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Deformation Estimates for the Berezin-Toeplitz Quantization

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Abstract. Deformation estimates for the Berezin-Toeplitz quantization of $\mathbb{C}^n$ are obtained. These estimates justify the description of CCR $+\mathscr{K}$ as a first-order quantum deformation of AP $+C_0$, where CCR is the usual C^*-algebra of (boson) canonical commutation relations, $\mathscr{K}$ is the full algebra of compact operators, AP is the algebra of almost-periodic functions and C_0 is the algebra of continuous functions which vanish at infinity.

1. Introduction

We consider the family of Gaussian probability measures

$$d\mu_r(z) = \left(\frac{r}{\pi}\right)^n e^{-r|z|^2} dv(z)\,, \qquad r > 0$$

for $z = (z_1, \ldots, z_n)$ in complex Euclidean space $\mathbb{C}^n$, $dv(z)$ ordinary Lebesgue measure, $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$. The space of entire $d\mu_r$-square-integrable functions is denoted by $H^2(d\mu_r) \equiv H^2(\mathbb{C}^n, d\mu_r)$. For g in $L^2(d\mu_r)$, the Berezin-Toeplitz operator $T_g^{(r)}$ is defined on a dense linear subspace of $H^2(d\mu_r)$ by

$$(T_g^{(r)} h)(z) = \int g(w) h(w) e^{rz \cdot w} d\mu_r(w)\,.$$

Here $z \cdot w \equiv z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n$ and $e^{rz \cdot w}$ is the Bergman reproducing kernel for $H^2(d\mu_r)$ so that, for gh in $L^2(d\mu_r)$, $T_g^{(r)} h$ is in $H^2(d\mu_r)$.

The map $g \to T_g^{(r)}$ has been considered by Berezin [Be] and others [Ba, G, Ho] as a "quantization" in which r plays the role of the reciprocal of Planck's constant.

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In this guise, with $[A, B] = AB - BA$, the "canonical commutation relations" are given by

$$[T_{\bar{z}_j}^{(r)}, T_{z_k}^{(r)}] = \frac{1}{r}\,\delta_{jk} I\,,$$

where

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases},$$

while

$$[T_{\bar{z}_j}^{(r)}, T_{\bar{z}_k}^{(r)}] = 0\,, \qquad [T_{z_j}^{(r)}, T_{z_k}^{(r)}] = 0\,.$$

There is an isometry $B_r : L^2(\mathbb{R}^n, dv) \to H^2(\mathbb{C}^n, d\mu_r)$, due to Bargmann [Ba], so that for sufficiently smooth g,

$$B_r^{-1} T_g^{(r)} B_r$$

is a Weyl pseudo-differential operator [F, H, Sh].

Here, we establish a first-order composition calculus for $T_f^{(r)}$, $T_g^{(r)}$ analogous to results of [H, Sh] for the Weyl calculus. To obtain such a calculus, it does not seem possible to simple apply conjugation by B_r to the results of [H, Sh] or the related results of [S]. Instead, we proceed by a combination of direct calculation and an asymptotic analysis analogous to that of [KL].

Our results are, in particular, sufficient to justify the description of $\mathrm{CCR}(\mathbb{C}^n) + \mathscr{K}$ as a first-order quantum-deformation of $AP(\mathbb{C}^n) + C_0(\mathbb{C}^n)$. Here, $\mathrm{CCR}(\mathbb{C}^n)$ is the standard simple C^*-algebra generated by the canonical commutation relations [BR, BC$_1$], $\mathscr{K}$ is the full algebra of compact operators on a separable infinite-dimensional Hilbert space, $AP(\mathbb{C}^n)$ is the supremum norm closed algebra of almost periodic functions, and $C_0(\mathbb{C}^n)$ is the supremum norm closure of the compactly supported continuous functions.

