ whose general properties were summarized in Eq. (1.16). To know the invariant vacuum is obviously important, e.g. for the possibility of writing the q-analog of the Wigner–Eckart theorem, which would provide a very useful tool for the calculation of matrix elements of covariant operators. On the other hand, it might turn out that $U_q(sl(2))$ is spontaneously broken, so that no invariant vacuum exists. Let us make a further general remark in this connection, namely, matrix realizations of our centrally extended algebra (Eq. (2.10)) do not have a general highest- (or lowest-) weight states in the usual sense. Indeed, since $C_\pm$ and $q^{\widetilde{J}_3}$ commute, we may diagonalize them simultaneously. A lowest-weight state $|j, c_\pm\rangle$ would have to satisfy (we use the notation of Eq. (2.10))

$$C_\pm|j,c_\pm\rangle = c_\pm|j,c_\pm\rangle, \qquad \widetilde{J}_-|j,c_\pm\rangle = 0, \qquad q^{\widetilde{J}_3}|j,c_\pm\rangle = q^j|j,c_\pm\rangle\,.$$

It then follows from Eqs. (2.10) that

$$[q^{\widetilde{J}_3}\widetilde{J}_- - q^{-1}\widetilde{J}_- q^{\widetilde{J}_3}]|j,c_\pm\rangle = 0 = c_-|j,c_\pm\rangle\,,$$

so that c_- should vanish. Thus, if $c_\pm \neq 0$, the representations of the algebra have neither lowest- nor highest-weight states. Of course, there is always the possibility to take the short distance limit of our FP discussion. Then $C_\pm$ tend to zero, and we can have highest- or lowest-weight representations (although a vacuum in the sense of Eq. (1.16) still does not exist). However we prefer to consider the general situation, as we feel that important information about the symmetry properties may be lost in the limit. The dependence of our generators upon the points reflects the fact that they do not commute with the Virasoro generators. Thus there exists an interplay between the two symmetries, which should play a basic role.

Another striking aspect deserves closer study: we have seen that the central terms may be expressed as series of polynomials in the stress-energy tensor (and its derivatives). Similar series already appeared in the derivation of the infinite set of commuting operators associated with the Virasoro algebra [21]. Thus there may be a deep connection between the present centrally extended $U_q(sl(2))$ algebra of our generators and the complete integrability of the Liouville theory. The fact that the generators have become dependent upon another variable (the position) is reminiscent of the transition from a Lie algebra to a Kac–Moody algebra. Thus the full symmetry of the theory may be ultimately much larger than known at present. One may hope that the understanding of this point will allow us to solve the dynamics of the full integrable structure obtained by including all of the conserved charges. Clearly, we are still somewhat far from this ideal situation, but we may be optimistic.

At a more immediate level, the present scheme may be used to understand the quantum group action on the Bloch wave operators ψ, whose monodromy is diagonal. This is interesting since so far, in sharp contrast with the ξ fields, their quantum group properties have remained a mystery. In particular, for them the role of $3j$ symbols is actually played by $6j$ symbols. By braiding our generators with the Bloch wave operators, we may define their quantum group transformations. In connection with our previous remarks about highest/lowest-weight states, we may mention that the change of basis from the ξ fields to the Bloch wave fields has its counterpart on the generators themselves. This leads to new generators where the central extension is only multiplicative and does not prevent the existence of highest- or lowest-weight states. This is described in another article [22]. Another direction is to consider higher-rank algebras. Our construction of the generators from the defining representation (and not from the adjoint, as one would expect a priori) used the fact that, for $U_q(sl(2))$, the dimension of the defining representation coincides with the dimension of each Borel algebra. For higher ranks the counting is completely different, the former being smaller than the latter. Of course now there is more than one representation of lowest dimension. Finally, it would be interesting to illuminate the connection of the present analysis with the general framework of Poisson–Lie symmetries [24, 25], and in particular with the dressing symmetries mentioned in the introduction. It is challenging to find a unified treatment which includes both types of symmetries, explaining how one and the same $U_q(sl(2))$ quantum group can manifest itself in apparently rather different guises.