Our main result can be simply stated. We write $\mathrm{TP}(\mathbb{C}^n)$ for the algebra of trigonometric polynomials on $\mathbb{C}^n = \mathbb{R}^{2n}$. This algebra is generated by the characters $\chi_a(w) \equiv \exp\{i \operatorname{Im} w \cdot a\}$, for a in $\mathbb{C}^n$. We let $C_c^m(\mathbb{C}^n)$ be the algebra of m times continuously differentiable functions with compact support. We have

Main Theorem. *For f, g in $\mathrm{TP} + C_c^{2n+6}$, $r > 0$,*

$$\left\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \frac{1}{r} T^{(r)}_{\sum_j (\partial_j f)(\bar{\partial}_j g)} \right\|_{(r)} \leq C(f, g) r^{-2}$$

holds for $C(f, g)$ independent of r.

2. Preliminary Results

We make use of the maps

$$t_a(z) = z - a\,, \qquad \gamma_a(z) = a - z\,.$$

These maps determine unitary operators on $H^2(d\mu_r)$ and $L^2(d\mu_r)$ given by

$$(U_a^{(r)} f)(z) = k_a^{(r)}(z) f(z - a)\,,$$
$$(V_a^{(r)} f)(z) = k_a^{(r)}(z) f(a - z)\,,$$

where

$$k_a^{(r)}(z) = e^{rz \cdot a - r|a|^2/2}$$

is the normalized reproducing kernel for $H^2(d\mu_r)$. Note that

$$(V_a^{(r)})^2 = I$$

and

$$V_a^{(r)} T_g^{(r)} V_a^{(r)} = T_{g \circ \gamma_a}^{(r)} .$$

We will need

Lemma 1. *For g bounded and uniformly continuous on $\mathbb{C}^n$ and $\varepsilon > 0$ given, there is an $R = R(\varepsilon)$, independent of w, so that*

$$\int |g(w) - g(w - z)| d\mu_r(z) < \varepsilon$$

whenever $r > R(\varepsilon)$.

Proof. Note that

$$\int\limits_{|z| \geq \delta} d\mu_r(z) \leq n e^{-r\delta^2/n} .$$

By uniform continuity, there is a $\delta = \delta(\varepsilon)$ so that $|g(z_1) - g(z_2)| < \frac{\varepsilon}{2}$ whenever $|z_1 - z_2| < \delta$. For this δ, write

$$\begin{aligned} \int |g(w) - g(w - z)| d\mu_r(z) &= \int\limits_{|z|<\delta} |g(w) - g(w - z)| d\mu_r(z) \\ &\quad + \int\limits_{|z| \geq \delta} |g(w) - g(w - z)| d\mu_r(z) \\ &< \frac{\varepsilon}{2} \int\limits_{|z|<\delta} d\mu_r(z) + 2\|g\|_\infty \int\limits_{|z| \geq \delta} d\mu_r(z) \\ &< \frac{\varepsilon}{2} + 2n\|g\|_\infty e^{-r\delta^2/n} . \end{aligned}$$

Thus, choosing

$$R(\varepsilon) = -\frac{n}{\delta^2} \ln \left[\frac{\varepsilon}{4n\|g\|_\infty} \right]$$

completes the proof.

We can now prove, similarly to [KL] for the disc, that

Theorem 1. *For g bounded and uniformly continuous on $\mathbb{C}^n$, we have*

$$\lim_{r \to \infty} \|T_g^{(r)}\|_{(r)} = \|g\|_\infty .$$

Proof. Write

$$g(w) = < T^{(r)}_{g\circ\gamma_w} 1, 1 >_{(r)} + \int [g(w) - g(w-z)] d\mu_r(z) .$$

Thus,

$$|g(w)| \le \|T^{(r)}_{g\circ\gamma_w}\|_{(r)} + \int |g(w) - g(w-z)| d\mu_r(z) .$$

Using $V^{(r)}_w T^{(r)}_g V^{(r)}_w = T^{(r)}_{g\circ\gamma_w}$, we have

$$|g(w)| \le \|T^{(r)}_g\|_{(r)} + \int |g(w) - g(w-z)| d\mu_r(z)$$

and, by Lemma 1,

$$|g(w)| < \|T^{(r)}_g\|_{(r)} + \varepsilon$$

for $r > R(\varepsilon)$. It follows that $\|g\|_\infty - \varepsilon \le \|T^{(r)}_g\|_{(r)}$. Since $\|T^{(r)}_g\|_{(r)} \le \|g\|_\infty$ is trivial and ε is arbitrary, the proof is complete.

We consider some differential identities which will be needed later. For f sufficiently smooth on $\mathbb{C}^n$, we write

$$\partial_1^{k_1} \ldots \partial_n^{k_n} \bar{\partial}_1^{l_1} \ldots \bar{\partial}_n^{l_n} f ,$$

where $\partial_j \equiv \frac{\partial}{\partial z_j}$, $\bar{\partial}_j \equiv \frac{\partial}{\partial \bar{z}_j}$; k_j, l_j are non-negative integers. For φ in $H^2(d\mu_r)$, we recall that

$$(U^{(r)}_{-w}\varphi)(a) = \varphi(a+w) k^{(r)}_{-w}(a) .$$

We have

Lemma 2. *For φ in $H^2(d\mu_r)$,*

$$\partial_1^{m_1} \ldots \partial_n^{m_n} (U^{(r)}_{-w}\varphi)(0) = e^{r|w|^2/2} \partial_1^{m_1} \ldots \partial_n^{m_n} \{\varphi(w) e^{-r|w|^2}\} .$$

Proof. Direct calculation.

Lemma 3. *For φ in $H^2(d\mu_r)$ and $m = m_1 + \ldots + m_n$,*

$$\partial_1^{m_1} \ldots \partial_n^{m_n} \varphi(0) = r^m \int \varphi(w) \bar{w}_1^{m_1} \ldots \bar{w}_n^{m_n} d\mu_r(w) .$$

Proof. Write

$$\varphi(a) = \int \varphi(w) e^{ra\cdot w} d\mu_r(w)$$

and check that differentiation "under the integral" is permissible.

Lemma 4. $\int |a|^{2k} d\mu_r(a) = b(k,n) r^{-k}$.

Proof. Easy calculation.

In what follows, we will write BC^m for the set of functions which are bounded and continuous, with all derivatives bounded and continuous up to order m. Clearly, C_c^m is contained in BC^m. For g in $\mathrm{BC}^{m+1}(\mathbb{C}^n)$, we will consider the Taylor series

$$\begin{aligned} g(a+w) &= g(w) + (\partial_1 g)(w)a_1 + \ldots + (\partial_n g)(w)a_n \\ &\quad + (\bar\partial_1 g)(w)\bar a_1 + \ldots + (\bar\partial_n g)(w)\bar a_n + \ldots \\ &\quad + \frac{1}{m!}(\bar\partial_n^m g)(w)\bar a_n^m + g_{m+1}(a,w), \end{aligned}$$

where

$$g_{m+1}(a,w) = \sum c(k_1, \ldots, l_n)(\partial_1^{k_1} \ldots \partial_n^{k_n} \bar\partial_1^{l_1} \ldots \bar\partial_n^{l_n} g)(w^*) a_1^{k_1} \ldots a_n^{k_n} \bar a_1^{l_1} \ldots \bar a_n^{l_n}$$

for $k_1 + \ldots + k_n + l_1 + \ldots + l_n = m+1$. For g in BC^{m+1}, the remainder term $g_{m+1}(a,w)$ can be estimated by using

Lemma 5. *We have*

$$|g_{m+1}(a,w)| \le \sum c(k_1, \ldots, l_n)\, \|\partial_1^{k_1} \ldots \bar\partial_n^{l_n} g\|_\infty |a|^{m+1}.$$

Proof. Immediate.

We can now establish the main analytic preliminary result.

Theorem 2. *Let f be in $C_c^{n+3}(\mathbb{C}^n)$ with g in $\mathrm{BC}^{2n+6}(\mathbb{C}^n)$. Then we have a constant $C(f,g)$ so that*

$$\left\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \frac{1}{r} T^{(r)}_{\sum_j (\partial_j f)(\bar\partial_j g)} \right\|_{(r)} \le C(f,g) r^{-2}$$

for all $r > 0$.