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A. More About the Central Terms

Here we supplement some details of Sect. 5. First, we give the derivation of Eq. (5.11). We start from Eq. (5.10) and use the following quadratic transformation of Goursat's table (see for instance ref. [17], p. 112, Eq. (26)):

$$(1-y)^{a/2}F(a,b;2b;y) = F\left(\frac{a}{2}, b-\frac{a}{2}, b+\frac{1}{2}; \frac{y^2}{4(y-1)}\right), \tag{A.1}$$

which leads to

$$\left(\frac{z'}{z}\right)^{(u_0-u)/2} F\left(u_0-u, u_0; 2u_0; 1-\frac{z'}{z}\right) = F\left(\frac{u_0-u}{2}, \frac{u_0+u}{2}, u_0+\frac{1}{2}; \frac{-(z-z')^2}{4zz'}\right).$$

Expanding the right-hand side, one sees that the expectation value of the central term between highest-weight states is given by

$$\begin{aligned}\langle\varpi|\xi^{[\frac{1}{2},\frac{1}{2}](0)}(z,z')|\varpi\rangle &= (z-z')^{1+3h/2\pi}\frac{\Gamma(-1-\frac{2h}{\pi})}{\Gamma(\frac{-h}{\pi})}\frac{\lfloor\varpi+1\rfloor+\lfloor\varpi-1\rfloor}{\lfloor\varpi\rfloor}\\ &\times\sum_{\nu=0}^{\infty}\left(\frac{(z-z')^2}{4zz'}\right)^{\nu}\left(\frac{h}{\pi}\right)^{\nu}\prod_{l=1}^{\nu}\frac{\Delta(\varpi_{2l})-\Delta(\varpi)}{(u_0+\frac{1}{2})_\nu\nu!}.\end{aligned} \tag{A.2}$$

Here $(a)_\nu = \prod_{r=0}^{\nu-1}(a+r)$, and $\varpi_{2l}\equiv\varpi_0+2l$. Using Definition 5.8 we may rewrite the final result under the form

$$\begin{aligned}\langle\varpi|\xi^{[\frac{1}{2},\frac{1}{2}](0)}(z,z')|\varpi\rangle &= (z-z')^{1+3h/2\pi}\frac{\Gamma(-1-\frac{2h}{\pi})}{\Gamma(\frac{-h}{\pi})}\frac{\lfloor\varpi+1\rfloor+\lfloor\varpi-1\rfloor}{\lfloor\varpi\rfloor}\\ &\times\sum_{\nu=0}^{\infty}\left(\frac{(z-z')^2}{4zz'}\right)^{\nu}\left(\frac{h}{\pi}\right)^{\nu}\prod_{l=1}^{\nu}\left\langle\varpi\left|\frac{\Delta(\varpi_{2l})-L_0}{(u_0+\frac{1}{2})_\nu\nu!}\right|\varpi\right\rangle,\end{aligned} \tag{A.3}$$

which is seen to agree with Eq. (5.11).

Being a Guass series, the hypergeometric function in Eq. (5.10) is known to square to another one of type ${}_3F_2$. The precise relation is

$$\left\{F\left(\frac{u_0-u}{2},\frac{u_0+u}{2},u_0+\frac{1}{2};t\right)\right\}^2 = {}_3F_2\left[\begin{matrix}u_0-u,u_0+u,u_0\\ 2u_0,u_0+\frac{1}{2}\end{matrix},t\right]. \qquad \text{(A.4)}$$

Furthermore, applying the so-called Watson theorem (see [18], p. 54) one deduces that

$${}_3F_2\left[\begin{matrix}u_0-u,u_0+u,u_0\\ 2u_0,u_0+\frac{1}{2}\end{matrix},1\right] = \left\{\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+u_0)}{\Gamma(\frac{1}{2}(1+u_0+u))\Gamma(\frac{1}{2}(1+u_0-u))}\right\}^2.$$

Thus we conclude that

$$F\left(\frac{u_0-u}{2},\frac{u_0+u}{2},u_0+\frac{1}{2};1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+u_0)}{\Gamma(\frac{1}{2}(1+u_0+u))\Gamma(\frac{1}{2}(1+u_0-u))}. \qquad \text{(A.5)}$$

Thus the above expectation value is especially simple at the point 1. In terms of the original variables this corresponds to $\sigma_+ - \sigma'_+ = \pi$.

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