Proof. Borrowing from [KL], we write for φ, ψ in $H^2(d\mu_r)$,

$$\begin{aligned} \langle T_f T_g \varphi, \psi\rangle_{(r)} &= \int f(w)\overline{\psi(w)} d\mu_r(w) \int e^{rw\cdot z} g(z)\varphi(z) d\mu_r(z) \\ &= \int f(w)\overline{\psi(w)} d\mu_r(w) \int e^{rw\cdot(a+w)} g(a+w)\varphi(a+w) d\mu_r(a+w) \\ &= \int f(w)\overline{\psi(w)} e^{r|w|^2/2} d\mu_r(w) \int g(a+w)\,(U_{-w}^{(r)}\varphi)(a) d\mu_r(a). \end{aligned}$$

Next, write

$$g(a+w) = \{g(a+w) - g_{m+1}(a,w)\} + g_{m+1}(a,w).$$

Using Lemmas 4 and 5, we check that for $m = n+3$,

$$\begin{aligned} &\left| \int f(w)\overline{\psi(w)} e^{r|w|^2/2} d\mu_r(w) \int g_{m+1}(a,w)\,(U_{-w}^{(r)}\varphi)(a) d\mu_r(a) \right| \\ &\quad \le \|\varphi\|_{(r)} \|\psi\|_{(r)} C(g) \pi^{-n} b(m+1,n)^{1/2} \left\{ \int |f(w)|^2 dv(w) \right\}^{1/2} r^{-2}. \end{aligned}$$

Thus, for $m = n + 3$, it remains to consider the expression

$$(\dagger) \qquad \int f(w)\overline{\psi(w)}e^{r|w|^2/2}d\mu_r(w)\int\{g(a+w) - g_{m+1}(a,w)\}\,(U_{-w}^{(r)}\varphi)\,(a)d\mu_r(a)\,.$$

For $k = k_1 + \ldots + k_n$, $l = l_1 + \ldots + l_n$, and $k + l \le m$ the typical term in the expansion of

$$\int\{g(a+w) - g_{m+1}(a,w)\}\,(U_{-w}^{(r)}\varphi)\,(a)d\mu_r(a)$$

has the form

$$(\dagger\dagger) \qquad a(k_1, \ldots, l_n)\,(\partial_1^{k_1}\ldots\bar{\partial}_n^{l_n}g)\,(w)\int \bar{a}_1^{l_1}\ldots\bar{a}_n^{l_n}a_1^{k_1}\ldots a_n^{k_n}(U_{-w}^{(r)}\varphi)\,(a)d\mu_r(a)\,.$$

Applying Lemmas 2 and 3, we see that $(\dagger\dagger) = 0$ unless

$$l_j \ge k_j$$

for all j. In this case, $(\dagger\dagger)$ is a sum of terms

$$r^{-l}a'(k_1, \ldots, l_n)e^{r|w|^2/2}(\partial_1^{k_1}\ldots\bar{\partial}_n^{l_n}g)\,(w)\partial_1^{t_1}\ldots\partial_n^{t_n}\{\varphi(w)e^{-r|w|^2}\}$$

with $l_j \ge t_j$.

It follows that $(\dagger)$ is a linear combination, with coefficients independent of r, of terms

$$(\dagger\,\dagger\,\dagger) \qquad r^{-l}\int f(w)\overline{\psi(w)}\,(\partial_1^{k_1}\ldots\partial_n^{k_n}\bar{\partial}_1^{l_1}\ldots\bar{\partial}_n^{l_n}g)\,(w) \times \partial_1^{t_1}\ldots\partial_n^{t_n}\{\varphi(w)e^{-r|w|^2}\}\left(\frac{r}{\pi}\right)^n dv(w)\,,$$

where

$$l_j \ge k_j\,, \qquad l_j \ge t_j\,, \qquad m \ge l + k\,.$$

Iterated application of Gauss' Theorem ("integration by parts") shows that $(\dagger\,\dagger\,\dagger)$ is a linear combination, with coefficients independent of r, of terms

$$r^{-l}\int(\partial_1^{u_1}\ldots\partial_n^{u_n}f)\,(w)\,(\partial_1^{k_1+s_1}\ldots\partial_n^{k_n+s_n}\bar{\partial}_1^{l_1}\ldots\bar{\partial}_n^{l_n}g)\,(w)\varphi(w)\overline{\psi(w)}d\mu_r(w)$$

where

$$t_j \ge u_j\,, \qquad t_j \ge s_j\,.$$

Thus, for $l > 1$, we have explicit estimates.

It remains to consider the cases $l = 0$, $l = 1$. Going back to $(\dagger)$, $(\dagger\dagger)$, we see that the only $l = 0$ term is

$$\begin{aligned}&\int f(w)\overline{\psi(w)}e^{r|w|^2/2}d\mu_r(w)\int g(w)\,(U_{-w}^{(r)}\varphi)\,(a)d\mu_r(a)\\&= \int f(w)g(w)\overline{\psi(w)}\varphi(w)d\mu_r(w)\\&= \langle T_{fg}^{(r)}\varphi, \psi\rangle_{(r)}\,.\end{aligned}$$

For $l = 1$, we can have, for some j with $1 \le j \le n$,

$$\begin{cases} l_j = 1,\ l_{j'} = 0 & j' \neq j \\ k_j = 1,\ k_{j'} = 0 & j' \neq j \end{cases}$$

or

$$\begin{cases} l_j = 1,\ l_{j'} = 0 & j' \neq j \\ k_j = 0,\ k_{j'} = 0 & j' \neq j\,. \end{cases}$$

In either case, $a(k_1, \ldots, l_n) = 1$ in (††). Direct calculation now shows that the sum of the $l = 1$ terms is

$$-r^{-1} \int \varphi(w)\overline{\psi(w)} \Big\{ \sum_j (\partial_j f)(\bar{\partial}_j g) \Big\} d\mu_r(w)\,.$$

This completes the proof!

3. Main Results

For each a in $\mathbb{C}^n$, we have the *character*

$$\chi_a(w) = \exp\{i \operatorname{Im} w \cdot a\}\,.$$

The algebra TP($\mathbb{C}^n$) consists of finite linear combinations of characters. The supremum norm closure of TP($\mathbb{C}^n$) is exactly AP($\mathbb{C}^n$). We also consider the algebra $\mathrm{TP} + C_c^{2n+6}$. Clearly,

$$\mathrm{TP} + C_c^{2n+6} \subset BC^{2n+6}\,.$$

Lemma 6. *For g in $\mathrm{TP} + C_c^{2n+6}$, the representation*

$$g = t + u$$

with t in TP and u in C_c^{2n+6} is unique.

Proof. On $\mathrm{TP} + C_c^{2n+6}$, we consider the functional

$$m(g) = \operatorname*{Lim}_{T\to\infty} (2T)^{-2n} \int_{-T}^{T} \cdots \int_{-T}^{T} g(x_1, y_1, x_2, y_2, \ldots) dx_1 dy_1 \ldots dy_n\,,$$

where $z_j = x_j + iy_j$, x_j, y_j real. It is easy to check that

$$m(u) = 0$$

while, for

$$t = \sum_{k=1}^{r} c_k \chi_{a_k}\,, m\{g\bar{\chi}_{a_k}\} = c_k\,.$$

Uniqueness follows.

Remark. It follows from the proof of Lemma 6 that

$$|c_k| \le \|g\|_\infty .$$

We can now complete the proof of the main theorem.

Theorem 3. *For f, g in* TP $+C_c^{2n+6}$ *there is a constant $C(f,g)$ so that for all $r > 0$,*

$$\left\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \frac{1}{r} T^{(r)}_{\sum(\partial_j f)(\bar\partial_j g)} \right\|_{(r)} \le C(f,g) r^{-2} . \tag{*}$$

Proof. For $f = t_1 + u_1$, $g = t_2 + u_2$ with t_j in TP and u_j in C_c^{2n+6}, it will suffice to check that each of the pairs (t_1, t_2), (t_1, u_2), (u_1, t_2), (u_1, u_2) satisfy $(^*)$.

The pairs (u_1, t_2), (u_1, u_2) are handled using Theorem 2. For (t_1, u_2), we note that $T_F^* = T_{\bar F}$ and $\bar\partial_j \bar F = \overline{\partial_j F}$ for F in BC^{2n+6} so that

$$\left(T_{t_1}^{(r)} T_{u_2}^{(r)} - T_{t_1 u_2}^{(r)} + \frac{1}{r} T^{(r)}_{\sum(\partial_j t_1)\bar\partial_j u_2)} \right)^* \\ = T_{\bar u_2}^{(r)} T_{\bar t_1}^{(r)} - T_{\bar u_2 \bar t_1}^{(r)} + \frac{1}{r} T^{(r)}_{\sum(\partial_j \bar u_2)(\bar\partial_j \bar t_1)} .$$

Since $\|A^*\|_{(r)} = \|A\|_{(r)}$ and $(\bar u_2, \bar t_1)$ has been handled using Theorem 2, $(^*)$ holds automatically for (t_1, u_2).

The proof is now reduced to checking $(^*)$ for (t_1, t_2). By linearity, this is, in turn, reduced to checking $(^*)$ in the case (χ_a, χ_b). Direct calculation shows that

$$T_{\chi_a}^{(r)} T_{\chi_b}^{(r)} = \exp\{b \cdot a / 4r\} T_{\chi_{a+b}}^{(r)} ,$$
$$\|T_{\chi_a}^{(r)}\|_{(r)} = \exp\{-|a|^2/8r\} .$$

It follows that

$$\left\| T_{\chi_a}^{(r)} T_{\chi_b}^{(r)} - T_{\chi_a \chi_b}^{(r)} + \frac{1}{r} T^{(r)}_{\sum_j (\partial_j \chi_a)(\bar\partial_j \chi_b)} \right\|_{(r)} \\ = \left| e^{b \cdot a/4r} - 1 - \frac{b \cdot a}{4r} \right| \exp\{-|a+b|^2/8r\}$$

and routine calculation now establishes $(^*)$.

4. Remarks

Direct calculation shows that

$$W_a^{(r)} = \exp\{|a|^2/8r\} T_{\chi_a}^{(r)}$$

is unitary, with

$$W_a^{(r)} W_b^{(r)} = \exp\{i \operatorname{Im} b \cdot a / 4r\} W_{a+b}^{(r)} .$$

It follows that the C^*-algebra generated by the $\{T^{(r)}_{\chi_a}: a \in \mathbb{C}^n\}$ is just the canonical commutation relation algebra $\mathrm{CCR}_r(\mathbb{C}^n)$ with respect to the symplectic form

$$\sigma(a,b) = \operatorname{Im} a \cdot b/2r$$

[BR, p. 20]. It is now easy to check that all the $\mathrm{CCR}_r(\mathbb{C}^n)$ for fixed n are *-isomorphic via the obvious spatial dilations. Following an argument in [BC$_2$], it is also easy to check that the C^*-algebra generated by the $\{T^{(r)}_f : f \in C_c^{2n+6}(\mathbb{C}^n)\}$ is exactly the full algebra $\mathscr{K}_r$ of compact operators on $H^2(\mathbb{C}^n, d\mu_r)$.

Using the fact that $\mathrm{CCR}_r(\mathbb{C}^n)$ is simple, we note that the sum

$$\mathrm{CCR}_r(\mathbb{C}^n) + \mathscr{K}_r$$

is closed, and, therefore, is a C^*-algebra. Combining this with the observations above, we have

Remark. The C^*-algebra generated by the

$$\{T^{(r)}_f : f \in \mathrm{TP} + C_c^{2n+6}\}$$

is just $\mathrm{CCR}_r(\mathbb{C}^n) + \mathscr{K}_r$ and these algebras are *-isomorphic for all $r > 0$.

We also note that, using the Poisson bracket on $\mathbb{C}^n$

$$\{f,g\} = i\sum_{j=1}^{n} (\partial_j f)(\bar{\partial}_j g) - i\sum_{j=1}^{n} (\bar{\partial}_j f)(\partial_j g),$$

we have, for $[A,B] = AB - BA$,

Corollary to Theorem 3. *For f, g in* $\mathrm{TP} + C_c^{2n+6}$,

$$(**) \qquad \left\| [T^{(r)}_f, T^{(r)}_g] - \frac{i}{r} T^{(r)}_{\{f,g\}} \right\|_{(r)} \le 2C(f,g) r^{-2}.$$

Considering the above results and the correponding results of [KL] for the hyperbolic disc, it is plausible that, in a very general framework involving Toeplitz operators on Bergman spaces, the appropriate generalization of (**) holds.

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